20

MATRICES

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In the middle of the 19th Century, Arthur Cayley (1821-1895), an English mathematician created a new discipline of mathematics, called **matrices**. He used matrices to represent simultaneous system of equations. As of now, theory of matrices has come to stay as an important area of mathematics. The matrices are used in game theory, allocation of expenses, budgeting for by-products etc. Economists use them in social accounting, input-output tables and in the study of inter-industry economics. Matrices are extensively used in solving the simultaneous system of equations. Linear programming has its base in matrix algebra. Matrices have found applications not only in mathematics, but in other subjects like Physics, Chemistry, Engineering, Linear Programming etc.

In this lesson we will discuss different types of matrices and algebraic operations on matrices in details.

OBJECTIVES

After studying this lesson, you will be able to:

- define a matrix, order of a matrix and cite examples thereof;
- define and cite examples of various types of matrices-square, rectangular, unit, zero, diagonal, row, column matrix;
- state the conditions for equality of two matrices;
- define transpose of a matrix;
- define symmetric and skew symmetric matrices and cite examples;
- find the sum and the difference of two matrices of the same order;
- multiply a matrix by a scalar;
- state the condition for multiplication of two matrices; and
- multiply two matrices whenever possible.
- use elementary transformations
- find inverse using elementary trnsformations

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of number system
- Solution of system of linear equations

20.1 MATRICES AND THEIR REPRESENTATIONS

Suppose we wish to express that Anil has 6 pencils. We may express it as [6] or (6) with the understanding that the number inside [ ] denotes the number of pencils that Anil has. Next suppose that we want to express that Anil has 2 books and 5 pencils. We may express it as [2 5] with the understanding that the first entry inside [ ] denotes the number of books; while the second entry, the number of pencils, possessed by Anil.

Let us now consider, the case of two friends Shyam and Irfan. Shyam has 2 books, 4 notebooks and 2 pens; and Irfan has 3 books, 5 notebooks and 3 pens.

A convenient way of representing this information is in the tabular form as follows:

	Books	Notebooks	Pens
Shyam	2	4	2
Irfan	3	5	3

We can also briefly write this as follows:

$$\begin{array}{lccc} & \text{First Column} & \text{Second Column} & \text{Third Column} \\ & \downarrow & \downarrow & \downarrow \\ \text{First Row} & 2 & 4 & 2 \\ \text{Second Row} & 3 & 5 & 3 \end{array}$$

This representation gives the following information:

(1) The entries in the first and second rows represent the number of objects (Books, Notebooks, Pens) possessed by Shyam and Irfan, respectively

(2) The entries in the first, second and third columns represent the number of books, the number of notebooks and the number of pens, respectively.

Thus, the entry in the first row and third column represents the number of pens possessed by Shyam. Each entry in the above display can be interpreted similarly.

The above information can also be represented as

	Shyam	Irfan
Books	2	3
Notebooks	4	5
Pens	2	3

which can be expressed in three rows and two columns as given below:

$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{bmatrix}$ The arrangement is called a **matrix.** Usually, we denote a matrix by a capital letter of English alphabets, i.e. *A*, *B*, *X*, etc. Thus, to represent the above information in the form of a matrix, we write

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{bmatrix} \text{ or } \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}$$

Note: Plural of matrix is matrices.

20.1.1 Order of a Matrix Observe the following matrices (arrangement of numbers):

(a) $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & i \\ i & 1+i \\ 1+i & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & -1 & -2 \\ 2 & 3 & 4 & 5 \\ 4 & -1 & -2 & 0 \end{pmatrix}$

In matrix (a), there are two rows and two columns, this is called a 2 by 2 matrix or a matrix of order 2×2. This is written as 2×2 matrix. In matrix (b), there are three rows and two columns. It is a 3 by 2 matrix or a matrix of order 3×2. It is written as 3×2 matrix. The matrix (c) is a matrix of order 3×4.

Note that there may be any number of rows and any number of columns in a matrix. If there are *m* rows and *n* columns in matrix *A*, its order is $m \times n$ and it is read as an $m \times n$ matrix.

Use of two suffixes *i* and *j* helps in locating any particular element of a matrix. In the above $m \times n$ matrix, the element a_{ij} belongs to the *i*th row and *j*th column.

$$A = \begin{bmatrix} a_{11}\, a_{12}\, a_{13} \cdots a_{1j} & \cdots a_{1n} \\ a_{21}\, a_{22}\, a_{23} \cdots a_{2j} & \cdots a_{2n} \\ a_{31}\, a_{32}\, a_{33} \cdots a_{3j} & \cdots a_{3n} \\ a_{i1}\, a_{i2}\, a_{i3} \cdots a_{ij} & \cdots a_{in} \\ a_{m1}\, a_{m2}\, a_{m3} \cdots a_{mj} & \cdots a_{mn} \end{bmatrix}$$

A matrix of order $m \times n$ can also be written as

$A = [a_{ij}]$, $i = 1, 2, ..., m$; and
$j = 1, 2, ..., n$

Example 20.1 Write the order of each of the following matrices:

(i) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 10 \end{bmatrix}$

Solution: The order of the matrix

(i) is 2×2 (ii) is 3×1

(iii) is 1×3 (iv) is 2×3

Example 20.2 For the following matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 3 & 2 & 5 \\ 3 & 2 & 3 & 6 \end{bmatrix}$$

(i) find the order of A

(ii) write the total number of elements in A

(iii) write the elements a_{23}, a_{32}, a_{14} and a_{34} of A

(iv) express each element 3 in A in the form a_{ij}.

Solution: The order of the matrix

(i) Since A has 3 rows and 4 columns, A is of order 3×4.

(ii) number of elements in $A = 3 \times 4 = 12$

(iii) $a_{23} = 2;\ a_{32} = 2;\ a_{14} = 4$ and $a_{34} = 6$

(iv) a_{22}, a_{31} and a_{33}

Example 20.3 If the element in the ith row and jth column of a 2×3 matrix A is given by $\frac{i+2j}{2}$, write the matrix A.

Solution: Here, $a_{ij} = \frac{i+2j}{2}$ (Given)

$$a_{11} = \frac{1+2\times1}{2} = \frac{3}{2};\quad a_{12} = \frac{1+2\times2}{2} = \frac{5}{2};\quad a_{13} = \frac{1+2\times3}{2} = \frac{7}{2}$$

$$a_{21} = \frac{2+2\times1}{2} = 2;\quad a_{22} = \frac{2+2\times2}{2} = 3;\quad a_{23} = \frac{2+2\times3}{2} = 4$$

Thus, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 2 & 3 & 4 \end{bmatrix}$

Example 20.4 There are two stores A and B. In store A, there are 120 shirts, 100 trousers and 50 cardigans; and in store B, there are 200 shirts, 150 trousers and 100 cardigans. Express this information in tabular form in two different ways and also in the matrix form.

Solution:

Tabular Form 1

	Shirts	Trousers	Cardigans
Store A	120	100	50
Store B	200	150	100

Matrix Form

$$\Rightarrow \begin{pmatrix} 120 & 100 & 50 \\ 200 & 150 & 100 \end{pmatrix}$$

Tabular Form 2

	Store A	Store B
Shirts	120	200
Trousers	100	150
Cardigans	50	100

Matrix Form

$$\Rightarrow \begin{pmatrix} 120 & 200 \\ 100 & 150 \\ 50 & 100 \end{pmatrix}$$

CHECK YOUR PROGRESS 20.1

1. Marks scored by two students A and B in three tests are given in the adjacent table. Represent this information in the matrix form, in two ways

	Test 1	Test 2	Test 3
A	56	65	71
B	29	37	57

2. Three firms X, Y and Z supply 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of sand respectively, to a contractor. Express this information in the matrix form in two ways.

3. In family P, there are 4 men, 6 women and 3 children; and in family Q, there are 4 men, 3 women and 5 children. Express this information in the form of a matrix of order 2×3.

4. How many elements in all are there in a

   (a) 2×3 matrix (b) 3×4 matrix (c) 4×2 matrix

   (d) 6×2 matrix (e) $a \times b$ matrix (f) $m \times n$ matrix

5. What are the possible orders of a matrix if it has

(a) 8 elements (b) 5 elements (c) 12 elements (d) 16 elements

6. In the matrix A,

$$A = \begin{bmatrix} 5 & 1 & 8 & 0 & 5 \\ 7 & 6 & 7 & 4 & 6 \\ 3 & 9 & 3 & -3 & 9 \\ 4 & 4 & 8 & 5 & 1 \end{bmatrix}$$

find: (a) number of rows;

(b) number of columns;

(c) the order of the matrix A;

(d) the total number of elements in the matrix A;

(e) $a_{14}, a_{23}, a_{34}, a_{45}$ and a_{33}

7. Construct a 3×3 matrix whose elements in the ith row and jth column is given by

(a) $i - j$ (b) $\frac{i^2}{j}$ (c) $\frac{(i+2j)^2}{2}$ (d) $3j - 2i$

8. Construct a 3×2 matrix whose elements in the ith row and jth column is given by

(a) $i + 3j$ (b) $5.i.j.$ (c) i^j (d) $i + j - 2$

20.2 TYPES OF MATRICES

Row Matrix : A matrix is said to be a *row matrix* if it has only one row, but may have any number of columns, e.g. the matrix $\begin{bmatrix} 1 & 6 & 0 & 1 & 2 \end{bmatrix}$ is a row matrix.

T[he order of a row matrix is $1 \times n$.

Column Matrix : A matrix is said to be a *column matrix* if it has only one column, but may have any number of rows, e.g. the matrix $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 7 \end{bmatrix}$ is a column matrix. The order of a column matrix is $m \times 1$

Square Matrix : A matrix is said to be a *square matrix* if number of rows is equal to the number of columns, e.g. the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 1 \\ 3 & 4 & 2 \end{bmatrix}$ having 3 rows and 3 columns is a square matrix. The order of a square matrix is $n \times n$ or simply n.

The diagonal of a square matrix from the top extreme left element to the bottom extreme right element is said to be the principal diagonal. The principal diagonal of the matrix $\begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 7 \\ 3 & 8 & 9 \end{bmatrix}$ contains elements 2, 1 and 9.

Note: In any given matrix $A = [a_{ij}]$ of order $m \times n$, the elements of the principal diagonal are $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$

Rectangular Matrix : A matrix is said to be a *rectangular matrix* if the number of rows is not equal to the number of columns, e.g. the matrix $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 0 \\ -1 & 2 & 1 & 3 \end{bmatrix}$ having 3 rows and 4 columns is a rectangular matrix.It may be noted that a row matrix of order $1 \times n \ (n \neq 1)$ and a column matrix of order $m \times 1 \ (m \neq 1)$ are rectangular matrix.

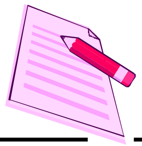
Zero or Null Matrix : A matrix each of whose element is zero is called a *zero or null matrix,* e.g. each of the matrix

$$[0 \quad 0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a zero matrix. Zero matrix is denoted by O.

Note: A zero matrix may be of any order $m \times n$.

Diagonal Matrix : A square matrix is said to be a *diagonal matrix*, if all elements other than those occuring in the principal diagonal are zero, i.e., if $A = [a_{ij}]$ is a square matrix of order m $\times$ n, then it is said to be a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.



For example, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ are diagonal matrices.

Note: A diagonal matrix $A = [a_{ij}]_{n \times n}$ is also written as $A =$ diag $[a_{11}, a_{12}, a_{13}, \ldots, a_{nn}]$

Scalar Matrix : A diagonal matrix is said to be a *scalar matrix* if all the elements in its principal diagonal are equal to some non-zero constant, say k e.g., the matrix $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a scalar matrix.

Note: A square zero matrix is not a scalar matrix.

Unit or Identity Matrix : A scalar matrix is said to be a *unit or identity matrix*, if all of its elements in the principal diagonal are unity. It is denoted by I_n, if it is of order n e.g., the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Note: A square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = \begin{cases} 0, \text{when } i \neq j \\ 1, \text{when } i = j \end{cases}$

Equal Matrices : Two matrices are said to be equal if they are of the same order and if their corresponding elements are equal.

If A is a matrix of order $m \times n$ and B is a matrix of order $p \times r$, then $A = B$ if

(1) $m = p$; $n = r$; and

(2) $a_{ij} = b_{ij}$ for all 3×2 and $j = 1, 2, 3, \ldots, n$

Two matrices X and Y given below are not equal, since they are of different orders, namely 2×3 and 3×2 respectively.

$$X = \begin{bmatrix} 7 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix}, \quad Y = \begin{bmatrix} 7 & 2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}$$

Also, the two matrices P and Q are not equal, since some elements of P are not equal to the corresponding elements of Q.

$$P = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & 3 & 6 \\ 0 & 2 & 1 \end{bmatrix}$$

Example 20.5 Find whether the following matrices are equal or not:

(i) $A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}$

(ii) $P = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $X = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}$

Solution:

(i) Matrices A and B are of the same order 2×2. But some of their corresponding elements are different. Hence, $A \neq B$.

(ii) Matrices P and Q are of different orders, So, $P \neq Q$.

(iii) Matrices X and Y are of the same order 3×3, and their corresponding elements are also equal.

So, $X = Y$.

Example 20.6 Determine the values of x and y, if

(i) $\begin{bmatrix} x & 5 \end{bmatrix} = \begin{bmatrix} 2 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ y \end{bmatrix}$ (iii) $\begin{bmatrix} x & 2 \\ 3 & -y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

Solution: Since the two matrices are equal, their corresponding elements should be equal.

(i) $x = 2$ (ii) $x = 4, \; y = 3$ (iii) $x = 1, \; y = -5$

Example 20.7 For what values of a, b, c and d, are the following matrices equal?

(i) $A = \begin{bmatrix} a & -2 & 2b \\ 6 & 3 & d \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 4 \\ 6 & 5c & 2 \end{bmatrix}$

(ii) $P = \begin{bmatrix} a & b-2d \\ -3 & 2b \\ a+c & 7 \end{bmatrix}, \quad Q = \begin{bmatrix} 5 & 1 \\ -3 & 6 \\ 4 & 7 \end{bmatrix}$

Solution:

(i) The given matrices A and B will be equal only if their corresponding elements are equal, i.e. if

$$a=1,\ 2b=4,\ 3=5c, \text{ and } d=2$$

$$\Rightarrow a=1,\ b=2,\ c=\frac{3}{5} \text{ and } d=2$$

Thus, for $a = 1,\ b = 2,\ c = \frac{3}{5}$ and $d = 2$ matrices A and B are equal.

(ii) The given matrices P and Q will be equal if their corresponding elements are equal, i.e. if

$$2b = 6,\ b-2d = 1,\ a = 5 \text{ and } a+c = 4$$

$$\Rightarrow a = 5,\ b = 3,\ c = -1 \text{ and } d = 1$$

Thus, for $a = 5,\ b = 3,\ c = -1$ and $d = 1$, matrices P and Q are equal.

CHECK YOUR PROGRESS 20.2

1. Which of the following matrices are

(a) row matrices (b) column matrices (c) square matrices (d) diagonal matrices (e) scalar matrices (f) identity matrices and (g) zero matrices

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 7 \\ 8 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$E = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 9 & 8 \\ 1 & 0 & 2 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G = [3 \quad 4 \quad 10 \quad 8], \; H = \begin{pmatrix} 2 & 3 & 7 \\ 1 & 4 & 9 \end{pmatrix}, \; I = \begin{pmatrix} 2 & -1 \\ 3 & 2 \\ -1 & 0 \end{pmatrix}$$

2. Find the values of a, b, c and d if

(a) $\begin{bmatrix} b & 2c \\ b+d & c-2a \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 8 & 2 \end{bmatrix}$ (b) $\begin{pmatrix} a+2 & 4 \\ b+3 & 25 \end{pmatrix} = \begin{pmatrix} 4 & 2c \\ 6 & 5d \end{pmatrix}$

(c) $\begin{pmatrix} 2a & b \\ -4 & 6 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ d & 3c \end{pmatrix}$

3. Can a matrix of order 1×2 be equal to a matrix of order 2×1?

4. Can a matrix of order 2×3 be equal to a matrix of order 3×3?

20.3 TRANSPOSE OF A MATRIX

Associated with each given matrix there exists another matrix called its *transpose*. The transpose of a given matrix A is formed by interchanging its rows and columns and is denoted by A' or A^t, e.g. if

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 3 \\ 7 & 6 & 1 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 0 & 6 \\ -3 & 3 & 1 \end{bmatrix}$$

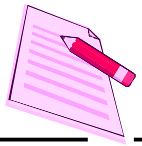
In general, If $A = [a_{ij}]$ is an $m \times n$ matrix, then the transpose A' of A is the $n \times m$ matrix; and, (a_{ij})th element of $A = (a_{ji})$th element of A'

20.3.1 Symmetric Matrix

A square matrix A is said to be a *symmetric matrix* if $A' = A$.

For example,

$$\text{If } A = \begin{bmatrix} 2 & 3i & 1-i \\ 3i & 4 & 2i \\ 1-i & 2i & 5 \end{bmatrix}, \text{ then } A' = \begin{pmatrix} 2 & 3i & 1-i \\ 3i & 4 & 2i \\ 1-i & 2i & 5 \end{pmatrix}$$



Notes

Since $A' = A$, A is a symmetric matrix.

Note: (1) In a symmetric matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = a_{ji}$ for all i and j

(2) A rectangular matrix can never be symmetric.

20.3.2 Skew-Symmetric Matrix

A square matrix A is said to be a *skew symmetric* if $A' = -A$, i.e. $a_{ij} = -a_{ji}$ for all i and j.

For example,

If $A = \begin{bmatrix} 0 & c & d \\ -c & 0 & f \\ -d & -f & 0 \end{bmatrix}$, *then* $A' = \begin{bmatrix} 0 & -c & -d \\ c & 0 & -f \\ d & f & 0 \end{bmatrix}$

But $A' = \begin{bmatrix} 0 & -c & -d \\ c & 0 & -f \\ d & f & 0 \end{bmatrix}$, which is the same as A'

$A' = -A$

Hence, A is a skew symmetric matrix

Note: In a skew symmetric matrix $A = \left[a_{ij}\right]_{n \times n}$, $a_{ij} = 0$, for $i = j$

i.e. all elements in the principal diagonal of a skew symmetric matrix are zeroes.

20.4 SCALAR MULTIPLICATION OF A MATRIX

Let us consider the following situation:

The marks obtained by three students in English, Hindi, and Mathematics are as follows:

	English	Hindi	Mathematics
Elizabeth	20	10	15
Usha	22	25	27
Shabnam	17	25	21

It is also given that these marks are out of 30 in each case. In matrix form, the above information can be written as

$$\begin{bmatrix} 20 & 10 & 15 \\ 22 & 25 & 27 \\ 17 & 25 & 21 \end{bmatrix}$$

(It is understood that rows correspond to the names and columns correspond to the subjects)

If the maximum marks are doubled in each case, then the marks obtained by these girls will also be doubled. In matrix form, the new marks can be given as:

$$\begin{pmatrix} 2\times 20 & 2\times 10 & 2\times 15 \\ 2\times 22 & 2\times 25 & 2\times 27 \\ 2\times 17 & 2\times 25 & 2\times 21 \end{pmatrix} \quad \text{which is equal to} \quad \begin{pmatrix} 40 & 20 & 30 \\ 44 & 50 & 54 \\ 34 & 50 & 42 \end{pmatrix}$$

So, we write that

$$2\times\begin{pmatrix} 20 & 10 & 15 \\ 22 & 25 & 27 \\ 17 & 25 & 21 \end{pmatrix} = \begin{pmatrix} 2\times 20 & 2\times 10 & 2\times 15 \\ 2\times 22 & 2\times 25 & 2\times 27 \\ 2\times 17 & 2\times 25 & 2\times 21 \end{pmatrix} = \begin{pmatrix} 40 & 20 & 30 \\ 44 & 50 & 54 \\ 34 & 50 & 42 \end{pmatrix}$$

Now consider another matrix

$$A = \begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{pmatrix}$$

Let us see what happens, when we multiply the matrix A by 5

$$\text{i.e.} \quad 5\times A = 5A = 5\times\begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 5\times 3 & 5\times 2 \\ 5\times(-2) & 5\times 0 \\ 5\times 1 & 5\times 6 \end{pmatrix} = \begin{pmatrix} 15 & 10 \\ -10 & 0 \\ 5 & 30 \end{pmatrix}$$

When *a* matrix is multiplied by a scalar, then each of its element is multiplied by the same scalar.

For example,

$$\text{if} \quad A = \begin{pmatrix} 2 & -1 \\ 6 & 3 \end{pmatrix} \quad \text{then} \quad kA = \begin{pmatrix} k\times 2 & k\times(-1) \\ k\times 6 & k\times 3 \end{pmatrix} = \begin{pmatrix} 2k & -k \\ 6k & 3k \end{pmatrix}$$

$$\text{When } k = -1,\ kA = (-1)A = \begin{pmatrix} -2 & 1 \\ -6 & -3 \end{pmatrix}$$

So, $(-1)A = -A$

Thus, if $A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & 1 \\ -6 & -3 \end{bmatrix}$

Example 20.8 If $A = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix}$, find

(i) $2A$ (ii) $\frac{1}{2}A$ (iii) $-A$ (iv) $\frac{2}{3}A$

Solution:

(i) Here, $2A = 2 \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\times(-2) & 2\times 3 & 2\times 4 \\ 2\times(-1) & 2\times 0 & 2\times 1 \end{bmatrix} = \begin{bmatrix} -4 & 6 & 8 \\ -2 & 0 & 2 \end{bmatrix}$

(ii) $\frac{1}{2}A = \frac{1}{2} \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\times(-2) & \frac{1}{2}\times 3 & \frac{1}{2}\times 4 \\ \frac{1}{2}\times(-1) & \frac{1}{2}\times 0 & \frac{1}{2}\times 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} & 2 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

(iii) $-A = (-1)\times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -1 \end{bmatrix}$

iv) $\frac{2}{3}A = \frac{2}{3} \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 2 & \frac{8}{3} \\ -\frac{2}{3} & 0 & \frac{2}{3} \end{bmatrix}$

CHECK YOUR PROGRESS 20.3

1. If $A = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$, find:

   (a) $4A$ (b) $-A$ (c) $\frac{1}{2}A$ (d) $-\frac{3}{2}A$

2. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, find:

   (a) $5A$ (b) $-3A$ (c) $\frac{1}{3}A$ (d) $-\frac{1}{2}A$

3. If A= $\begin{bmatrix} -1 & 0 \\ 4 & 2 \\ 0 & -1 \end{bmatrix}$, find (–7)$A$

4. If X = $\begin{pmatrix} 3 & 0 & 1 \\ 4 & -2 & 0 \\ -1 & 0 & 5 \end{pmatrix}$, find:

(a) 5X (b) $-4X$ (c) $\frac{1}{3}X$ (d) $-\frac{1}{2}X$

5. Find A' (transpose of A):

(a) $A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ (b) $A = \begin{pmatrix} 4 & 10 & 9 \\ 6 & 8 & 7 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & -2 \\ 4 & -1 \\ -6 & 9 \end{pmatrix}$ (d) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. For any matrix A, prove that $(A')' = A$

7. Show that each of the following matrices is a symmetric matrix:

(a) $\begin{bmatrix} 2 & -4 \\ -4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 2 & -3 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. Show that each of the following matrices is a skew symmetric matrix:

(a) $\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$ (b) $\begin{bmatrix} 0 & i & 4 \\ -i & 0 & 2-i \\ -4 & -2+i & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 5 \\ -7 & -5 & 0 \end{bmatrix}$

20.5 ADDITION OF MATRICES

Two students A and B compare their performances in two tests in Mathematics, Physics and English. The maximum marks in each test in each subject are 50. The marks scored by them are as follows:

First Test Second Test

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 50 & 38 & 33 \\ 47 & 40 & 36 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 45 & 32 & 30 \\ 42 & 30 & 39 \end{bmatrix} \end{array}$$

How can we find their total marks in each subject in the two tests taken together ?

Observe that the new matrix giving the combined information of two matrices

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 50+45 & 38+32 & 33+30 \\ 47+42 & 40+30 & 36+39 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 95 & 70 & 63 \\ 89 & 70 & 75 \end{bmatrix} \end{array}$$

This new matrix is called the *sum* of the given matrices.

If A and B are any two given matrices of the same order, then their sum is defined to be a matrix C whose respective elements are the sum of the corresponding elements of the matrices A and B and we write this as $C = A + B$.

1. The order of the matrix C will also be the same as that of A and B.
2. It is not possible to add two matrices of different orders.

Example 20.9 If 2×2 and $B = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}$, then find $A + B$.

Solution: Since the given matrices A and B are of the same order, i.e. 2×2, we can add them. So,

$$A + B = \begin{bmatrix} 1+5 & 3+2 \\ 4+1 & 2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 \\ 5 & 2 \end{bmatrix}$$

Example 20.10 If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 4 \\ 1 & 2 & 1 \end{bmatrix}$, then find $A + B$.

Solution: Since the given matrices A and B are of the same order, i.e. 2×2, we can add them. So,

$$A + B = \begin{bmatrix} 0+3 & 1+0 & -1+4 \\ 2+1 & 3+2 & 0+1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}$$

20.5.1 Properties of Addition

Recall that in case of numbers, we have

(i) $x + y = y + x,$ i.e., addition is commutative

(ii) $x + (y + z) = (x + y) + z,$ i.e., addition is associative

(iii) $x + 0 = x,$ i.e., additive identity exists

(iv) $x + (-x) = 0$, i.e., additive inverse exists

Let us now find if these properties hold true in case of matrices too:

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$, Then,

$$A + B = \begin{bmatrix} 1+0 & 2-2 \\ -1+1 & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

and

$$B + A = \begin{bmatrix} 0+1 & -2+2 \\ 1+(-1) & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

We see that $A + B$ and $B + A$ denote the same matrix. Thus, in general,

For any two matrices A and B of the same order, $A + B = B + A$

i.e. matrix addition is commutative

Let $A = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$. Then,

$$A+(B+C) = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 1+1 & -4+0 \\ 0+2 & 2+3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2 & 3+(-4) \\ -2+2 & 1+5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}$$

and $$(A+B)+C = \begin{bmatrix} 0+1 & 3+(-4) \\ -2+0 & 1+2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & -1+0 \\ -2+2 & 3+3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}$$

We see that $A + (B + C)$ and $(A + B) + C$ denote the same matrix. Thus, in general

For any three matrices A, B and C of the same order,

$A + (B + C) = (A + B) + C$ i.e., matrix addition is associative.

Recall that we have talked about zero matrix. A zero matrix is that matrix, all of whose elements are zeroes. It can be of any order.

Let $A = \begin{bmatrix} 2 & -2 \\ 4 & 5 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then,

$$A+O = \begin{bmatrix} 2+0 & -2+0 \\ 4+0 & 5+0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & 5 \end{bmatrix} = A$$

and $$O+A = \begin{bmatrix} 0+2 & 0-2 \\ 0+4 & 0+5 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & 5 \end{bmatrix} = A$$

We see that $A + O$ and $O + A$ denote the same matrix A.

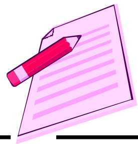
Thus, we find that $A + O = A = O + A$, where O is a zero matrix.

The matrix O, which is a zero matrix, is called the additive identity.

Additive identity is a zero matrix, which when added to a given matrix, gives the same given matrix, i.e., $A + O = A = O + A$.

Example 20.11 If $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, then

find: (a) $A + B$ (b) $B + C$ (c) $(A + B) + C$ (d) $A + (B + C)$



Solution:

(a) $A+B=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}2+(-3) & 0+1\\ 1+1 & 3+2\end{bmatrix}=\begin{bmatrix}-1 & 1\\ 2 & 5\end{bmatrix}$

(b) $B+C=\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}(-3)+(-1) & 1+0\\ 1+0 & 2+3\end{bmatrix}=\begin{bmatrix}-4 & 1\\ 1 & 5\end{bmatrix}$

(c) $(A+B)+C=\begin{bmatrix}-1 & 1\\ 2 & 5\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$ ... [From (a)]

$= \;=\begin{bmatrix}(-1)+(-1) & 1+0\\ 2+0 & 5+3\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 2 & 8\end{bmatrix}$

(d) $A+(B+C)=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+\begin{bmatrix}-4 & 1\\ 1 & 5\end{bmatrix}$ ... [From (b)]

$=\begin{bmatrix}2+(-4) & 0+1\\ 1+1 & 3+5\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 2 & 8\end{bmatrix}$

Example 20.12 If $A=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$ and $O=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

then find (a) $A+O$ (b) $O+A$

What do you observe?

Solution: (a) $A+O=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

$=\begin{bmatrix}-2+0 & 3+0 & 5+0\\ 1+0 & -1+0 & 0+0\end{bmatrix}=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$

(b) $O+A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$

$=\begin{bmatrix}0+(-2) & 0+3 & 0+5\\ 0+1 & 0+(-1) & 0+0\end{bmatrix}=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$

From (a) and (b), we see that

$A+O=O+A=A$

MODULE - VI
Algebra -II

Notes

20.6 SUBTRACTION OF MATRICES

Let A and B two matrices of the same order. Then the matrix A–B is defined as the subtraction of B from A. A–B is obtained by subtracting corresponding elements of B from the corresponding elements of A.

We can write $A-B = A+(-B)$

Note : $A-B$ and $B-A$ do not denote the same matrix, except when $A = B$.

Example 20.13 If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ then find

(a) $A-B$ (b) $B-A$

Solution : (a) We know that

$$A-B = A+(-B)$$

Since $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, we have $-B = \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$

Substituting it in (i), we get

$$A-B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & 0+(-2) \\ 2+(-1) & (-1)+(-4) \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$$

(b) Similarly,

$$B-A = B+(-A)$$

$$B-A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3+(-1) & 2+0 \\ 1+(-2) & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$$

Remarks : To obtain $A-B$, we can subtract directly the elements of B from the corresponding elements of A. Thus,

$$A-B = \begin{bmatrix} 1-3 & 0-2 \\ 2-1 & -1-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$$

and $$B-A = \begin{bmatrix} 3-1 & 2-0 \\ 1-2 & 4-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$$

Example 20.14 If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$; $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A+B = O$, find B.

Solution : Here, it is given that A+B = O

$$\therefore \quad \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2+a & 3+b \\ -1+c & 4+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad 2+a = 0 \quad ; \quad 3+b=0$$

$$-1+c=0 \quad ; \quad 4+d=0$$

$$\Rightarrow \quad a = -2; \quad b = -3; \quad c = 1 \text{ and } d = -4$$

$$\therefore \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & -2 \end{bmatrix}$$

In general, given a matrix *A*, there exists another matrix *B* = (− 1) A such that *A* + *B*= *O*, then such a matrix *B* is called the additive inverse of the matrix of *A*.

CHECK YOUR PROGRESS 20.4

1. If $A = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$ then find :

   (a) $A+B$ (b) $2A+B$ (c) $A+3B$ (d) $2A+3B$

2. If $P = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 1 & -5 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 1 & -5 \end{bmatrix}$, then find :

   (a) P−Q (b) Q−P (c) P− 2Q (d) 2Q−3P

3. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & -1 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -4 & 0 \\ 1 & 6 & 1 \\ 2 & 0 & 7 \end{bmatrix}$, then find:

   (a) $A+B$ (b) $A-B$ (c) $-A+B$ (d) $3A+2B$

4. If $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$, find the zero matrix O satisfying A+O = A.

5. If $A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \\ -4 & 0 & 1 \end{bmatrix}$ then find :

   (a) −A (b) A + (−A) (c) (−A)+A

6. If $A = \begin{bmatrix} 1 & 9 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 7 & 9 \end{bmatrix}$, then find :

   (a) 2A (b) 3B (c) 2A+3B (d) If 2A+3B+5X=O, what is X ?

7. If $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -4 & 0 \end{bmatrix}$, then find :

   (a) A′ (b) B′ (c) A+B (d) (A+B)′ (e) A′+B′

   What do you observe ?

8. If $A = \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, then find :

   (a) A−B (b) B−C (c) A−C (d) 3B−2C (e) A−B−C (f) 2A−B−3C

20.7 MULTIPLICATION OF MATRICES

Salina and Rakhi are two friends. Salina wants to buy 17 kg wheat, 3 kg pulses and 250gm ghee; while Rakhi wants to buy 15 kg wheat, 2kg pulses and 500 gm ghee. The prices of wheat, pulses and ghee per kg respectively are Rs. 8.00, Rs. 27.00 and Rs. 90.00.How much money will each spend? Clearly, the money needed by Salina and Rakhi will be :

Salina

Cost of 17 kg wheat	$\Rightarrow 17 \times$ Rs. 8	= Rs. 136.00
Cost of 3 kg pulses	$\Rightarrow 3 \times$ Rs. 27	= Rs. 81.00
Cost of 250 gm ghee	$\Rightarrow \frac{1}{4} \times$ Rs. 90	= Rs. 22.50
	Total	= Rs. 239.50

Rakhi

Cost of 15 kg wheat $\Rightarrow 15 \times$ Rs. 8 $=$ Rs. 120.00

Cost of 2 kg pulses $\Rightarrow 2 \times$ Rs. 27 $=$ Rs. 54.00

Cost of 500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 90 $=$ Rs. 45.00

Total $=$ Rs. 219.00

In matrix form, the above information can be represented as follows:

Requirements Price Money Needed

$$\begin{bmatrix} \text{Wheat} & \text{pulses} & \text{ghee} \\ 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 8 \\ 27 \\ 90 \end{bmatrix} = \begin{bmatrix} 17 \times 8 + 3 \times 27 + 0.250 \times 90 \\ 15 \times 8 + 2 \times 27 + 0.500 \times 90 \end{bmatrix} = \begin{bmatrix} 239.50 \\ 219.00 \end{bmatrix}$$

Another shop in the same locality quotes the following prices.
Wheat : Rs. 9 per kg.; pulses : Rs.26 per kg; ghee : Rs. 100 per kg.

The money needed by Salina and Rakhi to buy the required quantity of articles from this shop will be

Salina

17 kg wheat $\Rightarrow 17 \times$ Rs. 9 $=$ Rs. 153.00

3 kg pulses $\Rightarrow 3 \times$ Rs. 26 $=$ Rs. 78.00

250 gm ghee $\Rightarrow \frac{1}{4} \times$ Rs. 100 $=$ Rs. 25.00

Total $=$ Rs. 256.00

Rakhi

15 kg wheat $\Rightarrow 15 \times$ Rs. 9 $=$ Rs. 135.00

2 kg pulses $\Rightarrow 2 \times$ Rs. 26 $=$ Rs. 52.00

500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 100 $=$ Rs. 50.00

Total $=$ Rs. 237.00

In matrix form, the above information can be written as follows :

Requirements Price Money needed

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 9.00 \\ 26.00 \\ 100.00 \end{bmatrix} = \begin{bmatrix} 17 \times 9.00 + 3 \times 26.00 + 0.250 \times 100 \\ 15 \times 900 + 2 \times 26.00 + 0.500 \times 100 \end{bmatrix} = \begin{bmatrix} 256.00 \\ 237.00 \end{bmatrix}$$

To have a comparative study, the two information can be combined in the following way:

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 27 & 26 \\ 90 & 100 \end{bmatrix} = \begin{bmatrix} 239.50 & 256.00 \\ 219.00 & 237.00 \end{bmatrix}$$

Let us see how and when we write this product :

i) The three elements of first row of the first matrix are multiplied respectively by the corresponding elements of the first column of the second matrix and added to give element of the first row and the first column of the product matrix. In the same way, the product of the elements of the second row of the first matrix to the corresponding elements of the first column of the second matrix on being added gives the element of the second row and the first column of the product matrix; and so on.

ii) The number of column of the first matrix is equal to the number of rows of the second matrix so that the first matrix is compatible for multiplication with the second matrix.

Thus, If $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ and $B = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix}$, then

$$A \times B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \times \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 \end{pmatrix}$$

Definition : If *A* and *B* are two matrices of order $m \times p$ and $p \times n$ respectively, then their product will be a matrix *C* of order $m \times n$; and if a_{ij}, b_{ij} and c_{ij} are the elements of the ith row and jth column of the matrices *A*, *B* and *C* respectively, then

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Example 20.15 If $A = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, then find:

(a) *AB* (b) *BA*

Is *AB*= *BA*?

Solution : Order of *A* is 1×3

Order of *B* is 3×1

$\therefore$ Number of columns of *A* = Number of rows of *B*

$\therefore$ *AB* exists

$$\text{Now, } AB = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$= [1 \times (-2) + (-1) \times 0 + 2 \times 2] = [-2 + 0 + 4] = [2]$$

Thus, $AB = [2]$, a matix of order 1×1

Again, number of columns of B = number of rows of A.

$\therefore$ BA exists

Now,

$$BA = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} [1 \quad -1 \quad 2]$$

$$= \begin{bmatrix} -2 \times 1 & (-2) \times (-1) & (-2) \times 2 \\ 0 \times 1 & 0 \times (-1) & 0 \times 2 \\ 2 \times 1 & 2 \times (-1) & 2 \times 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$$

Thus, $BA = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$, a matrix of order 3×3

From the above, we find that $AB \neq BA$

Example 20.16 Find AB and BA, if possible for the matrices A and B:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Solution : Here, Number of columns of A $\neq$ Number of rows of B

$\therefore$ AB does not exist.

Further, Number of columns of B $\neq$ Number of rows of A

$\therefore$ BA does not exist.

Example 20.17 If $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, then find AB and BA. Also find if AB=BA.

Solution : Here, Number of columns of A = Number of rows of B

$\therefore$ AB exists.

Further,Number of columns of B = Number of rows of A

$\therefore$ BA also exists.

Now, $AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1\times 2 + 2\times 2 & 1\times 1 + 2\times 2 \\ -1\times 2 + 0\times 2 & -1\times 1 + 0\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & 1+4 \\ -2+0 & -1+0 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -2 & -1 \end{bmatrix}_{2\times 2}$$

and $BA = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2\times 1 + 1\times(-1) & 2\times 2 + 1\times 0 \\ 2\times 1 + 2\times(-1) & 2\times 2 + 2\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 4+0 \\ 2-2 & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}_{2\times 2}$$

Thus, $AB \neq BA$

Remarks : We observe that AB and BA are of the same order 2×2, but still $AB \neq BA$.

Example 20.18 If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, find AB and BA. Is $AB = BA$?

Solution : Here, both A and B are of order 2×2. So, both AB and BA exist. Now

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix}_{2\times 2} \text{ and}$$

$$BA = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix}_{2\times 2}$$

Here, both AB and BA are of the same order and $AB = BA$.

Hence, if two matrcies A and B are multiplied, then the following five cases arise:

(i) Both AB and BA exist, but are of different orders

(ii) Only one of the products AB or BA exists.

(iii) Neither AB nor BA exist.

(iv) Both AB and BA exist and are of the same order, but $AB \neq BA$.

(v) Both AB and BA exist and are of the same order. Also, $AB = BA$.

Example 20.19 If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, verify that $A^2 - 2A - 3I = O$

Solution: Here,

$$A^2 = AA = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9+0 & 0+0 \\ 0+0 & 0+9 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

and $\quad 3I \quad = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$A^2 - 2A - 3I = -\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \left(\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9-9 & 0-0 \\ 0-0 & 9-9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence, verified.

Example 20.20 Solve the matrix equation :

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Solution : Here,

$$\text{L.H.S.} = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ x + y \end{bmatrix} \Rightarrow \begin{bmatrix} 2x - 3y \\ x + y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow 2x - 3y = 1; x + y = 3$$

Solving these equations, we get

$x = 2$ and $y = 1$

Notes

Example 20.21 If $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$, then find AB

Soution : Here,

$$A-B=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}-\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}1\times(-1)+1\times 1 & 1\times 1+(-1)\\ 1\times(-1)+1\times 1 & 1\times 1+(-1)\end{bmatrix}$$

$$=\begin{bmatrix}-1+1 & 1-1\\ -1+1 & 1-1\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$$

Hence, we conclude that the product of two non-zero matrices can be a zero matrix, whereas in numbers, the product of two non-zero numbers is always non-zero.

Example 20.22 For $A=\begin{bmatrix}1 & -2\\ 3 & 5\end{bmatrix}$, $B=\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}$ and $C=\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$, find

(a) $(AB)C$ (b) $A(BC)$

Is $(AB)C=A(BC)$?

Solution : (a) $$(AB)C=\left(\begin{bmatrix}1 & -2\\ 3 & 5\end{bmatrix}\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}\right)\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}4+2 & 0-4\\ 12-5 & 0+10\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}6 & -4\\ 7 & 10\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-6+0 & 0-12\\ -7+0 & 0+30\end{bmatrix}=\begin{bmatrix}-6 & -12\\ -7 & 30\end{bmatrix}$$

(b) $$A(BC)=\begin{bmatrix}1 & -2\\ 3 & 5\end{bmatrix}\left(\begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}\right)$$

$$= \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -4+0 & 0+0 \\ 1+0 & 0+6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 1 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -4-2 & 0-12 \\ -12+5 & 0+30 \end{pmatrix} = \begin{pmatrix} -6 & -12 \\ -7 & 30 \end{pmatrix}$$

From (a) and (b), we find that $(AB)C = A(BC)$, i.e., matrix multiplication is associative.

CHECK YOUR PROGRESS 20.5

1. If $A = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, find *AB* and *BA*. Is *AB*=*BA*?

2. If A= $B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ and B = $\begin{pmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -2 \end{pmatrix}$, find AB and BA. Is AB = BA?

3. If A = $\begin{pmatrix} a \\ b \end{pmatrix}$ and B= $\begin{bmatrix} x & y & z \end{bmatrix}$, find AB and BA, whichever exists.

4. If A = $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}$, find BA. Does AB exists?

5. If A= $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

   (a) Does AB exist? Why? (b) Does BA exist? Why?

6. If A = $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & 0 \\ 2 & 5 \end{pmatrix}$, find AB and BA. Is AB=BA?

7. If A = $\begin{bmatrix} 1 & 2 & 0 \\ -3 & 5 & 4 \\ 5 & 3 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find AB and BA . Is AB=BA?

8. If A= $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and B = $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$, find AB and BA. Is AB=BA?

9. Find the values of *x* and *y* if

(a) $\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

10. For $A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$, verify that AB=O

11. For A= $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, verify that $A^2 - 5A + I = O$, where I is a unit matrix of order 2.

12. If A = $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, B= $B = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$, and C = = $\begin{bmatrix} 4 & -3 \\ -2 & 3 \end{bmatrix}$, find :

(a) A(BC) (b) (AB)C (c) (A+B)C

(d) AC+BC (e) $A^2 - B^2$ (f) (A−B) (A+B)

13. If A = $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, B= $A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$ and C = $C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, find : (a) AC (b) BC

Is AC = BC ? What do you conclude?

14. If A = $\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$, B = $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ and C= $\begin{bmatrix} 3 & 8 \\ 7 & -1 \end{bmatrix}$, find :

(a) B+C (b) A (B+C) (c) AB (d) AC (e) AB+AC

What do you observe?

15. For matices A = $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & -3 \\ -1 & 0 \end{bmatrix}$, verify that $(AB)' = B'A'$

16. If $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, find X such that AX = B.

17. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, show that, $A^2 - (a+d)A = (bc - ad)I$

18 If $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, is it true that

(a) $(A+B)^2 = A^2+B^2+2AB$? (b) $(A-B)^2 = A^2+B^2-2AB$?

(c) $(A+B)(A-B) = A^2-B^2$?

20.8 INVERTIBLE MATRICES

Definition : A square matrix of order *n* is invertible if there exists a square matrix B of the same order such that

$AB = I_n = BA$, Where I_n is identify matrix of order *n*.

In such a case, we say that the inverse of A is B and we write, $A^{-1} = B$.

Theorem 1 : Every invertible matrix possesses a unique inverse.

Proof : Let A be an invertible matrix of order

Let B and C be two inverses of A.

Then,

$$AB = BA = I_n \quad \text{...(1)}$$

and $$AC = CA = I_n \quad \text{...(2)}$$

Now, $AB = I_n$

$\Rightarrow$ $C(AB) = C\,I_n$ [Pre-multiplying by C]

$\Rightarrow$ $(CA)B = C\,I_n$ [by associativity]

$\Rightarrow$ $I_n B = C\,I_n$ $(\because CA = I_n$ from (*ii*)]

$\Rightarrow$ $B = C$ $[\because I_n B = B, C\,I_n = C]$

Hence, an invertible matrix possesses a unique inverse.

CORROLLARY **If A is an invertible matrix then $(A^{-1})^{-1} = A$**

Proof : We have, $A\,A^{-1} = I = A^{-1}A$

$\Rightarrow$ A is the inverse of A^{-1} i.e., $A = (A^{-1})^{-1}$

Theorem 2 : A square matrix is invertible iff it is non-singular.

Proof : Let A be an invertible matrix. Then, there exists a matrix B such that

$$AB = I_n = BA$$

$\Rightarrow \quad |AB| = |I_n|$

$\Rightarrow \quad |A|\,|B| = 1$

$[\because\ |AB| = |A|\,|B|]$

$\Rightarrow \quad |A| \neq 0$

$\Rightarrow$ A is a non-singular matrix.

Conversely, let A be a non-singular square matrix of order n, then,

$$\Rightarrow \quad A\left(\frac{1}{|A|}adj\ A\right) = I_n = \left(\frac{1}{|A|}adj\ A\right)A \quad \left[\because |A| \neq 0 \therefore \frac{1}{|A|} \text{exists}\right]$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}adj\ A \qquad \text{[By def. of inverse]}$$

Hence, A is an invertible matrix.

Remark : This theorem provides us a formula for finding the inverse of a non-singular square matrix.

The inverse of A is given by

$$A^{-1} = \frac{1}{|A|}adj\ A$$

20.9 ELEMENTARY TRANSFORMATIONS OR ELEMENTARY OPERATIONS OF A MATRIX

The following three operations applied on the rows (columns) of a matrix are called elementary row (column) transformations.

(i) Interchange of any two rows (columns)

If i^{th} row (column) of a matrix is interchanged with the jth row (column), it is dennoted by $R_i \leftrightarrow R_j$ or $(C_i \leftrightarrow C_j)$.

for example, $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, then by applying $R_2 \leftrightarrow R_3$

we get $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

(ii) Multiplying all elements of any row (column) of a matrix by a non-zero scalar

If the elements of ith row (column) are multiplied by a non-zero scalar k, it is denoted by $R_i \to k\ R_i\ [C_i \to k\ C_i]$

For example

If $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, then by applying $R_1 \to 2R_1$ we get $B = \begin{bmatrix} 6 & 4 & -2 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$

(iii) Adding to the elements of a row (column), the corresponding elements of any other row (column) multiplied by any scalar *k*

If k times the elements of jth row (column) are added to the corresponding elements of the ith row (column), it is denoted by $R_i \to R_i + kR_j$ $(C_i \to C_i + k\ C_j)$.

If $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$, then the application of elementary operation $R_3 \to R_3 + 2R_1$, gives the matrix

$$B = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 4 & 3 & 9 & 3 \end{bmatrix}$$

20.9.1 INVERSE OF A MATRIX BY ELEMENTARY OPERATIONS

We can find the inverse of a matrix, if it exists, by using either elementary row operations or column operations but not both simultaneously.

Let A be an invertible square matrix of order n, if we want to find A^{-1} by using elementary raw operations then we write

$$A = I_n\ A \qquad ...(i)$$

As an elementary row operation on the product of two matrices can be affected by subjecting the pre factor to the same elementary row operation, we shall use elementary row operations on (i) so that its L.H.S reduces to In and R.H.S (after applying corresponding elementary row operations on the prefactor I_n), we get

$$I_n = BA \qquad ...(ii)$$

Which means matrix B and matrix A are inverse of each other i.e. $A^{-1} = B$

Similarly if we want to find A^{-1} by using elementary column operations, we write

$$A = A\, I_n \qquad \text{...(iii)}$$

Now use elementary column operations on (iii) so that its L.H.S reduces to I_n and R.H.S (after applying corresponding elementary column operations on the post factor I_n) takes the shape

$$I_n = AB$$

$$\text{Then} \quad A^{-1} = B$$

The method is explained below with the help of some examples.

Example 20.23 Find the inverse of matrix A, using elementary column operations where,

$$A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

Solution : Writing

$$A = A\,I_2 \quad \Rightarrow \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} & 3 \\ 0 & 1 \end{bmatrix} \quad \text{Operating } C_2 \to C_2 + 3C_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{Operating } C_1 \to \frac{1}{2} C_1$$

$$\Rightarrow I_2 = AB, \text{ where } B = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{Operating } C_1 \to C_1 - \frac{1}{2} C_2$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Example 20.24 Find the inverse of the matrix A using elementary row operations, where

$$A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

Solution : Writing

$$A = I_2A$$

$$\Rightarrow \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix} A \quad \text{Operating } R_1 \to \frac{1}{10} R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} A \quad \text{Operating } R_2 \to R_2 + 5\,R_1,$$

As the matrix in L.H.S contain, a row in which all elements are 0. So inverse of this matrix does not exist. Because in such case the matrix in L.H.S can not be conversed into a unit matrix.

Example 20.25 Find the inverse of the matrix A, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

Solution : We have

$$A = I\,A$$

or
$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \text{Operating } R_1 \to R_1 - R_2,$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad \text{Operating } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1,$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad \text{Operating } R_2 \to \frac{1}{2} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ -5 & 6 & 1 \end{bmatrix} A \quad \text{Operating } R_1 \to R_1 + R_2,\ R_3 \to R_3 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ \frac{-5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} A \quad \text{Operating } R_3 \to \frac{1}{4} R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix} A \quad \text{Operating } R_1 \to R_1 + \frac{1}{2} R_3,\ R_2 \to R_2 - \frac{1}{2} R_3$$

Hence $A^{-1} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} & \frac{1}{8} \\ -\frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$

CHECK YOUR PROGESS 20.6

1. Find inverse of the following matrices using elementary operations :

(a) $\begin{bmatrix} 7 & 1 \\ 4 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & 10 \\ 3 & 6 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

LET US SUM UP

- A rectangular array of numbers, arranged in the form of rows and columns is called a *matrix*. Each number is called an *element* of the matrix.
- The *order* of a matrix having 'm' rows and 'n' columns is $m \times n$.
- If the number of rows is equal to the number of columns in a matrix, it is called a square matrix.
- A diagonal matrix is a square matrix in which all the elements, except those on the diagonal, are zeroes.
- A unit matrix of any order is a diagonal matrix of that order whose all the diagonal elements are 1.
- Zero matrix is a matrix whose all the elements are zeroes.
- Two matrices are said to be equal if they are of the same order and their corresponding elements are equal.
- A transpose of a matrix is obtained by interchanging its rows and columns.
- Matrix A is said to be symmetric if $A' = A$ and skew symmetric if $A' = -A$.
- Scalar multiple of a matrix is obtained by multiplying each elements of the matrix by the scalar.
- The sum of two matrices (of the same order) is a matrix obtained by adding corresponding elements of the given matrices.
- Difference of two matrices A and B is nothing but the sum of matrix A and the negative of matrix B.
- Product of two matrices A of order $m \times n$ and B of order $n \times p$ is a matrix of order $m \times p$, whose elements can be obtained by multiplying the rows of A with the columns of B element wise and then taking their sum.
- Product of a matrix and its inverse is equal to identity matrix of same order.
- Inverse of a matrix is always unique.
- All matrices are not necessarily invertible.
- Three points are collinear if the area of the triangle formed by these three points is zero.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=xZBbfLLfVV4

http://www.youtube.com/watch?v=ArcrdMkEmKo

http://www.youtube.com/watch?v=S4n-tQZnU6o

http://www.youtube.com/watch?v=obts_JDS6_Q

http://www.youtube.com/watch?v=01c12NaUQDw

http://www.math.odu.edu/~bogacki/cgi-bin/lat.cgi?c=sys

TERMINAL EXERCISE

1. How many elements are there in a matrix of order

   (a) 2×1 (b) 3×2 (c) 3×3 (d) 3×4

2. Construct a matrix of order 3×2 whose elements a_{ij} are given by

   (a) $a_{ij} = i - 2j$ (b) $a_{ij} = 3i - j$ (c) $a_{ij} = i + \frac{3}{2}j$

3. What is the order of the matrix?

   (a) $A = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & -1 & 5 \\ 7 & 6 & 1 \end{bmatrix}$

4. Find the value of *x*, *y* and *z* if

   (a) $\begin{bmatrix} x & y \\ z & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} x+y & z \\ 6 & x-y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 6 & 4 \end{bmatrix}$

   (c) $\begin{bmatrix} x-2 & 3 \\ 0 & y+5 \end{bmatrix} = \begin{bmatrix} 1 & z \\ y+z & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

5. If $A = \begin{pmatrix} 1 & -2 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -1 & 4 \end{pmatrix}$, find :

(a) A+B (b) 2A (c) 2A−B

6. Find X, if

(a) $\begin{pmatrix} 4 & 5 \\ -3 & 6 \end{pmatrix} + X = \begin{pmatrix} 10 & -2 \\ 1 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} + X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

7. Find the values of a and b so that

$$\begin{pmatrix} 3 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} a-b & 2 & -2 \\ 4 & a & b \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 5 & 2a+b & 5 \end{pmatrix}$$

8. For matrices A, B and C

$A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 5 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 3 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 5 & 6 \\ 7 & 1 \\ 4 & 1 \end{pmatrix}$

verify that A+(B+C) = (A+B)+C

9. If $A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix}$, find AB and BA. Is AB= BA?

10. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$, find AB and BA. Is AB = BA?

11. If $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$, find A^2.

12. Find A (B+C), if

$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $B = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 0 & 3 \\ 4 & 0 & -3 \end{pmatrix}$

13. If $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} x & 1 \\ y & -1 \end{pmatrix}$ and $(A+B)^2 = A^2 + B^2$, find the values of x and y.

14. Show that $A = \begin{pmatrix} -8 & 5 \\ 2 & 4 \end{pmatrix}$ satisfies the matrix equation $A^2 + 4A - 2I = O$.

Find inverse of the following matrices using elementary transformations.

15. $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

18. $\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

20. $\begin{bmatrix} \cos x & \sin x \\ \sin x & \cos x \end{bmatrix}$

21. $\begin{bmatrix} 1 & \tan\frac{x}{2} \\ -\tan\frac{x}{2} & 1 \end{bmatrix}$

22. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 0 & 1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

24. $\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

ANSWERS

CHECK YOUR PROGRESS 20.1

1. $\begin{bmatrix} 56 & 65 & 71 \\ 29 & 37 & 57 \end{bmatrix}$; $\begin{bmatrix} 56 & 29 \\ 65 & 37 \\ 71 & 57 \end{bmatrix}$

2. $\begin{bmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix}$; $\begin{bmatrix} 40 & 10 \\ 35 & 5 \\ 25 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 6 & 3 \\ 4 & 3 & 5 \end{bmatrix}$

4. (a) 6 (b) 12 (c) 8 (d) 12 (e) ab (f) *mn*

5. (a) 1×8; 2×4; 4×2; 8×1 (b) 1×5; 5×1
   (c) 1×12; 2×6; 3×4 ; 4×3; 6×2; 12×1
   (d) 1×16; 2×8; 4×4; 8×2; 16×1

6. (a) 4 (b) 5 (c) 4×5 (d) 20
   (e) $a_{14} = 0$; $a_{23} = 7$; $a_{34} = -3$; $a_{45} = 1$ and $a_{33} = 3$

7. (a) $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 4 & 2 & \frac{4}{3} \\ 9 & \frac{9}{2} & 3 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32 \\ \frac{25}{2} & \frac{49}{2} & \frac{81}{2} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4 & 7 \\ -1 & 2 & 5 \\ -3 & 0 & 3 \end{bmatrix}$

8. (a) $\begin{bmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 10 \\ 10 & 20 \\ 15 & 30 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$

CHECK YOUR PROGRESS 20.2

1. (a) G (b) B (c) A, D, E and F (d) A, D and F
   (e) D and F (f) F (g) C

2. (a) $a = 2$, $b = 10$, $c = 6$, $d = -2$
   (b) $a = 2$, $b = 3$, $c = 2$, $d = 5$
   (c) $a = \frac{3}{2}$, $b = -2$, $c = 2$, $d = -4$

MODULE - VI
Algebra -II

Notes

3. No 4. No

CHECK YOUR PROGRESS 20.3

1. (a) $\begin{bmatrix} 28 & 8 \\ 8 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} -7 & -2 \\ -2 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{7}{2} & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-21}{2} & -3 \\ -3 & \frac{-9}{2} \end{bmatrix}$

2. (a) $\begin{bmatrix} 0 & -5 & 10 \\ 15 & 5 & 20 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3 & -6 \\ -9 & -3 & -12 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & \frac{-1}{3} & \frac{2}{3} \\ 1 & \frac{1}{3} & \frac{4}{3} \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & \frac{1}{2} & -1 \\ \frac{-3}{2} & \frac{-1}{2} & -2 \end{bmatrix}$

3. $\begin{bmatrix} 7 & 0 \\ -28 & -14 \\ 0 & 7 \end{bmatrix}$

4. (a) $\begin{bmatrix} 15 & 0 & 5 \\ 20 & -10 & 0 \\ -5 & 0 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} -12 & 0 & -4 \\ -16 & 8 & 0 \\ 4 & 0 & -20 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ \frac{4}{3} & \frac{-2}{3} & 0 \\ \frac{-1}{3} & 0 & \frac{5}{3} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-3}{2} & 0 & \frac{-1}{2} \\ -2 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-5}{2} \end{bmatrix}$

5. (a) $\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 10 & 8 \\ 9 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 & -6 \\ -2 & -1 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

CHECK YOUR PROGRESS 20.4

1. (a) $\begin{bmatrix} 3 & -2 \\ 8 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -3 \\ 13 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -4 \\ 14 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} 6 & -5 \\ 19 & 10 \end{bmatrix}$

2. (a) $\begin{pmatrix} 1 & 3 & 6 \\ -5 & 3 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -3 & -6 \\ 5 & -3 & -5 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 9 \\ -9 & 2 & 10 \end{pmatrix}$ (d) $\begin{pmatrix} -4 & -11 & -15 \\ 11 & -10 & -10 \end{pmatrix}$

3. (a) $\begin{pmatrix} 0 & -6 & 3 \\ 5 & 5 & 3 \\ 6 & 5 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 2 & 3 \\ 3 & -7 & 1 \\ 2 & 5 & -7 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & -2 & -3 \\ -3 & 7 & -1 \\ -2 & -5 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -14 & 9 \\ 14 & 9 & 8 \\ 16 & 15 & 14 \end{pmatrix}$

4. (a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

5. (a) $\begin{pmatrix} 2 & 1 & 0 \\ -1 & -2 & -3 \\ 4 & 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

6. (a) $\begin{pmatrix} 2 & 18 \\ 6 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 15 & 3 \\ 21 & 27 \end{pmatrix}$ (c) $\begin{pmatrix} 17 & 21 \\ 27 & 31 \end{pmatrix}$ (d) $\begin{bmatrix} \frac{-17}{5} & \frac{-21}{5} \\ \frac{-27}{5} & \frac{-31}{5} \end{bmatrix}$

7. (a) $\begin{pmatrix} 2 & 4 \\ 0 & 3 \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 2 \\ 0 & -4 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 \\ 6 & -1 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{pmatrix}$

We observe that $(A+B)' = B' + A'$

8. (a) $\begin{pmatrix} 0 & 2 \\ -5 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 3 \\ 0 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 5 \\ -5 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 8 \\ 3 & -7 \end{pmatrix}$

(e) $\begin{pmatrix} -2 & 3 \\ -8 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} -5 & 9 \\ -16 & -3 \end{pmatrix}$

CHECK YOUR PROGRESS 20.5

1. $AB = \begin{bmatrix} -6 \end{bmatrix}$; $BA = \begin{bmatrix} 0 & 0 & 0 \\ -4 & -6 & 0 \\ 2 & 3 & 0 \end{bmatrix}$ $AB \neq BA$

2. $AB = \begin{bmatrix} 2 & 2 \\ 1 & -6 \end{bmatrix}$; $BA = \begin{bmatrix} -3 & 13 & -4 \\ 1 & -1 & -2 \\ 2 & -6 & 0 \end{bmatrix}$ $AB \neq BA$

3. $AB = \begin{bmatrix} ax & ay & az \\ bx & by & bz \end{bmatrix}$; BA does not exist.

4. $BA = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$; AB does not exist.

5. Both AB and BA do not exist. AB does not exist since the number of columns of A is not equal to the number of rows of B. BA also does not exist since number of coluumns of B is not equal to the number of rows of A.

6. $AB = \begin{bmatrix} 0 & 5 \\ 6 & 15 \end{bmatrix}$; $BA = \begin{bmatrix} -2 & -1 \\ 4 & 17 \end{bmatrix}$; $AB \neq BA$

7. $AB = \begin{bmatrix} 4 & -3 & 7 \\ 3 & 17 & 24 \\ 14 & -13 & 17 \end{bmatrix}$; $BA = \begin{bmatrix} 16 & -8 & -11 \\ 16 & 11 & 3 \\ 10 & 21 & 11 \end{bmatrix}$; $AB \neq BA$.

8. $AB = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$; $BA = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$; $AB \neq BA$.

9. (a) $x = 3, \ y = -1$ (b) $x = -1, \ y = 2$

12. (a) $\begin{bmatrix} -14 & 18 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} -14 & 18 \\ 2 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 5 & 0 \\ 7 & 8 \end{bmatrix}$ (f) $\begin{bmatrix} -2 & -3 \\ 9 & 15 \end{bmatrix}$

13. (a) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$; AC= BC

Here, A≠B and C≠O, yet AC= BC
i.e. cancellation law does not hold good for matrices.

14. (a) $\begin{pmatrix} 4 & 7 \\ 9 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & -7 \\ -14 & 9 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 1 \\ -3 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} -3 & -8 \\ -11 & 10 \end{pmatrix}$ (e) $\begin{pmatrix} -4 & -7 \\ -14 & 9 \end{pmatrix}$

We observe that A(B+C) = AB+AC

16. x= $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ 18. (a) No (b) No (c) No

CHECK YOUR PROGRESS 20.6

1. (*a*) $\frac{1}{25}\begin{bmatrix} 3 & 1 \\ 4 & -7 \end{bmatrix}$ (*b*) $\frac{1}{23}\begin{bmatrix} 5 & -6 \\ 3 & 1 \end{bmatrix}$ (*c*) does not exist

(*d*) $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$ (*e*) $\begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & 12 & 9 \end{bmatrix}$

TERMIAL EXERCISE

1. (a) 2 (b) 6 (c) 9 (d) 12

2. (a) $\begin{pmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ 5 & 4 \\ 8 & 7 \end{pmatrix}$ (c) $\begin{pmatrix} \frac{5}{2} & 4 \\ \frac{7}{2} & 5 \\ \frac{9}{2} & 6 \end{pmatrix}$

3. (a) 3×1 (b) 1×3

(c) 3×2 (d) 2×3

4. (a) $x = 1, y = 2, z = 3$ (b) $x = 5, y = 1, z = 5$

(c) $x = 3, y = -3, z = 3$ (d) $x = 2, y = 1, z = 5$

5. (a) $\begin{pmatrix} 3 & 2 \\ 3 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -4 \\ 8 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -8 \\ 9 & 0 \end{pmatrix}$

6. (a) $\begin{pmatrix} 6 & -7 \\ 4 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 4 & -3 \\ -3 & 0 & -1 \end{pmatrix}$

7. $a = \frac{3}{2}$ $b = -\frac{3}{2}$

9. $AB = \begin{pmatrix} 13 & 11 \\ 38 & 43 \end{pmatrix}$; $BA = \begin{pmatrix} 5 & 10 & 17 \\ 6 & 14 & 24 \\ 4 & 21 & 37 \end{pmatrix}$; $AB \neq BA$

10. $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $AB = BA$

11. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 12. $\begin{pmatrix} 9 & 1 & 1 \\ -1 & -4 & 10 \end{pmatrix}$

13. $x = 1, y = -4.$

15. $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ 16. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$ 18. $\frac{1}{22}\begin{bmatrix} -4 & +5 \\ +2 & +3 \end{bmatrix}$

19. $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ 20. $\begin{bmatrix} \cos x & -\sin x \\ -\sin x & \cos x \end{bmatrix}$

21. $\cos^2\frac{x}{2}\begin{bmatrix} 1 & -\tan x/2 \\ \tan x/2 & 1 \end{bmatrix}$ 22. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

23. $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ 24. $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix}$