211en06

6

QUADRATIC EQUATIONS

In this lesson, you will study about quadratic equations. You will learn to identify quadratic equations from a collection of given equations and write them in standard form. You will also learn to solve quadratic equations and translate and solve word problems using quadratic equations.

OBJECTIVES

After studying this lesson, you will be able to

- *identify a quadratic equation from a given collection of equations;*
- *write quadratic equations in standard form;*
- *solve quadratic equations by (i) factorization and (ii) using the quadratic formula;*
- *solve word problems using quadratic equations.*

EXPECTED BACKGROUND KNOWLEDGE

- Polynomials
- Zeroes of a polynomial
- Linear equations and their solutions
- Factorisation of a polynomial

6.1 QUADRATIC EQUATIONS

You are already familiar with a polynomial of degree two. A polynomial of degree two is called a quadratic polynomial. When a quadratic polynomial is equated to zero, it is called **a quadratic equation**. In this lesson, you will learn about quadratic equations in one variable only. Let us consider some examples to identify a quadratic equation from a collection of equations

Example 6.1: Which of the following equations are quadratic equations?

(i) $3x^2 = 5$ (ii) $x^2 + 2x + 3 = 0$

(iii) $x^3 + 1 = 3x^2$ (iv) $(x + 1)(x + 3) = 2x + 1$

(v) $x + \frac{1}{x} = \frac{5}{2}$ (v) $x^2 + \sqrt{x} + 1 = 0$

Solution:

(i) It is a quadratic equation since $3x^2 = 5$ can be written as $3x^2 - 5 = 0$ and $3x^2 - 5$ is a quadratic polynomial.

(ii) $x^2 + 2x + 3 = 0$ is a quadratic equation as $x^2 + 2x + 3$, is a polynomial of degree 2.

(iii) $x^3 + 1 = 3x^2$ can be written as $x^3 - 3x^2 + 1 = 0$. LHS is not a quadratic polynomial since highest exponent of x is 3. **So, the equation is not a quadratic equation.**

(iv) $(x + 1)(x + 3) = 2x + 1$ is a quadratic equation, since $(x + 1)(x + 3) = 2x + 1$ can be written as

$$x^2 + 4x + 3 = 2x + 1$$

or $x^2 + 2x + 2 = 0$

Now, LHS is a polynomial of degree 2, hence $(x + 1)(x + 3) = 2x + 1$ is a quadratic equation.

(v) $x + \frac{1}{x} = \frac{5}{2}$ is not a quadratic equation.

However, it can be reduced to quadratic equation as shown below:

$$x + \frac{1}{x} = \frac{5}{2}$$

or $\frac{x^2 + 1}{x} = \frac{5}{2}, x \neq 0$

or $2(x^2 + 1) = 5x, x \neq 0$

or $2x^2 - 5x + 2 = 0, x \neq 0$

(vi) $x^2 + \sqrt{x} + 1 = 0$ is not a quadratic equation as $x^2 + \sqrt{x} + 1$ is not a quadratic polynomial (Why?)

CHECK YOUR PROGRESS 6.1

1. Which of the following equations are quadratic equations?

(i) $3x^2 + 5 = x^3 + x$ (ii) $\sqrt{3}\,x^2 + 5x + 2 = 0$

(iii) $(5y + 1)(3y - 1) = y + 1$ (iv) $\dfrac{x^2+1}{x+1} = \dfrac{5}{2}$

(v) $3x + 2x^2 = 5x - 4$

6.2 STANDARD FORM OF A QUADRATIC EQUATION

A quadratic equation of the form $ax^2 + bx + c = 0$, $a > 0$ where a, b, c, are constants and x is a variable is called a quadratic equation in the standard form. Every quadratic equation can always be written in the standard form.

Example 6.2: Which of the following quadratic equations are in standard form? Those which are not in standard form, express them in standard form.

(i) $2 + 3x + 5x^2 = 0$ (ii) $3x^2 - 5x + 2 = 0$

(iii) $7y^2 - 5y = 2y + 3$ (iv) $(z + 1)(z + 2) = 3z + 1$

Solution: (i) It is not in the standard form. Its standard form is $5x^2 + 3x + 2 = 0$

(ii) It is in standard form

(iii) It is not in the standard form. It can be written as

$7y^2 - 5y = 2y + 3$

or $7y^2 - 5y - 2y - 3 = 0$

or $7y^2 - 7y - 3 = 0$

which is now in the standard form.

(iv) It is not standard form. It can be rewritten as

$(z + 1)(z + 2) = 3z + 1$

or $z^2 + 3z + 2 = 3z + 1$

or $z^2 + 3z - 3z + 2 - 1 = 0$

or $z^2 + 1 = 0$

or $z^2 + 0z + 1 = 0$

which is now in the standard form.

CHECK YOUR PROGRESS 6.2

1. Which of the following quadratic equations are in standard form? Those, which are not in standard form, rewrite them in standard form:

   (i) $3y^2 - 2 = y + 1$ (ii) $5 - 3x - 2x^2 = 0$

   (iii) $(3t - 1)(3t + 1) = 0$ (iv) $5 - x = 3x^2$

6.3 SOLUTION OF A QUADRATIC EQUATION

You have learnt about the zeroes of a polynomial. A zero of a polynomial is that real number, which when substituted for the variable makes the value of the polynomial zero. In case of a quadratic equation, the value of the variable for which LHS and RHS of the equation become equal is called a root or solution of the quadratic equation. You have also learnt that if α is a zero of a polynomial p(x), then $(x - \alpha)$ is a factor fo p(x) and conversely, if $(x - \alpha)$ is a factor of a polynomial, then α is a zero of the polynomial. You will use these results in finding the solution of a quadratic equation. There are two algebraic methods for finding the solution of a quadratic equation. These are (i) Factor Method and (ii) Using the Quadratic Formula.

Factor Method

Let us now learn to find the solutions of a quadratic equation by factorizing it into linear factors. The method is illustrated through examples.

Example 6.3: Solve the equation $(x - 4)(x + 3) = 0$

Solution: Since, $(x - 4)(x + 3) = 0$, therefore,

either $x - 4 = 0$, or $x + 3 = 0$

or $x = 4$ or $x = -3$

Therefore, $x = 4$ and $x = -3$ are solutions of the equation.

Example 6.4: Solve the equation $6x^2 + 7x - 3 = 0$ by factorisation.

Solution: Given $6x^2 + 7x - 3 = 0$

By breaking middle term, we get

$6x^2 + 9x - 2x - 3 = 0$ [since, $6 \times (-3) = -18$ and $-18 = 9 \times (-2)$ and $9 - 2 = 7$]

or $3x(2x + 3) - 1(2x + 3) = 0$

or $(2x + 3)(3x - 1) = 0$

This gives $2x + 3 = 0$ or $3x - 1 = 0$

or $x = -\frac{3}{2}$ or $x = \frac{1}{3}$

Therefore, $x = -\frac{3}{2}$ and $x = \frac{1}{3}$ are solutions of the given equation.

Example 6.5: Solve $x^2 + 2x + 1 = 0$

Solution: We have $\quad x^2 + 2x + 1 = 0$

or $\quad (x + 1)^2 = 0$

or $\quad x + 1 = 0$

which gives $\quad x = -1$

Therefore, $x = -1$ is the only solution.

Note: In Examples 6.3 and 6.4, you saw that equations had two distinct solutions. However, in Example 6.5, you got only one solution. We say that it has two solutions and these are coincident.

CHECK YOUR PROGRESS 6.3

1. Solve the following equations using factor method.

   (i) $(2x + 3)(x + 2) = 0$ (ii) $x^2 + 3x - 18 = 0$

   (iii) $3x^2 - 4x - 7 = 0$ (iv) $x^2 - 5x - 6 = 0$

   (v) $25x^2 - 10x + 1 = 0$ (vi) $4x^2 - 8x + 3 = 0$

Quadratic Formula

Now you will learn to find a formula to find the solution of a quadratic equation. For this, we will rewrite the general quadratic equation $ax^2 + bx + c = 0$ by completing the square.

We have $ax^2 + bx + c = 0$

Multiplying both sides by '4a' to make the coefficient of x^2 a perfect square, of an even number, we get

$$4a^2x^2 + 4abx + 4ac = 0$$

or $\quad (2ax)^2 + 2(2ax)b + (b)^2 + 4ac = b^2 \qquad$ [adding b^2 to both sides]

or $\quad (2ax)^2 + 2(2ax)b + (b)^2 = b^2 - 4ac$

or $\quad (2ax + b)^2 = \left\{\pm\sqrt{b^2 - 4ac}\right\}^2$

or $\quad 2ax + b = \pm\sqrt{b^2 - 4ac}$

or $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

This gives two solutions of the quadratic equation $ax^2 + bx + c = 0$. The solutions (roots) are:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Here, the expression $(b^2 - 4ac)$, denoted by D, is called **Discriminant**, because it determines the number of solutions or nature of roots of a quadratic equation.

For a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, if

(i) $D = b^2 - 4ac > 0$, the equation has two real distinct roots, which are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

(ii) $D = b^2 - 4ac = 0$, then equation has two real equal roots, each equal to $\dfrac{-b}{2a}$

(iii) $D = b^2 - 4ac < 0$, the equation will not have any real root, since square root of a negative real number is not a real number.

Thus, a quadratic equation will have at the most two roots.

Example 6.6: Without determining the roots, comment on the nature (number of solutions) of roots of the following equations:

(i) $3x^2 - 5x - 2 = 0$

(ii) $2x^2 + x + 1 = 0$

(iii) $x^2 + 2x + 1 = 0$

Solution: (i) The given equation is $3x^2 - 5x - 2 = 0$. Comparing it with $ax^2 + bx + c = 0$, we get $a = 3$, $b = -5$ and $c = -2$.

Now $D = b^2 - 4ac \quad = (-5)^2 - 4 \times 3 \times (-2)$

$\qquad\qquad\qquad\quad = 25 + 24 = 49$

Since, $\quad D > 0$, the equation has two real distinct roots.

(ii) Comparing the equation $2x^2 + x + 1 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 2$, $b = 1$, $c = 1$

Now $D = b^2 - 4ac = (1)^2 - 4 \times 2 \times 1 = 1 - 8 = -7$

Since, $D = b^2 - 4ac < 0$, the equation does not have any real root.

(iii) Comparing the equation $x^2 + 2x + 1 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 1$, $b = 2$, $c = 1$

Now $D = b^2 - 4ac = (2)^2 - 4 \times 1 \times 1 = 0$

Since, $D = 0$, the equation has two equal roots.

Example 6.7: Using quadratic formula, find the roots of the equation $6x^2 - 19x + 15 = 0$

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

We get, $a = 6$, $b = -19$, $c = 15$

Now $\quad D = b^2 - 4ac \quad = (-19)^2 - 4 \times 6 \times 15$

$= 361 - 360 = 1$

Therefore, roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{19 \pm \sqrt{1}}{12} = \frac{19 \pm 1}{12}$$

So, roots are $\frac{19+1}{12} = \frac{5}{3}$ and $\frac{19-1}{12} = \frac{3}{2}$

Thus, the two roots are $\frac{5}{3}$ and $\frac{3}{2}$.

Example 6.8: Find the value of m sothat the equation $3x^2 + mx - 5 = 0$ has equal roots.

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

We have, $a = 3$, $b = m$, $c = -5$

For equal roots

$D = b^2 - 4ac \quad = 0$

or $\quad m^2 - 4 \times 3 \times (-5) = 0$

or $\quad m^2 = 60$

This gives $\qquad m = \pm 2\sqrt{15}$

Hence, for $m = \pm 2\sqrt{15}$, the equation will have equal roots.

CHECK YOUR PROGRESS 6.4

1. Without determining the roots, comment on nature of roots of following equations:

   (i) $3x^2 - 7x + 2 = 0$ (ii) $4x^2 - 12x + 9 = 0$ (iii) $25x^2 + 20x + 4 = 0$

   (iv) $x^2 - x + 1$

2. Solve the following equations using quadratic formula:

   (i) $y^2 - 14y - 12 = 0$ (ii) $x^2 - 5x = 0$ (iii) $x^2 - 15x + 50 = 0$

3. Find the value of m so that the following equations have equal roots:

   (i) $2x^2 - mx + 1 = 0$ (ii) $mx^2 + 3x - 5 = 0$

   (iii) $3x^2 - 6x + m = 0$ (iv) $2x^2 + mx - 1 = 0$

6.4 WORD PROBLEMS

We will now solve some problems which involve the use of quadratic equations.

Example 6.9: The sum of squares of two consecutive odd natural numbers is 74. Find the numbers.

Solution: Let two consecutive odd natural numbers be x and x + 2. Since, sum of their squares is 74. we have

$$x^2 + (x + 2)^2 = 74$$

or $x^2 + x^2 + 4x + 4 = 74$

or $2x^2 + 4x - 70 = 0$

or $x^2 + 2x - 35 = 0$

or $x^2 + 7x - 5x - 35 = 0$

or $x(x + 7) - 5(x + 7) = 0$

or $(x + 7)(x - 5) = 0$

Therefore $x + 7 = 0$ or $x - 5 = 0$

or $x = -7$ or $x = 5$

Now, x can not be negative as it is a natural number. Hence x = 5

So, the numbers are 5 and 7.

Example 6.10: The sum of the areas of two square fields is 468 m^2. If the difference of their perimeter is 24 m, find the sides of the two squares.

Solution: Let the sides of the bigger square be x and that of the smaller square be y.

Hence, perimeter of bigger square = 4x

and perimeter of smaller square = 4y

Therefore, $4x - 4y = 24$

or $x - y = 6$

or $x = y + 6$ ....(1)

Also, since sum of areas of two squares is 468 m^2

Therefore, $x^2 + y^2 = 468$ ....(2)

Substituting value of x from (1) into (2), we get

$(y + 6)^2 + y^2 = 468$

or $y^2 + 12y + 36 + y^2 = 468$

or $2y^2 + 12y - 432 = 0$

or $y^2 + 6y - 216 = 0$

Therefore $y = \dfrac{-6 \pm \sqrt{36+864}}{2} = \dfrac{-6 \pm \sqrt{900}}{2}$

or $y = \dfrac{-6 \pm 30}{2}$

Therefore, $y = \dfrac{-6+30}{2}$ or $\dfrac{-6-30}{2}$

or $y = 12$ or -18

Since, side of square can not be negative, so $y = 12$

Therefore, $x = y + 6 = 12 + 6 = 18$

Hence, sides of squares are 18 m and 12 m.

Example 6.11: The product of digits of a two digit number is 12. When 9 is added to the number, the digits interchange their places. Determine the number.

Solution: Let the digit at ten's place be x

and digit at unit's place be y

Therefore, number = $10x + y$

When digits are interchanged, the number becomes $10y + x$

Therefore $10x + y + 9 = 10y + x$

or $\quad 10x - x + y - 10y = -9$

or $\quad 9x - 9y = -9$

or $\quad x - y = -1$

or $\quad x = y - 1 \qquad \ldots(1)$

Also, product of digits is 12

Hence, $\quad xy = 12 \qquad \ldots(2)$

Substituting value of x from (1) into (2), we get

$$(y - 1)y = 12$$

or $\quad y^2 - y - 12 = 0$

or $\quad (y - 4)(y + 3) = 0$

Hence, $\quad y = 4$ or $y = -3$

Since, digit can not be negative, $y = 4$

Hence $\; x = y - 1 = 4 - 1 = 3$

Therefore, the number is 34.

Example 6.12: The sum of two natural numbers is 12. If sum of their reciprocals is $\frac{4}{9}$, find the numbers.

Solution: Let one number be x

Therefore, other number $= 12 - x$

Since, sum of their reciprocals is $\frac{4}{9}$, we get

$$\frac{1}{x} + \frac{1}{12 - x} = \frac{4}{9}, \quad x \neq 0, \; 12 - x \neq 0$$

or $\quad \dfrac{12 - x + x}{x(12 - x)} = \dfrac{4}{9}$

or $\quad \dfrac{12}{12x - x^2} = \dfrac{4}{9}$

or $\quad \dfrac{12 \times 9}{4} = 12x - x^2$

or $\quad 27 = 12x - x^2$

or $\quad x^2 - 12x + 27 = 0$

or $\quad (x - 3)(x - 9) = 0$

It gives $x = 3$ or $x = 9$

When first number x is 3, other number is $12 - 3 = 9$ and when first number x is 9, other number is $12 - 9 = 3$.

Therefore, the required numbers are 3 and 9.

CHECK YOUR PROGRESS 6.5

1. The sum of the squares of two consecutive even natural numbers is 164. Find the numbers.
2. The length of a rectangular garden is 7 m more than its breadth. If area of the garden is 144 m^2, find the length and breadth of the garden.
3. The sum of digits of a two digit number is 13. If sum of their squares is 89, find the number.
4. The digit at ten's place of a two digit number is 2 more than twice the digit at unit's place. If product of digits is 24, find the two digit number.
5. The sum of two numbers is 15. If sum of their reciprocals is $\frac{3}{10}$, find the two numbers.

LET US SUM UP

- An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ and a, b, c are real numbers is called a quadratic equation in standard form.
- The value(s) of the variable which satisfy a quadratic equation are called it roots or solutions.
- The zeros of a quadratic polynomial are the roots or solutions of the corresponding quadratic equation.
- If you can factorise $ax^2 + bx + c = 0$, $a \neq 0$, into product of linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$, can be obtained by equating each factor to zero.
- Roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- $b^2 - 4ac$ is called discriminant of the quadratic equation. $ax^2 + bx + c = 0$, $a \neq 0$ It is usually denoted by D.

  (i) If $D > 0$, then the quadratic equation has two real unequal (distinct) roots.

  (ii) If $D = 0$, then the quadratic equation has two equal (coincident) roots.

  (iii) If $D < 0$, then the quadratic equation has no real root.

TERMINAL EXERCISE

1. Which of the following are quadratic equations?

   (i) $y\left(\sqrt{5}y - 3\right) = 0$ (ii) $5x^2 - 3\sqrt{x} + 8 = 0$

   (iii) $3x - \frac{1}{x} = 5$ (iv) $x(2x + 5) = x^2 + 5x + 7$

2. Solve the following equations by factorisation method:

   (i) $(x - 8)(x + 4) = 13$ (ii) $3y^2 - 7y = 0$

   (iii) $x^2 + 3x - 18 = 0$ (iv) $6x^2 + x - 15 = 0$

3. Find the value of m for which $5x^2 - 3x + m = 0$ has equal roots.
4. Find the value of m for which $x^2 - mx - 1 = 0$ has equal roots.
5. Solve the following quadratic equations using quadratic formula:

   (i) $6x^2 - 19x + 15 = 0$ (ii) $x^2 + x - 1 = 0$

   (iii) $21 + x = 2x^2$ (iv) $2x^2 - x - 6 = 0$

6. The sides of a right angled triangle are $x - 1$, x and $x + 1$. Find the value of x and hence the sides of the triangle.
7. the sum of squares of two consecutive odd integers is 290. Find the integers.
8. The hypotenuse of a right angled triangle is 13 cm. If the difference of remaining two sides is 7 cm, find the remaining two sides.
9. The sum of the areas of two squares is 41 cm^2. If the sum of their perimeters is 36 cm, find the sides of the two squares.
10. A right angled isosceles triangle is inscribed in a circle of radius 5 cm. Find the sides of the triangle.

ANSWERS TO CHECK YOUR PROGRESS

6.1

1. (ii), (iii), (v)

6.2

1. (i) No, $3y^2 - y - 3 = 0$ (ii) No, $2x^2 + 2x - 5 = 0$

   (iii) No, $6t^2 + t - 1 = 0$ (iv) No, $3x^2 + x - 5 = 0$

6.3

1. (i) $\frac{3}{2}, -2$ (ii) $3, -6$ (iii) $\frac{7}{3}, -1$

   (iv) $2, 3$ (v) $\frac{1}{5}, \frac{1}{5}$ (vi) $\frac{3}{2}, \frac{1}{2}$

6.4

1. (i) Two real distinct roots

   (ii) Two real equal roots

   (iii) Two real equal roots

   (iv) No real roots

2. (i) $7 \pm \sqrt{37}$ (ii) 0, 5 (iii) 5, 10

3. (i) $\pm 2\sqrt{2}$ (ii) $\frac{9}{20}$ (iii) 3 (iv) For no value of m

6.5

1. 8, 10 2. 16m, 9m 3. 85, 58

4. 83 (v) 5, 10

ANSWERS TO TERMINAL EXERCISE

1. (i), (iv)

2. (i) 8, 4 (ii) $0, \frac{7}{3}$ (iii) $3, -6$ (iv) $\frac{3}{2}, -\frac{5}{3}$

3. $\frac{9}{20}$

4. For no value of m

5. (i) $\frac{3}{2}, \frac{5}{3}$ (ii) $\frac{-1 \pm \sqrt{5}}{2}$ (iii) $\frac{7}{2}, -3$ (iv) $2, \frac{3}{2}$

6. 3, 4, 5

7. 11, 13 or –13 , –11

8. 5 cm, 12 cm

9. 5 cm, 4 cm

10. $5\sqrt{2}$ cm, $5\sqrt{2}$ cm, 10 cm