311en22

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INVERSE OF A MATRIX AND ITS APPLICATIONS

Let us Consider an Example:

Abhinav spends Rs. 120 in buying 2 pens and 5 note books whereas Shantanu spends Rs. 100 in buying 4 pens and 3 note books.We will try to find the cost of one pen and the cost of one note book using matrices.

Let the cost of 1 pen be Rs. x and the cost of 1 note book be Rs. y. Then the above information can be written in matrix form as:

$$\begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 120 \\ 100 \end{pmatrix}$$

This can be written as $AX = B$

where $A = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 120 \\ 100 \end{pmatrix}$

Our aim is to find $X = \begin{pmatrix} x \\ y \end{pmatrix}$

In order to find X, we need to find a matrix A^{-1} so that $X = A^{-1}B$

This matrix A^{-1} is called the inverse of the matrix A.

In this lesson, we will try to find the existence of such matrices.We will also learn to solve a system of linear equations using matrix method.

OBJECTIVES

After studying this lesson, you will be able to :

- define a minor and a cofactor of an element of a matrix;
- find minor and cofactor of an element of a matrix;

- find the adjoint of a matrix;
- define and identify singular and non-singular matrices;
- find the inverse of a matrix, if it exists;
- represent system of linear equations in the matrix form $AX = B$; and
- solve a system of linear equations by matrix method.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a determinant.
- Determinant of a matrix.
- Matrix with its determinant of value 0.
- Transpose of a matrix.
- Minors and Cofactors of an element of a matrix.

22.1 DETERMINANT OF A SQUARE MATRIX

We have already learnt that with each square matrix, a determinant is associated. For any given matrix, say $A = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$

its determinant will be $\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix}$. It is denoted by $|A|$.

Similarly, for the matrix $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{pmatrix}$, the corresponding determinant is

$$|A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{vmatrix}$$

A square matrix A is said to be singular if its determinant is zero, i.e. $|A| = 0$

A square matrix A is said to be non-singular if its determinant is non-zero, i.e. $|A| \neq 0$

Example 22.1 Determine whether matrix A is singular or non-singular where

(a) $A = \begin{pmatrix} -6 & -3 \\ 4 & 2 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{pmatrix}$

Solution: (a) Here, $|A| = \begin{vmatrix} -6 & -3 \\ 4 & 2 \end{vmatrix}$

$$= (-6)(2) - (4)(-3)$$
$$= -12 + 12 = 0$$

Therefore, the given matrix A is a singular matrix.

(b)

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= -7 + 4 - 3 = -6 \neq 0$$

Therefore, the given matrix is non-singular.

Example 22.2 Find the value of x for which the following matrix is singular:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{pmatrix}$$

Solution: Here,

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix}$$

$$= 1(-6-2) + 2(-3-x) + 3(2-2x)$$
$$= -8 - 6 - 2x + 6 - 6x$$
$$= -8 - 8x$$

Since the matrix A is singular, we have $|A| = 0$

$$|A| = -8 - 8x = 0$$
$$or \quad x = -1$$

Thus, the required value of x is -1.

Example 22.3 Given $A = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix}$. Show that $|A| = |A'|$, where A' denotes the transpose of the matrix.

Solution: Here, $A = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix}$

This gives $A' = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}$

Now, $$|A| = \begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 6 = -16 \qquad ...(1)$$

and $$|A'| = \begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 6 = -16 \qquad ...(2)$$

From (1) and (2), we find that $|A| = |A'|$

22.2 MINORS AND COFACTORS OF THE ELEMENTS OF SQUARE MATRIX

Consider a matrix $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of the matrix obtained by deleting the ith row and jth column of A, is called the minor of a_{ij} and is denoted by M_{ij}.

Cofactor C_{ij} of a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, M_{23} = Minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

and C_{23} = Cofactor of a_{23}

$$= (-1)^{2+3} M_{23} = (-1)^5 M_{23} = -M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 22.4 Find the minors and the cofactors of the elements of matrix $A = \begin{pmatrix} 2 & 5 \\ 6 & 3 \end{pmatrix}$

Solution: For matrix A, $|A| = \begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix} = 6 - 30 = -24$

M_{11} (minor of 2) = 3; $C_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = 3$

M_{12} (minor of 5) = 6; $C_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -6$

M_{21} (minor of 6) = 5; $C_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21} = -5$

M_{22} (minor of 3) = 2; $C_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22} = 2$

Example 22.5 Find the minors and the cofactors of the elements of matrix

$$A = \begin{pmatrix} -1 & 3 & 6 \\ 2 & 5 & -2 \\ 4 & 1 & 3 \end{pmatrix}$$

Solution: Here, $M_{11} = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$; $C_{11} = (-1)^{1+1} M_{11} = 17$

$$M_{12} = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 + 8 = 14;\ C_{12} = (-1)^{1+2} M_{12} = -14$$

$$M_{13} = \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = 2 - 20 = -18;\ C_{13} = (-1)^{1+3} M_{13} = -18$$

$$M_{21} = \begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix} = 9 - 6 = 3;\ C_{21} = (-1)^{2+1} M_{21} = -3$$

$$M_{22} = \begin{vmatrix} -1 & 6 \\ 4 & 3 \end{vmatrix} = (-3 - 24) = -27;\ C_{22} = (-1)^{2+2} M_{22} = -27$$

$$M_{23} = \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} = (-1-12) = -13;\ C_{23} = (-1)^{2+3} M_{23} = 13$$

$$M_{31} = \begin{vmatrix} 3 & 6 \\ 5 & -2 \end{vmatrix} = (-6-30) = -36;\ C_{31} = (-1)^{3+1} M_{31} = -36$$

$$M_{32} = \begin{vmatrix} -1 & 6 \\ 2 & -2 \end{vmatrix} = (2-12) = -10;\ C_{32} = (-1)^{3+2} M_{32} = 10$$

and $M_{33} = \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = (-5-6) = -11;\ C_{33} = (-1)^{3+3} M_{33} = -11$

CHECK YOUR PROGRESS 22.1

1. Find the value of the determinant of following matrices:

(a) $A = \begin{pmatrix} 0 & 6 \\ 2 & 5 \end{pmatrix}$ (b) $B = \begin{pmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

2. Determine whether the following matrix are singular or non-singular.

(a) $A = \begin{pmatrix} 3 & 2 \\ -9 & -6 \end{pmatrix}$ (b) $Q = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & -1 \end{pmatrix}$

3. Find the minors of the following matrices:

(a) $A = \begin{pmatrix} 3 & -1 \\ 7 & 4 \end{pmatrix}$ (b) $B = \begin{pmatrix} 0 & 6 \\ 2 & 5 \end{pmatrix}$

4. (a) Find the minors of the elements of the 2^{nd} row of matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ -2 & -3 & 1 \end{pmatrix}$$

(b) Find the minors of the elements of the 3rd row of matrix

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 5 & 4 & 1 \\ -2 & 0 & -3 \end{pmatrix}$$

5. Find the cofactors of the elements of each the following matrices:

(a) $A = \begin{pmatrix} 3 & -2 \\ 9 & 7 \end{pmatrix}$ (b) $B = \begin{pmatrix} 0 & 4 \\ -5 & 6 \end{pmatrix}$

6. (a) Find the cofactors of elements of the 2nd row of matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 0 \\ 4 & 1 & -2 \end{pmatrix}$$

(b) Find the cofactors of the elements of the 1st row of matrix

$$A = \begin{pmatrix} 2 & -1 & 5 \\ 6 & 4 & -2 \\ -5 & -3 & 0 \end{pmatrix}$$

7. If $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 3 \\ 7 & 4 \end{pmatrix}$, verify that

(a) $|A| = |A'|$ and $|B| = |B'|$ (b) $|AB| = |A|\,|B| = |BA|$

22.3 ADJOINT OF A SQUARE MATRIX

Let $A = \begin{pmatrix} 2 & 1 \\ 5 & 7 \end{pmatrix}$ be a matrix. Then $|A| = \begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix}$

Let M_{ij} and C_{ij} be the minor and cofactor of a_{ij} respectively. Then

$$M_{11} = |7| = 7;\ C_{11} = (-1)^{1+1}\,|7| = 7$$

$$M_{12} = |5| = 5;\ C_{12} = (-1)^{1+2}\,|5| = -5$$

$$M_{21} = |1| = 1;\ C_{21} = (-1)^{2+1}\,|1| = -1$$

$$M_{22} = |2| = 2; \; C_{22} = (-1)^{2+2} \, |2| = 2$$

We replace each element of A by its cofactor and get

$$B = \begin{pmatrix} 7 & -5 \\ -1 & 2 \end{pmatrix} \qquad \ldots(1)$$

The transpose of the matrix B of cofactors obtained in (1) above is

$$B' = \begin{pmatrix} 7 & -1 \\ -5 & 2 \end{pmatrix} \qquad \ldots(2)$$

The matrix B' obtained above is called the adjoint of matrix A. It is denoted by Adj A.

Thus, adjoint of a given matrix is the transpose of the matrix whose elements are the cofactors of the elements of the given matrix.

Working Rule: To find the Adj A of a matrix A:

(a) replace each element of A by its cofactor and obtain the matrix of cofactors; and

(b) take the transpose of the marix of cofactors, obtained in (a).

Example 22.6 Find the adjoint of

$$A = \begin{pmatrix} -4 & 5 \\ 2 & -3 \end{pmatrix}$$

Solution: Here, $|A| = \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix}$ Let A_{ij} be the cofactor of the element a_{ij}.

Then,

$$A_{11} = (-1)^{1+1} \, (-3) = -3 \qquad A_{21} = (-1)^{2+1} \, (5) = -5$$

$$A_{12} = (-1)^{1+2} \, (2) = -2 \qquad A_{22} = (-1)^{2+2} \, (-4) = -4$$

We replace each element of A by its cofactor to obtain its matrix of cofators as

$$\begin{pmatrix} -3 & -2 \\ -5 & -4 \end{pmatrix} \qquad \ldots(1)$$

Transpose of matrix in (1) is Adj A.

Thus, Adj $A = \begin{pmatrix} -3 & -5 \\ -2 & -4 \end{pmatrix}$

Example 22.7 Find the adjoint of $A = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{pmatrix}$

Solution: Here,

$$A = \begin{vmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{vmatrix}$$

Let A_{ij} be the cofactor of the element a_{ij} of $|A|$

Then $\quad A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} = (-4-2) = -6\,;\quad A_{12} = (-1)^{1+2}\begin{vmatrix} -3 & 1 \\ 5 & -1 \end{vmatrix} = -(3-5) = 2$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 4 \\ 5 & 2 \end{vmatrix} = (-6-20) = -26\,;\quad A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = (-1-4) = 3$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} = (-1-10) = -11\,;\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} = -(2+5) = -7$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = (-1-8) = -9\,;\quad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = -(1+6) = -7$$

and $\quad A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} = (4-3) = 1$

Replacing the elements of A by their cofactors, we get the matrix of cofactors as

$$\begin{pmatrix} -6 & 2 & -26 \\ 3 & -11 & -7 \\ -9 & -7 & 1 \end{pmatrix} \qquad \text{Thus, Adj } A = \begin{pmatrix} -6 & 3 & -9 \\ 2 & -11 & -7 \\ -26 & -7 & 1 \end{pmatrix}$$

If A is any square matrix of order n, then A(Adj A) = (Adj A) A = |A| I_n

where I_n is the unit matrix of order n.

Verification:

(1) Consider $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix}$ or $|A| = 2 \times 3 - (-1) \times (4) = 10$

Here, $A_{11}=3;\ A_{12}=1;\ A_{21}=-4$ and $A_{22}=2$

Therefore, $\text{Adj } A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$

Now, $A\,(\text{Adj}A) = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A| I_2$

(2) Consider, $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Then, $|A| = 3(-6-1) - 5(4-1) + 7(2+3) = -1$

Here, $A_{11} = -7;\ A_{12} = -3;\ A_{13} = 5$

$A_{21} = -3;\ A_{22} = -1;\ A_{23} = 2$

$A_{31} = 26;\ A_{32} = 11;\ A_{33} = -19$

Therefore, $\text{Adj } A = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

Now $(A)(\text{Adj } A) = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3$$

Also, $(\text{Adj A})\,A = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix} \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3$$

Note : If A is a singular matrix, i.e. $|A|=0$, then $A\,(\text{Adj A}) = O$

CHECK YOUR PROGRESS 22.2

1. Find adjoint of the following matrices:

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (c) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

2. Find adjoint of the following matrices :

(a) $\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & -i \\ i & i \end{bmatrix}$

Also verify in each case that $A(\text{Adj A}) = (\text{Adj A})A = |A|I_2$.

3. Verify that

$A(\text{Adj A}) = (\text{Adj A})\,A = |A|\,I_3$, where A is given by

(a) $\begin{bmatrix} 6 & 8 & -1 \\ 0 & 5 & 4 \\ -3 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 7 & 9 \\ 0 & -1 & 2 \\ 3 & -7 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix}$

22.4 INVERSE OF A MATRIX

Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We will find, if possible, a matrix

$$B = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \text{ such that } AB = BA = I$$

i.e., $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or $$\begin{pmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

On comparing both sides, we get

$$ax + bu = 1 \quad ay + bv = 0$$

$$cx + du = 0 \qquad cy + dv = 1$$

Solving for x, y, u and v, we get

$$x = \frac{d}{ad - bc}, \; y = \frac{-b}{ad - bc}, \; u = \frac{-c}{ad - bc}, \; v = \frac{a}{ad - bc}$$

provided $ad - bc \neq 0$, i.e., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

Thus, $$B = \begin{pmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{pmatrix}$$

or $$B = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

It may be verified that $BA = I$.

It may be noted from above that, we have been able to find a matrix.

$$B = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{|A|} Adj\ A \qquad ...(1)$$

This matrix B, is called the inverse of A and is denoted by A^{-1}.

For a given matrix A, if there exists a matrix B such that AB = BA = I, then B is called the multiplicative inverse of A. We write this as B= A^{-1}.

Note: Observe that if $ad - bc = 0$, i.e., $|A| = 0$, the R.H.S. of (1) does not exist and $B\ (=A^{-1})$ is not defined. This is the reason why we need the matrix A to be non-singular in order that A possesses multiplicative inverse. Hence only non-singular matrices possess multiplicative inverse. Also B is non-singular and $A=B^{-1}$.

Example 22.8 Find the inverse of the matrix

$$A = \begin{pmatrix} 4 & 5 \\ 2 & -3 \end{pmatrix}$$

Solution : $$A = \begin{pmatrix} 4 & 5 \\ 2 & -3 \end{pmatrix}$$

Therefore, $|A| = -12 - 10 = -22 \neq 0$

$\therefore A$ is non-singular. It means A has an inverse. i.e. A^{-1} exists.

Now, $$\text{Adj } A = \begin{pmatrix} -3 & -5 \\ -2 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-22}\begin{bmatrix} -3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{22} & \frac{5}{22} \\ \frac{1}{11} & -\frac{2}{11} \end{bmatrix}$$

Note : Verify that $AA^{-1} = A^{-1}A = I$

Example 22.9 Find the inverse of matrix

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{pmatrix}$$

Solution : Here, $$A = \begin{pmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{pmatrix}$$

$$\begin{aligned} \therefore \quad |A| &= 3(5 - 24) - 2(-5 - 30) - 2(4 + 5) \\ &= 3(-19) - 2(-35) - 2(9) \\ &= -57 + 70 - 18 \\ &= -5 \neq 0 \end{aligned}$$

$\therefore \quad A^{-1}$ exists.

Let A_{ij} be the cofactor of the element a_{ij}.

Then,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 6 \\ 4 & -5 \end{vmatrix} = 5 - 24 = -19,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 6 \\ 5 & -5 \end{vmatrix} = -(-5 - 30) = 35.$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -1 \\ 5 & 4 \end{vmatrix} = 4 - 5 = 9,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -2 \\ 4 & -5 \end{vmatrix} = -(-10 + 8) = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & -2 \\ 5 & -5 \end{vmatrix} = -15 + 10 = -5,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = -(12 - 10) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -2 \\ -1 & 6 \end{vmatrix} = 12 - 2 = 10,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -2 \\ 1 & 6 \end{vmatrix} = -(18 + 2) = -20$$

and
$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -3 - 2 = -5$$

Matrix of cofactors $= \begin{pmatrix} -19 & 35 & 9 \\ 2 & -5 & -2 \\ 10 & -20 & -5 \end{pmatrix}$. Hence AdjA$= \begin{pmatrix} -19 & 2 & 10 \\ 35 & -5 & -20 \\ 9 & -2 & -5 \end{pmatrix}$

$$\therefore\ A^{-1} = \frac{1}{|A|}.Adj\ A = \frac{1}{-5}\begin{pmatrix} -19 & 2 & 10 \\ 35 & -5 & -20 \\ 9 & -2 & -5 \end{pmatrix} = \begin{bmatrix} \frac{19}{5} & \frac{-2}{5} & -2 \\ -7 & 1 & 4 \\ \frac{-9}{5} & \frac{2}{5} & 1 \end{bmatrix}$$

Note : Verify that $A^{-1}A = AA^{-1} = I_3$

Example 22.10 If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$; find

(i) $(AB)^{-1}$ (ii) $B^{-1}A^{-1}$ (iii) Is $(AB)^{-1} = B^{-1}A^{-1}$?

Solution : (i) Here, $AB = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -2+0 & 1+0 \\ -4+0 & 2+1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$$

$\therefore$ $$|AB| = \begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} = -6+4 = -2 \neq 0.$$

Thus, $(AB)^{-1}$ exists.

Let us denote AB by C_{ij}

Let C_{ij} be the cofactor of the element C_{ij} of |C|.

Then, $C_{11} = (-1)^{1+1}(3) = 3$ $C_{21} = (-1)^{2+1}(1) = -1$

$C_{12} = (-1)^{1+2}(-4) = 4$ $C_{22} = (-1)^{2+2}(-2) = -2$

Hence, $\text{Adj}(C) = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$

$$C^{-1} = \frac{1}{|C|}\text{Adj}(C) = \frac{1}{-2}\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$C^{-1} = (AB)^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

(ii) To find $B^{-1}A^{-1}$, first we will find B^{-1}.

Now. $B = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$ $\therefore |B| = \begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2 - 0 = 2 \neq 0$

$\therefore$ B^{-1} exists.

Let B_{ij} be the cofactor of the element bij of |B|

then $B_{11} = (-1)^{1+1}(-1) = -1$ $B_{21} = (-1)^{2+1}(1) = -1$

$B_{12} = (-1)^{1+2}(0) = 0$ and $B_{22} = (-1)^{2+2}(-2) = -2$

Hence, Adj $B = \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}$

$$\therefore \; B^{-1} = \frac{1}{|B|}.AdjB = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 \end{bmatrix}$$

Also, $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ Therefore, $|A| = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = 1 - 0 = -1 \quad \neq 0$

Therefore, A^{-1} exists.

Let A_{ij} be the cofactor of the element a_{ij} of |A|

then $A_{11} = (-1)^{1+1}(-1) = -1$ $A_{21} = (-1)^{2+1}(0) = 0$

$A_{12} = (-1)^{1+2}(2) = -2$ and $A_{22} = (-1)^{2+2}(1) = 1$

Hence, Adj $A = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$

$$\Rightarrow \; A^{-1} = \frac{1}{|A|} Adj\ A = \frac{1}{-1}\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

Thus, $$B^{-1}A^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} - 1 & 0 + \frac{1}{2} \\ 0 - 2 & 0 + 1 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

(iii) From (i) and (ii), we find that

$$(AB)^{-1} = B^{-1}A^{-1} = = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

Hecne, $(AB)^{-1} = B^{-1}A^{-1}$

CHECK YOUR PROGRESS 22.3

1. Find, if possible, the inverse of each of the following matrices:

(a) $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

2. Find, if possible, the inverse of each of the following matrices :

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 & 2 \\ 5 & 2 & 4 \\ 1 & -3 & -2 \end{bmatrix}$

Verify that $A^{-1}A = AA^{-1} = I$ for (a) and (b).

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

4. Find $(A')^{-1}$ if $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$

5. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$

show that ABA^{-1} is a diagonal matrix.

6. If $\phi(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $[\phi(x)]^{-1} = \phi(-x)$.

7. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

8. If $A=\begin{pmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{pmatrix}$, show that $aA^{-1}=(a^2+bc+1)I-aA$

9. If $A=\begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, show that $A^{-1}=A^2$

10. If $A=\frac{1}{9}\begin{pmatrix} -8 & 1 & 4 \\ 4 & -4 & 7 \\ 1 & -8 & 4 \end{pmatrix}$, show that $A^{-1}=A'$

22.5 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

In earlier classes, you have learnt how to solve linear equations in two or three unknowns (simultaneous equations). In solving such systems of equations, you used the process of elimination of variables. When the number of variables invovled is large, such elimination process becomes tedious.

You have already learnt an alternative method, called Cramer's Rule for solving such systems of linear equations.

We will now illustrate another method called the matrix method, which can be used to solve the system of equations in large number of unknowns. For simplicity the illustrations will be for system of equations in two or three unknowns.

22.5.1 MATRIX METHOD

In this method, we first express the given system of equation in the matrix form $AX = B$, where A is called the co-efficient matrix.

For example, if the given system of equation is $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, we express them in the matrix equation form as :

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Here, $A=\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \end{pmatrix}$ and $B=\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

If the given system of equations is $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$, then this system is expressed in the matrix equation form as:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Where, $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

Before proceding to find the solution, we check whether the coefficient matrix A is non-singular or not.

Note: If A is singular, then $|A|=0$. Hence, A^{-1} does not exist and so, this method does not work.

Consider equation $AX = B$, where $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

When $|A| \neq 0$, i.e. when $a_1b_2 - a_2b_1 \neq 0$, we multiply the equation $AX = B$ with A^{-1} on both side and get

$$A^{-1}(AX) = A^{-1}B$$

$\Rightarrow \quad (A^{-1}A)\, X = A^{-1}B$

$\Rightarrow \quad IX = A^{-1}B \qquad (\because A^{-1}A = I)$

$\Rightarrow \quad X = A^{-1}B$

Since $A^{-1} = \dfrac{1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$, we get

$$X = \frac{1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a_1b_2 - a_2b_1}\begin{pmatrix} b_2c_1 - b_1c_2 \\ -a_2c_1 + a_1c_2 \end{pmatrix} = \begin{bmatrix} \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \\ \dfrac{-a_2c_1 + a_1c_2}{a_1b_2 - a_2b_1} \end{bmatrix}$$

Hence, $x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$ and $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

Example 22.11 Using matrix method, solve the given system of linear equations.

$$\left.\begin{array}{r} 4x - 3y = 11 \\ 3x + 7y = -1 \end{array}\right\} \qquad \text{......(i)}$$

Solution: This system can be expressed in the matrix equation form as

$$\begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad \text{......(ii)}$$

Here, $A = \begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$

so, (ii) reduces to

$$AX = B \qquad \text{......(iii)}$$

Now, $|A| = \begin{vmatrix} 4 & -3 \\ 3 & 7 \end{vmatrix} = 28 + 9 = 37 \neq 0$

Since $|A| \neq 0$, A^{-1} exists.

Now, on multiplying the equation AX = B with A^{-1} on both sides, we get

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

i.e. $IX = A^{-1}B$

$$X = A^{-1}B$$

Hence, $X = \dfrac{1}{|A|}(Adj\ A)\ B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 7 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 11 \\ -1 \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 77 - 3 \\ -33 - 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 74 \\ -37 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So, $x = 2$, $y = -1$ is unique solution of the system of equations.

Example 22.12 Solve the following system of equations, using matrix method.

$$\left.\begin{array}{r} x + 2y + 3z = 14 \\ x - 2y + z = 0 \\ 2x + 3y - z = 5 \end{array}\right\}$$

Solution : The given equations expressed in the matrix equation form as :

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \\ 5 \end{pmatrix} \qquad \text{... (i)}$$

which is in the form $AX = B$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 14 \\ 0 \\ 5 \end{pmatrix}$$

$\therefore$ $X = A^{-1} B$ ... (ii)

Here, $|A| = 1(2-3) - 2(-1-2) + 3(3+4)$

$= 26 \neq 0$

$\therefore$ A^{-1} exists.

Also, Adj $A = \begin{bmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{bmatrix}$

Hence, from (ii), we have $X = A^{-1}B = \dfrac{1}{|A|} AdjA.B$

$$X = \frac{1}{26} \begin{pmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{pmatrix} \begin{pmatrix} 14 \\ 0 \\ 5 \end{pmatrix} = \frac{1}{26} \begin{pmatrix} 26 \\ 52 \\ 78 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ or, } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Thus, $x = 1$, $y = 2$ and $z = 3$ is the solution of the given system of equations.

22.6 CRITERION FOR CONSISTENCY OF A SYSTEM OF EQUATIONS

Let $AX = B$ be a system of two or three linear equations.

Then, we have the following criteria :

(1) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution, given by $X = A^{-1}B$.

(2) If $|A| = 0$, then the system may or may not be consistent and if consistent, it does not have a unique solution. If in addition,

(a) (Adj A) B $\neq$ O, then the system is inconsistent.

(b) (Adj A) B = O, then the system is consistent and has infinitely many solutions.

Note : These criteria are true for a system of '*n*' equations in '*n*' variables as well.

We now, verify these with the help of the examples and find their solutions wherever possible.

(a)
$$5x + 7y = 1$$
$$2x - 3y = 3$$

This system is consistent and has a unique solution, because $\begin{vmatrix} 5 & 7 \\ 2 & -3 \end{vmatrix} \neq 0$ Here, the matrix equation is $\begin{pmatrix} 5 & 7 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

i.e. $AX = B$ .... (i)

where, $A = \begin{pmatrix} 5 & 7 \\ 2 & -3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Here, $|A| = 5 \times (-3) - 2 \times 7 = -15 - 14 = -29 \neq 0$

$$\textit{and } A^{-1} = \frac{1}{|A|} \textit{Adj } A = \frac{1}{-29}\begin{pmatrix} -3 & -7 \\ -2 & 5 \end{pmatrix} \quad \text{.... (ii)}$$

From (i), we have $X = A^{-1}B$

i.e., $$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-29}\begin{pmatrix} -3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{24}{29} \\ -\frac{13}{29} \end{pmatrix} \quad \text{[From (i) and (ii)]}$$

Thus, $x = \frac{24}{29}$, and $y = \frac{-13}{29}$ is the unique solution of the given system of equations.

(b) $\begin{aligned} 3x + 2y &= 7 \\ 6x + 4y &= 8 \end{aligned}$

In the matrix form the system can be written as

$$\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

or, $AX = B$

where $A = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$

Here, $|A| = 3 \times 4 - 6 \times 2 = 12 - 12 = 0$

$$\text{Adj } A = \begin{pmatrix} 4 & -6 \\ -6 & 3 \end{pmatrix}$$

Also, $(\text{Adj } A)\, B = \begin{bmatrix} 4 & -6 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} -20 \\ -18 \end{bmatrix} \neq 0$

Thus, the given system of equations is inconsistent.

(c) $\left.\begin{aligned} 3x - y &= 7 \\ 9x - 3y &= 21 \end{aligned}\right\}$

In the matrix form the system can be written as

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 21 \end{pmatrix}$$

or, $AX = B$, where

$$A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix};\ X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 \\ 21 \end{pmatrix}$$

Here, $|A| = \begin{vmatrix} 3 & -1 \\ 9 & -3 \end{vmatrix} = 3 \times (-3) - 9 \times (-1) = -9 + 9 = 0$

$$\text{Adj A} = \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix}$$

Also, $(\text{Adj A})B = \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

$\therefore$ The given system has an infinite number of solutions.

Let us now consider another system of linear equations, where |A| = 0 and (Adj A) B $\neq$ O.

Consider the following system of equations

$$x + 2y + z = 5$$
$$2x + y + 2z = -1$$
$$x - 3y + z = 6$$

In matrix equation form, the above system of equations can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$$

i.e., $AX = B$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 0$ $\quad (\because C_1 = C_3)$

Also, $(\text{Adj A})B = \begin{bmatrix} 7 & -5 & 3 \\ 0 & 0 & 0 \\ -7 & 5 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$ [Verify (Adj A) yourself]

$$= \begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix} \neq O$$

Since |A| = 0 and (Adj A) B $\neq$ O,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|}(Adj \quad A)B$$

$$= \frac{1}{0}\begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix} \text{ which is undefined.}$$

The given system of linear equation will have no solution.

Thus, we find that if $|A| = 0$ and (Adj A) $B \neq$ O then the system of equations will have no solution.

We can summarise the above findings as:

(i) If $|A| \neq 0$ and (Adj A) $B \neq$ O then the system of equations will have a non-zero, unique solution.

(ii) If $|A| \neq 0$ and (Adj A) B = O, then the system of equations will have trivial solutions.

(iii) If $|A| = 0$ and (Adj A) B = O, then the system of equations will have infinitely many solutions.

(iv) If $|A| = 0$ and (Adj A) $B \neq$ O, then the system of equations will have no solution Inconsistent.

CHECK YOUR PROGRESS 22.4

1. Solve the following system of equations, using the matrix inversion method:

(a) $2x + 3y = 4$
$x - 2y = 5$

(b) $x + y = 7$
$3x - 7y = 11$

2. Solve the following system of equations using matrix inversion method:

(a) $x + 2y + z = 3$
$2x - y + 3z = 5$
$x + y - z = 7$

(b) $2x + 3y + z = 13$
$3x + 2y - z = 12$
$x + y + 2z = 5$

(c) $-x + 2y + 5z = 2$
$2x - 3y + z = 15$
$-x + y + z = -3$

(d) $2x + y - z = 2$
$x + 2y - 3z = -1$
$5x - y - 2z = -1$

3. Determine whether the following system of equations are consistent or not. If consistent, find the solution:

(a) $2x - 3y = 5$
$x + y = 7$

(b) $2x - 3y = 5$
$4x - 6y = 10$

(c) $3x + y + 2z = 3$
$-2y - z = 7$
$x + 15y + 3z = 11$

LET US SUM UP

- A square matrix is said to be non-singular if its corresponding determinant is non-zero.
- The determinant of the matrix A obtained by deleting the i[th] row and j[th] column of A, is called the minor of a_{ij}. It is usually denoted by M_{ij}.
- The cofactor of a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$
- Adjoint of a matrix A is the transpose of the matrix whose elements are the cofactors of the elements of the determinat of given matrix. It is usually denoted by Adj A.
- If A is any square matrix of order n, then

  $A\,(\text{Adj}\,A) = (\text{Adj}\,A)\,A = |A|\; I_n$ where I_n is the unit matrix of order n.
- For a given non-singular square matrix A, if there exists a non-singular square matrix B such that $AB = BA = I$, then B is called the multiplicative inverse of A. It is written as $B = A^{-1}$.
- Only non-singular square matrices have multiplicative inverse.
- If $a_1 x + b_1 y = c_1$ and $a_2 x + b_2 y = c_2$, then we can express the system in the matrix equation form as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus, if $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, then

$$X = A^{-1}B = \frac{1}{a_1 b_2 - a_2 b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- A system of equations, given by $AX = B$, is said to be consistent and has a unique solution, if $|A| \neq 0$.
- A system of equations, given by $AX = B$, is said to be inconsistent, if $|A| = 0$ and $(\text{Adj } A)\, B \neq O$.
- A system of equations, given by $AX = B$, is said be be consistent and has infinitely many solutions, if $|A| = 0$, and $(\text{Adj } A)\, B = O$.

SUPPORTIVE WEB SITES

http://www.mathsisfun.com/algebra/matrix-inverse.html

http://www.sosmath.com/matrix/coding/coding.html

TERMINAL EXERCISE

1. Find $|A|$, if

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -1 & 0 \\ -2 & 5 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} -1 & 3 & 4 \\ 7 & 5 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

2. Find the adjoint of A, if

(a) $A = \begin{bmatrix} -2 & 3 & 7 \\ -1 & 4 & 5 \\ -1 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix}$

Also, verify that $A(\text{Adj } A) = |A| I_3 = (\text{Adj } A)\, A$, for (a) and (b)

3. Find A^{-1}, if exists, when

(a) $\begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

Also, verify that $(A')^{-1} = (A^{-1})'$, for (a), (b) and (c)

4. Find the inverse of the matrix A, if

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{pmatrix}$

5. Solve, using matrix inversion method, the following systems of linear equations

(a) $\begin{aligned} x + 2y &= 4 \\ 2x + 5y &= 9 \end{aligned}$ (b) $\begin{aligned} 6x + 4y &= 2 \\ 9x + 6y &= 3 \end{aligned}$

(c) $\begin{aligned} 2x + y + z &= 1 \\ x - 2y - z &= \frac{3}{2} \\ 3y - 5z &= 9 \end{aligned}$ (d) $\begin{aligned} x - y + z &= 4 \\ 2x + y - 3z &= 0 \\ x + y + z &= 2 \end{aligned}$

(e) $\begin{aligned} x + y - 2z &= -1 \\ 3x - 2y + z &= 3 \\ 2x + y - z &= 0 \end{aligned}$

6. Solve, using matrix inversion method

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4; \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1; \quad \frac{8}{x} + \frac{9}{y} - \frac{20}{z} = 3$$

7. Find the value of λ for which the following system of equation becomes consistent

$$\begin{aligned} 2x - 3y + 4 &= 0 \\ 5x - 2y - 1 &= 0 \\ 21x - 8y + \lambda &= 0 \end{aligned}$$

ANSWERS

CHECK YOUR PROGRESS 22.1

1. (a) –12 (b) 10
2. (a) singular (b) non-singular
3. (a) $M_{11} = 4; M_{12} = 7; M_{21} = -1; M_{22} = 3$ (b) $M_{11} = 5; M_{12} = 2; M_{21} = 6; M_{22} = 0$
4. (a) $M_{21} = 11; M_{22} = 7; M_{23} = 1$ (b) $M_{31} = -13; M_{32} = -13; M_{33} = 13$
5. (a) $C_{11} = 7; C_{12} = -9; C_{21} = 2; C_{22} = 3$ (b) $C_{11} = 6; C_{12} = 5; C_{21} = -4; C_{22} = 0$
6. (a) $C_{21} = 1; C_{22} = -8; C_{23} = -2$ (b) $C_{11} = -6; C_{12} = 10; C_{33} = 2$

CHECK YOUR PROGRESS 22.2

1. (a) $\begin{bmatrix} 6 & 1 \\ -3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (c) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & i \\ -i & i \end{bmatrix}$

CHECK YOUR PROGRESS 22.3

1. (a) $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{4}{10} & -\frac{2}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

2. (a) $\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \\ -\frac{8}{5} & \frac{6}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{7}{12} & \frac{1}{3} & \frac{1}{12} \\ \frac{17}{24} & -\frac{1}{3} & -\frac{11}{24} \end{bmatrix}$

4. $(A')^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

CHECK YOUR PROGRESS 22.4

1. (a) $x = \frac{23}{7},\ y = \frac{-6}{7}$ (b) $x = 6, y = 1$

2. (a) $x = \frac{58}{11},\ y = -\frac{2}{11},\ z = -\frac{21}{11}$ (b) $x = 2,\ y = 3,\ z = 0$

(c) $x = 2,\ y = -3,\ z = 2$ (d) $x = 1,\ y = 2,\ z = 2$

3. (a) Consistent; $x = \frac{26}{5},\ y = \frac{9}{5}$ (b) Consistent; infinitely many solutions

(c) Inconsistent

TERMINAL EXERCISE

1. (a) –31 (b) –24

2. (a) $\begin{bmatrix} 4 & -3 & -13 \\ -4 & 5 & 3 \\ 4 & -3 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 8 & -7 \\ -13 & 13 & 13 \\ 5 & 1 & 4 \end{bmatrix}$

3. (a) $\begin{bmatrix} -\frac{1}{18} & \frac{1}{6} \\ \frac{7}{36} & -\frac{1}{12} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{5}{7} & -\frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{bmatrix}$ (c) $\begin{bmatrix} -\frac{1}{7} & -\frac{5}{14} \\ -\frac{2}{7} & -\frac{3}{14} \end{bmatrix}$

4. (a) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{3} & 0 \\ 3 & \frac{2}{3} & -1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

5. (a) $x = 2,\ y = 1$ (b) $x = k,\ y = \frac{1}{2} - \frac{3}{2}k$

(c) $x = 1,\ y = \frac{1}{2},\ z = -\frac{3}{2}$ (d) $x = 2,\ y = -1,\ z = 1$

(e) $x = \frac{1}{2},\ y = -\frac{1}{2},\ z = \frac{1}{2}$

6. $x = 2,\ y = 3,\ z = 5$

7. $\lambda = -5$