311en23

23

RELATIONS AND FUNCTIONS-II

We have learnt about the basic concept of Relations and Functions. We know about the ordered pair, the cartesian product of sets, relation, functions, their domain, Co-doman and range. Now we will extend our knowledge to types of relations and functions, composition of functions, invertible functions and binary operations.

OBJECTIVES

After studying this lesson, you will be able to :

- verify the equivalence relation in a set
- verify that the given function is one-one, many one, onto/ into or one one onto
- find the inverse of a given function
- determine whether a given operation is binary or not.
- check the commutativity and associativity of a binary operation.
- find the inverse of an element and identity element in a set with respest to a binary operation.

EXPECTED BACKGROUND KNOWLEDGE

Before studying this lesson, you should know :

- Concept of set, types of sets, operations on sets
- Concept of ordered pair and cartesian product of set.
- Domain, co-domain and range of a relation and a function

23.1 RELATION

23.1.1 Relation :

Let A and B be two sets. Then a relation R from Set A into Set B is a subset of A × B.

Thus, R is a relation from A to B $\Leftrightarrow$ R $\subseteq$ A × B

- If $(a, b) \in$R then we write aRb which is read as 'a' is related to b by the relation R, if $(a, b) \notin$R, then we write $a \not\mathrm{R}\, b$ and we say that a is not related to b by the relation R.
- If $n(A) = m$ and $n(B) = n$, then A × B has mn ordered pairs, therefore, total number of relations form A to B is 2^{mn}.

23.1.2 Types of Relations

(i) Reflexive Relation :

A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow$ $(a, a) \in R$ for all $a \in A$

A relation R is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Let A = {1, 2, 3} be a set. Then

R = {(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)} is a reflexive relation on A.

but R_1 = {(1, 1), (3, 3) (2, 1) (3, 2)} is not a reflexive relation on A, because $2 \in A$ but $(2,2) \notin R$.

(ii) Symmetric Relation

A relation R on a set A is said to be symmetric relation if

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $(a, b) \in A$

i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$.

Let A = {1, 2, 3, 4} and R_1 and R_2 be relations on A given by

R_1 = {(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)}

and R_2 = {(1, 1), (2, 2), (3, 3), (1, 3)}

- R_1 is symmetric relation on A because $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$

  or $aR_1b \Rightarrow bR_1a$ for all $a,b \in A$

  but R_2 is not symmetric because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$.

A reflexive relation on a set A is not necessarily symmetric. For example, the relation R ={(1, 1), (2, 2), (3, 3), (1, 3)} is a reflexive relation on set A = {1, 2, 3} but it is not symmetric.

(iii) Transitive Relation:

Let A be any set. A relation R on A is said to be transitive relation if

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$

For example :

On the set N of natural numbers, the relation R defined by xRy

$\Rightarrow$ 'x is less than y', is transitive, because for any $x, y, z \in N$

$x < y$ and $y < z \Rightarrow x < z$

i.e. xRy and $yRz \Rightarrow xRz$

Take another example

Let A be the set of all straight lines in a plane. Then the relation 'is parallel to' on A is a transitive relation, because for any $l_1, l_2, l_3 \in A$

$l_1 \parallel l_2$ and $l_2 \parallel l_3 \Rightarrow l_1 \parallel l_3$

Example 23.1 Check the relation R for reflexivity, symmetry and transitivity, where R is defined as $l_1 R l_2$ iff $l_1 \perp l_2$ for all $l_1, l_2 \in A$

Solution : Let A be the set of all lines in a plane. Given that $l_1 R l_2 \Leftrightarrow l_1 \perp l_2$ for all $l_1, l_2 \in A$

Reflexivity : R is not reflexive because a line cannot be perpendicular to itself i.e. $l \perp l$ is not true.

Symmetry : Let $l_1, l_2 \in A$ such that $l_1 R l_2$

Then $l_1 R l_2 \Rightarrow l_1 \perp l_2 \Rightarrow l_2 \perp l_1 \Rightarrow l_2 R l_1$

So, R is symmetric on A

Transitive

R is not transitive, because $l_1 \perp l_2$ and $l_2 \perp l_3$ does not impty that $l_1 \perp l_3$

23.2 EQUIVALENCE RELATION

A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

For example the relation 'is congruent to' is an equivalence relation because

(i) it is reflexive as $\Delta \cong \Delta \Rightarrow (\Delta, \Delta) \in R$ for all $\Delta \in S$ where S is a set of triangles.

(ii) it is symmetric as $\Rightarrow \Delta_1 R \Delta_2 \Rightarrow \Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1$

$$\Rightarrow \Delta_2 R \Delta_1$$

(iii) it is transitive as $\Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3 \Rightarrow \Delta_1 \cong \Delta_3$

it means $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow (\Delta_1, \Delta_3) \in R$

Example 23.2 Show that the relation R defined on the set A of all triangles in a plane as $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$ is an equivalence relation.

Solution : We observe the following properties of relation R;

Reflexivity we know that every triangle is similar to itself. Therefore, $(T, T) \in R$ for all $T \in A \Rightarrow R$ is reflexive.

Symmetricity Let $(T_1, T_2) \in R$, then

$(T_1, T_2) \in R \Rightarrow T_1$ is similar to T_2

$\Rightarrow T_2$ is similar to T_1

$\Rightarrow (T_2, T_1) \in R$, So, R is symmetric.

Transitivity : Let $T_1, T_2, T_3 \in A$ such that $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$.

Then $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow T_1$ is similar to T_2 and T_2 is similar to T_3

$\Rightarrow T_1$ is similar to T_3

$\Rightarrow (T_1, T_3) \in R$

Hence, R is an equivalence relation.

CHECK YOUR PROGRESS 23.1

1. Let R be a relation on the set of all lines in a plane defined by $(l_1, l_2) \in R \Rightarrow$ line l_1 is parallel to l_2. Show that R is an equivalence relation.
2. Show that the relation R on the set A of points in a plane, given by

   R = {(P, Q) : Distance of the point P from the origin is same as the distance of the point Q from the origin} is an equivalence relation.
3. Show that each of the relation R in the set $A = \{x \in z : 0 \le x \le 12\}$, given by

   (i) $R = \{(a,b) : |a-b| \text{ is multiple of } 4\}$

   (ii) $R = \{(a,b) : a = b\}$ is an equivalence relation
4. Prove that the relation 'is a factor of' from R to R is reflexive and transitive but not symmetric.
5. If R and S are two equivalence relations on a set A then $R \cap S$ is also an equivalence relation.
6. Prove that the relation R on set $N \times N$ defined by (a,b) R (c,d) $\Leftrightarrow$ a+d = b + c for all (a,b), (c,d) $\in N \times N$ is an equivalence relation.

23.3 CLASSIFICATION OF FUNCTIONS

Let f be a function from A to B. If every element of the set B is the image of at least one element of the set A i.e. if there is no unpaired element in the set B then we say that the **function f maps the set *A* onto the set *B*. Otherwise we say that the function maps the set *A* into the set *B*.**

Functions for which each element of the set A is mapped to a different element of the set B are said to be ***one-to-one.***

One-to-one function

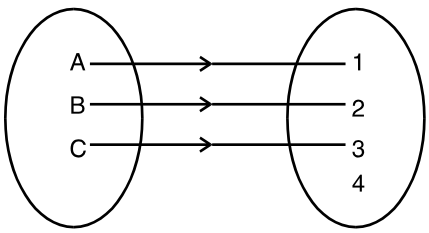

Fig.23.27

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 2, 3\}$

A function can map more than one element of the set A to the same element of the set B. Such a type of function is said to be ***many-to-one.***

Many-to-one function

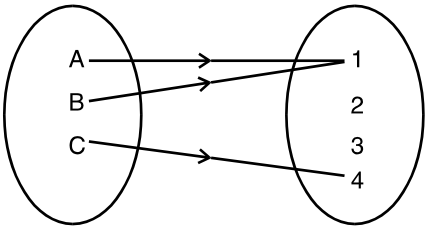

Fig. 23.2

The domain is $\{A, B, C\}$

The co-domain is $\{1, 2, 3, 4\}$

The range is $\{1, 4\}$

A function which is both one-to-one and onto is said to be a bijective function.

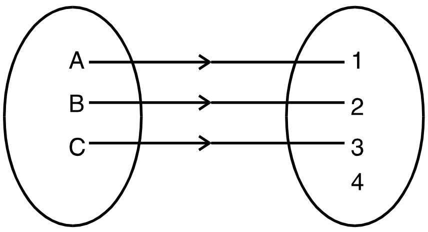

Fig. 23.3

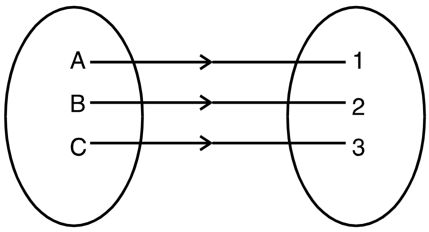

Fig. 23.4

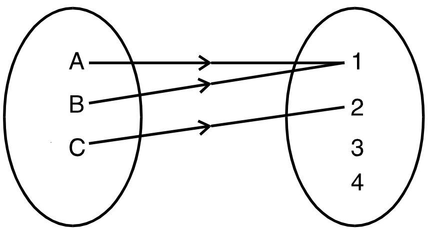

Fig. 23.5

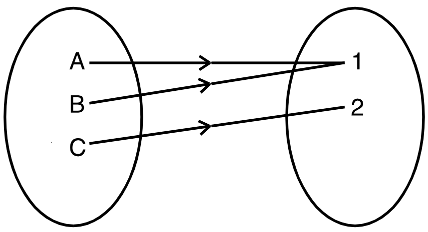

Fig. 23.6

Fig. 23.3 shows a one-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 23.4 shows a one-to-one function mapping $\{A, B, C\}$ onto $\{1, 2, 3\}$.

Fig. 23.5shows a many-to-one function mapping $\{A, B, C\}$ into $\{1, 2, 3, 4\}$.

Fig. 23.6 shows a many-to-one function mapping $\{A, B, C\}$ onto $\{1, 2\}$.

Function shown in Fig. 23.4 is also a bijective Function.

Note : Relations which are one-to-many can occur, but they are not functions. The following figure illustrates this fact.

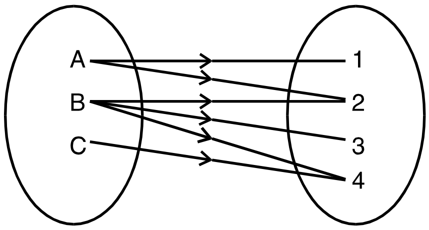

Fig. 23.7

Example 23.3 Without using graph prove that the function

$$f : R \to R \text{ defiend by } f(x) = 4 + 3x \text{ is } \textbf{\textit{one-to-one}}.$$

Solution : For a function to be one-one function

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall \quad x_1, x_2 \in \text{domain}$$

$\therefore$ Now $f(x_1) = f(x_2)$ gives

$$4 + 3x_1 = 4 + 3x_2 \quad \text{or} \quad x_1 = x_2$$

$\therefore$ f is a ***one-one function***.

Example 23.4 Prove that

$$f : R \to R \text{ defined by } f(x) = 4x^3 - 5 \text{ is a bijection}$$

Solution : Now $f(x_1) = f(x_2) \quad \forall \quad x_1, x_2 \in \text{Domain}$

$\therefore \quad 4x_1^3 - 5 = 4x_2^3 - 5$

$\Rightarrow \quad x_1^3 = x_2^3$

$\Rightarrow \quad x_1^3 - x_2^3 = 0 \Rightarrow \quad (x_2 - x_1)\left(x_1^2 + x_1x_2 + x_2^2\right) = 0$

$\Rightarrow \quad x_1 = x_2$ or

$x_1^2 + x_1x_2 + x_2^2 = 0$ (rejected). It has no real value of x_1 and x_2.

$\therefore$ f is a ***one-one function***.

Again let $y = (x)$ where $y \in$ codomain, $x \in$ domain.

We have $y = 4x^3 - 5$ or $x = \left(\dfrac{y+5}{4}\right)^{1/3}$

$\therefore$ For each $y \in$ codomain $\exists\, x \in$ domain such that $f(x) = y$.

Thus f is ***onto function***.

$\therefore$ f is a bijection.

Example 23.5 Prove that $f : R \to R$ defined by $f(x) = x^2 + 3$ is neither ***one-one*** nor ***onto function***.

Solution : We have $f(x_1) = f(x_2) \ \forall \ x_1, x_2 \in$ domain giving

$$x_1^2 + 3 = x_2^2 + 3 \Rightarrow x_1^2 = x_2^2$$

or $\quad x_1^2 - x_2^2 = 0 \Rightarrow x_1 = x_2 \quad$ or $\quad x_1 = -x_2$

or $\quad$ f is not ***one-one function***.

Again let $y = f(x)$ where $y \in$ codomain

$x \in$ domain.

$\Rightarrow \quad y = x^2 + 3 \quad \Rightarrow \quad x = \pm\sqrt{y-3}$

$\Rightarrow \quad \forall \ y < 3 \ \exists$ no real value of x in the domain.

$\therefore \quad$ f is not an ***onto fiction***.

23.4 GRAPHICAL REPRESENTATION OF FUNCTIONS

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

$$y = x^2$$

x	0	1	–1	2	–2	3	–3	4	–4
y	0	1	1	4	4	9	9	16	16

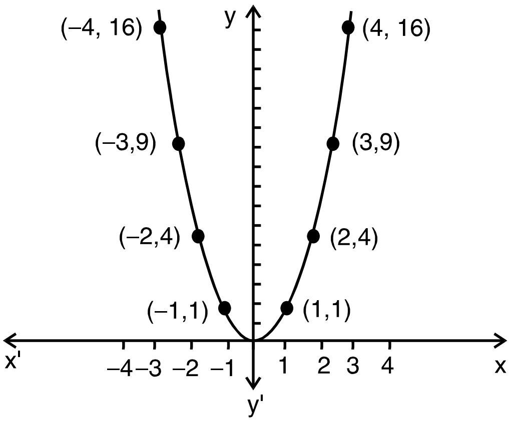

Fig. 23.8

Does this represent a function?

Yes, this represent a function because corresponding to each value of $x \ \exists$ a unique value of y.

Now consider the equation $x^2 + y^2 = 25$

$$x^2 + y^2 = 25$$

x	0	0	3	3	4	4	5	−5	−3	−3	−4	−4
y	5	−5	4	−4	3	−3	0	0	4	−4	3	−3

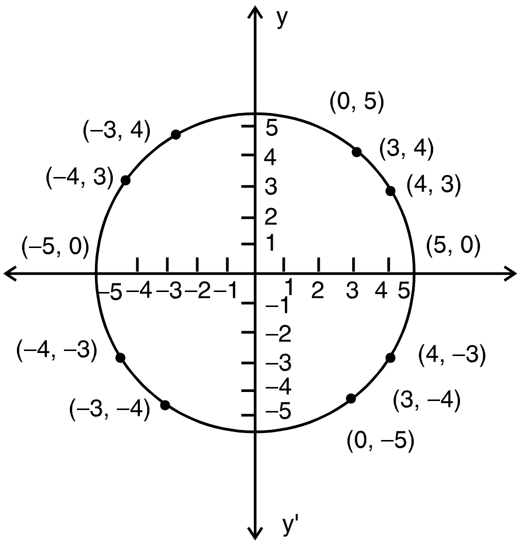

Fig. 23.9

This graph represents a circle.

Does it represent a function ?

No, this does not represent a function because corresponding to the same value of x, there does not exist a unique value of y.

CHECK YOUR PROGRESS 23.2

1. (i) Does the graph represent a function?

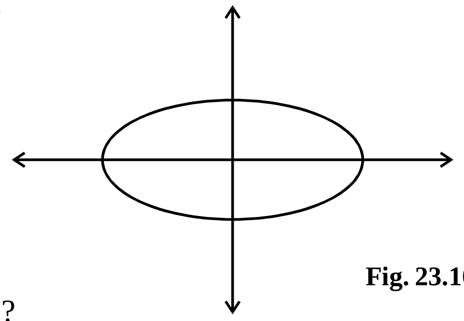

Fig. 23.10

(ii) Does the graph represent a function ?

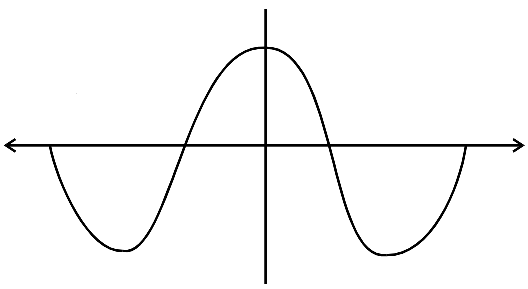

Fig. 23.11

2. Which of the following functions are into function ?

(a)

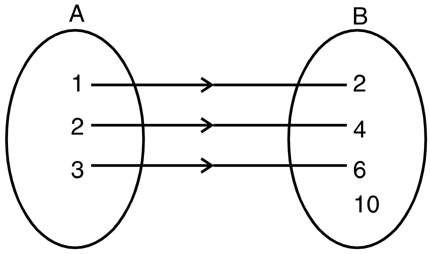

Fig.23.12

(b) $f : N \to N$, defined as $f(x) = x^2$

Here N represents the set of natural numbers.

(c) $f : N \to N$, defined as $f(x) = x$

3. Which of the following functions are onto function if $f : R \to R$

(a) $f(x) = 115x + 49$ (b) $f(x) = |x|$

4. Which of the following functions are one-to-one functions ?

(a) $f : \{20, 21, 22\} \to \{40, 42, 44\}$ defined as $f(x) = 2x$

(b) $f : \{7, 8, 9\} \to \{10\}$ defined as $f(x) = 10$

(c) $f : I \to R$ defined as $f(x) = x^3$

(d) $f : R \to R$ defined as $f(x) = 2 + x^4$

(d) $f : N \to N$ defined as $f(x) = x^2 + 2x$

5. Which of the following functions are many-to-one functions ?

(a) $f : \{-2, -1, 1, 2\} \to \{2, 5\}$ defined as $f(x) = x^2 + 1$

(b) $f : \{0, 1, 2\} \to \{1\}$ defined as $f(x) = 1$

(c)

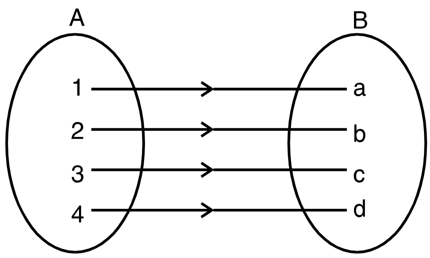

Fig.23.13

(d) $f : N \to N$ defined as $f(x) = 5x + 7$

23.5 COMPOSITION OF FUNCTIONS

Consider the two functions given below:

$$y = 2x + 1, \qquad x \in \{1, 2, 3\}$$

$$z = y + 1, \qquad y \in \{3, 5, 7\}$$

Then *z* is the composition of two functions *x* and *y* because *z* is defined in terms of *y* and *y* in terms of *x*.

Graphically one can represent this as given below :

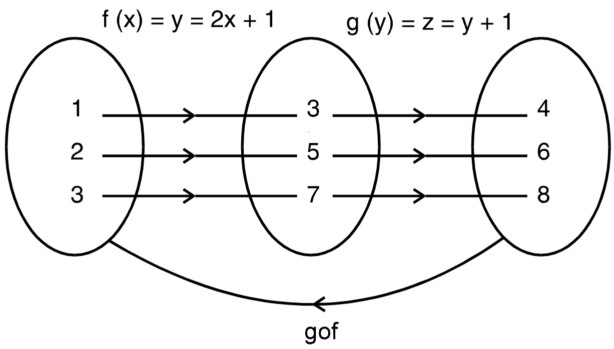

Fig. 23.18

The composition, say, gof of function g and f is defined as function g of function f.

If $\quad f : A \to B$ and $g : B \to C$

then $\quad g \, o \, f : A$ to C

Let $\qquad f(x) = 3x + 1$ and $\qquad g(x) = x^2 + 2$

Then $\qquad fog(x) = f(g(x)) = f(x^2 + 2)$

$$= 3(x^2 + 2) + 1 = 3x^2 + 7 \qquad \text{(i)}$$

and $\qquad (gof)(x) = g(f(x)) = g(3x + 1)$

$$= (3x + 1)^2 + 2 = 9x^2 + 6x + 3 \qquad \text{(ii)}$$

Check from (i) and (ii), if

$$fog = gof$$

Evidently, $\qquad fog \neq gof$

Similarly, $(fof)(x) = f(f(x)) = f(3x + 1)$ $\qquad$ [Read as function of function f].

$$= 3(3x + 1) + 1 = 9x + 3 + 1 = 9x + 4$$

$(gog)(x) = g(g(x)) = g(x^2 + 2)$ [Read as function of function g]

$$= (x^2 + 2)^2 + 2 = x^4 + 4x^2 + 4 + 2 = x^4 + 4x^2 + 6$$

Example 23. 6 If $f(x) = \sqrt{x+1}$ and $g(x) = x^2 + 2$, calculate *fog* and *gof*.

Solution :

$$fog(x) = f(g(x))$$

$$= f(x^2 + 2) = \sqrt{x^2 + 2 + 1} = \sqrt{x^2 + 3}$$

$$(gof)(x) = g(f(x))$$

$$= g(\sqrt{x+1}) = (\sqrt{x+1})^2 + 2 = x + 1 + 2 = x + 3.$$

Here again, we see that $(fog) \neq gof$

Example 23. 7 If $f(x) = x^3$, $f : R \to R$ and $g(x) = \frac{1}{x}$, $g : R - \{0\} \to R - \{0\}$

Find *fog* and *gof*.

Solution : $(fog)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 = \frac{1}{x^3}$

$$(gof)(x) = g(f(x)) = g(x^3) = \frac{1}{x^3}$$

Here we see that $fog = gof$

CHECK YOUR PROGRESS 23.3

1. Find *fog*, *gof*, *fof* and *gog* for the following functions :

   $f(x) = x^2 + 2$, $g(x) = 1 - \frac{1}{1-x}$, $x \neq 1$.

2. For each of the following functions write *fog*, *gof*, *fof* and *gog*.

   (a) $f(x) = x^2 - 4$, $g(x) = 2x + 5$

   (b) $f(x) = x^2$, $g(x) = 3$

   (c) $f(x) = 3x - 7$, $g(x) = \frac{2}{x}$, $x \neq 0$

3. Let $f(x) = |x|$, $g(x) = [x]$. Verify that $fog \neq gof$.

4. Let $f(x) = x^2 + 3$, $g(x) = x - 2$

   Prove that $fog \neq gof$ and $f\left(f\left(\frac{3}{2}\right)\right) = g\left(f\left(\frac{3}{2}\right)\right)$

5. If $f(x) = x^2$, $g(x) = \sqrt{x}$. Show that $fog = gof$.

6. Let $f(x) = |x|$, $g(x) = (x)^{\frac{1}{3}}$, $h(x) = \frac{1}{x}$; $x \neq 0$.

   Find (a) *fog* (b) *goh* (c) *foh* (d) *hog* (e) *fogoh*

23.6 INVERSE OF A FUNCTION

(A) Consider the relation

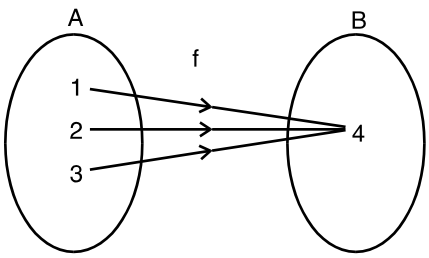

Fig. 23.19

This is a many-to-one function. Now let us find the inverse of this relation. Pictorially, it can be represented as

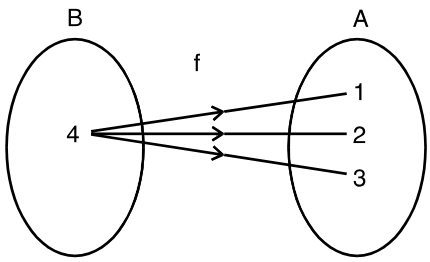

Fig 23.20

Clearly this relation does not represent a function. (Why ?)

(B) Now take another relation

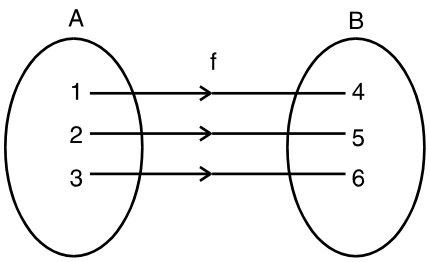

Fig.23.21

It represents one-to-one onto function. Now let us find the inverse of this relation, which is represented pictorially as

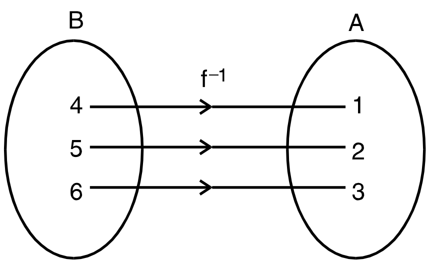

Fig. 23.22

This represents a function. (C) Consider the relation

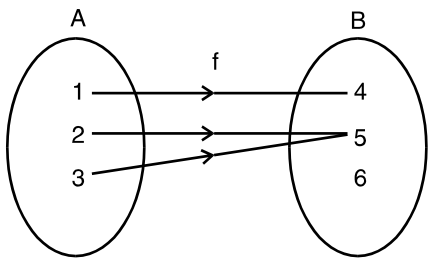

Fig. 23.23

Ir represents many-to-one function. Now find the inverse of the relation.

Pictorially it is represented as

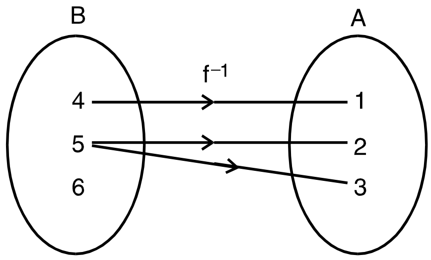

Fig. 23.24

This does not represent a function, because element 6 of set B is not associated with any element of A. Also note that the elements of B does not have a unique image.

(D) Let us take the following relation

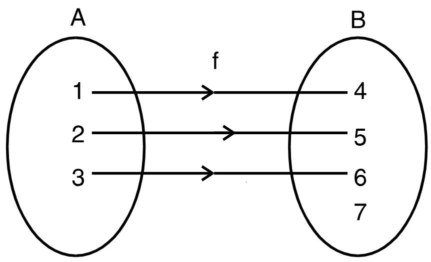

Fig. 23.25

It represent one-to-one into function. Find the inverse of the relation.

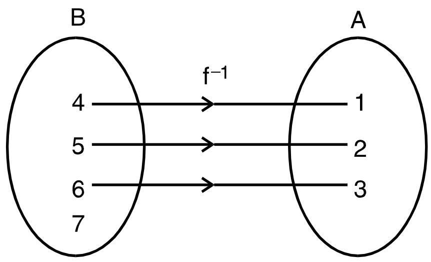

Fig. 23.26

It does not represent a function because the element 7 of *B* is not associated with any element of *A*. From the above relations we see that we may or may not get a relation as a function when we find the inverse of a relation (function).

We see that the inverse of a function exists only if the function is one-to-one onto function i.e. only if it is a bijective function.

CHECK YOUR PROGRESS 23.4

1 (i) Show that the inverse of the function

$$y = 4x - 7 \text{ exists.}$$

(ii) Let f be a one-to-one and onto function with domain A and range B. Write the domain and range of its inverse function.

2. Find the inverse of each of the following functions (if it exists) :

(a) $f(x) = x + 3 \quad \forall\ x \in R$

(b) $f(x) = 1 - 3x \quad \forall\ x \in R$

(c) $f(x) = x^2 \quad \forall\ x \in R$

(d) $f(x) = \dfrac{x+1}{x}, \quad x \neq 0 \quad x \in R$

23.7 BINARY OPERATIONS :

Let A, B be two non-empty sets, then a function from A × A to A is called a binary operation on A.

If a binary operation on A is denoted by '*', the unique element of A associated with the ordered pair (a, b) of A × A is denoted by $a * b$.

The order of the elements is taken into consideration, i.e. the elements associated with the pairs (a, b) and (b, a) may be different i.e. $a * b$ may not be equal to $b * a$.

Let A be a non-empty set and '*' be an operation on A, then

1. A is said to be closed under the operation * iff for all $a, b \in A$ implies $a * b \in A$.
2. The operation is said to be commutative iff $a * b = b * a$ for all $a, b \in A$.
3. The operation is said to be associative iff $(a * b) * c = a * (b * c)$ for all $a, b, c \in A$.
4. An element $e \in A$ is said to be an identity element iff $e * a = a = a * e$
5. An element $a \in A$ is called invertible iff these exists some $b \in A$ such that $a * b = e = b * a$, b is called inverse of a.

Note : If a non empty set A is closed under the operation *, then operation * is called a binary operation on A.

For example, let A be the set of all positive real numbers and '*' be an operation on A defined by $a * b = \frac{ab}{3}$ for all $a, b \in A$

For all $a, b, c \in A$, we have

(i) $a * b = \frac{ab}{3}$ is a positive real number $\Rightarrow$ A is closed under the given operation.

$\therefore$ * is a binary operation on A.

(ii) $a*b = \frac{ab}{3} = \frac{ba}{3} = b*a \Rightarrow$ the operation * is commutative.

(iii) $(a*b)*c = \frac{ab}{3}*c = \frac{\frac{ab}{3}.c}{3} = \frac{abc}{9}$ and $a*(b*c) = a*\frac{bc}{3} = \frac{a}{3}.\frac{bc}{3} = \frac{abc}{9}$ –

$\Rightarrow (a*b)*c = a*(b*c) \Rightarrow$ the operation * is associative.

(iv) There exists $3 \in A$ such that $3 * a = 3.\frac{a}{3} = a = \frac{a}{3}.3 = a*3$

$\Rightarrow$ 3 is an identity element.

(v) For every $a \in A$, there exists $\frac{9}{a} \in A$ such that $a*\frac{9}{a} = \frac{a.\frac{9}{a}}{3} = 3$ and $\frac{9}{a}*a = \frac{\frac{9}{a}.a}{3} = 3$

$\Rightarrow a*\frac{9}{a} = 3 = \frac{9}{a}*a \Rightarrow$ every element of A is invertible, and inverse of a is $\frac{9}{a}$

CHECK YOUR PROGRESS 23.5

1. Determine whether or not each of operation * defined below is a binary operation.

(i) $a*b = \frac{a+b}{2}, \forall a,b \in Z$

(ii) $a*b = a^b, \forall a,b \in Z$

(iii) $a*b = a^2 + 3b^2, \forall a,b \in R$

2. If $A = \{1,2\}$ find total number of binary operations on A.

3. Let a binary operation '*' on Q (set of all rational numbers) be defined as $a * b = a + 2b$ for all $a, b \in$ Q.

Prove that

(i) The given operation is not commutative.

(ii) The given operation is not associative.

4. Let * be the binary operation difined on $Q^{+} by\, a * b = \frac{ab}{3}$ for all $a, b \in Q^{+}$ then find the inwrse of 4*6.

5. Let $A = N \times N$ and * be the binary operation on A defined by (a,b)*(c,d)=(a+c, b+d). Show that * is commutative and associative. Find the identity element of on A if any

6. A binary operation * on Q - {-1} is defined by a * b = a+b+ab; for all $a, b \in Q - \{-1\}$. Find identity element on Q. Also find the inverse of an element in Q-{-1}.

LET US SUM UP

- Reflexive relation R in X is a relation with $(a, a) \in$ R $\forall\, a \in$ X.
- Symmetric relation R in X is a relation satisfying $(a, b) \in$ R implies $(b, a) \in$ R.
- Transitive relation R in X is a relation satisfying $(a, b) \in$ R and $(b, c) \in$ R implies that $(a, c) \in$ R.
- Equivalence relation R in X is a relation which is reflexive, symmetric and transitive.
- If range is a subset of co-domain that function is called on into function.
- If f: $A \to B$, and f (x) = f (y) $\to$ x = y that function is called one-one function.
- Any function is inuertible if it is one-one-onto or bijective.
- If more than one element of A has only one image in to than function is called many one function .
- A binary operation * on a set A is a function * from A × A to A.
- If $a * b = b * a$ for all $a, b \in$ A, then the operation is said to be commutative.
- If $(a * b) * c = a * (b * c)$ for all $a, b, \in$ A, then the operation is said to be associative.
- If $e * a = a = a * e$ for all $a \in$ A, then element $e \in$ A is said to be an identity element.
- If $a * b = e = b * a$ then a and b are inverse of each other
- A pair of elements grouped together in a particular order is called an a ordered pair.
- If n(A) = p, n(B) = q then n(A × B) = pq
- R × R = $\{(x, y) : x, y \in$ R$\}$ and R × R × R = $\{(x, y, z) : x, y, z \in$ R$\}$

- In a function $f : A \to B$, B is the codomain of f.
- $f, g : X \to R$ and $X \subset R$, then

  $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$

  $(f \cdot g)x = f(x) \cdot g(x)$, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
- A real function has the set of real number or one of its subsets both as its domain and as its range.

SUPPORTIVE WEBSITES

http://www.bbc.co.uk/education/asguru/maths/13pure/02functions/06composite/ index.shtml

http://mathworld.wolfram.com/Composition.html

http://www.cut-the-knot.org/Curriculum/Algebra/BinaryColorDevice.shtml

http://mathworld.wolfram.com/BinaryOperation.html

TERMINAL EXERCISE

1. Write for each of the following functions fog, gof, fof, gog.

   (a) $f(x) = x^3$ $\quad$ $g(x) = 4x - 1$

   (b) $f(x) = \frac{1}{x^2}, x \neq 0$ $\quad$ $g(x) = x^2 - 2x + 3$

   (c) $f(x) = \sqrt{x - 4}, x \geq 4$ $\quad$ $g(x) = x - 4$

   (d) $f(x) = x^2 - 1$ $\quad$ $g(x) = x^2 + 1$

2. (a) Let $f(x) = |x|$, $g(x) = \frac{1}{x}, x \neq 0$, $h(x) = x^{\frac{1}{3}}$. Find fogoh

   (b) $f(x) = x^2 + 3$, $g(x) = 2x^2 + 1$

   Find fog (3) and gof (3).

3. Which of the following equations describe a function whose inverse exists :

   (a) $f(x) = |x|$ $\quad$ (b) $f(x) = \sqrt{x}, x \geq 0$

   (c) $f(x) = x^2 - 1, x \geq 0$ $\quad$ (d) $f(x) = \frac{3x - 5}{4}$ $\quad$ (e) $f(x) = \frac{3x + 1}{x - 1}$ $x \neq 1$.

4. If $gof(x) = |\sin x|$ and $gof(x) = \left(\sin\sqrt{x}\right)^2$ then find f (x) and g (x)

5. Let $*$ be a binary operation on Q defined by $a*b=\dfrac{a+b}{3}$ for all a,b $\in Q$, prove that $*$ is commutative on Q.

6. Let $*$ be a binary operation on on the set Q of rational numbers define by $a*b=\dfrac{ab}{5}$ for all a,b $\in Q$, show that $*$ is associative on Q.

7. Show that the relation R in the set of real numbers, defined as R = $\{(a, b)\} : a \le b^2\}$ is neither reflexive, nor symmetric nor transitive.

8. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as $R=\{(a,b): b=a+1\}$ is reflexive, symmetric and transitive.

9. Show that the relation R in the set A defined as R = $\{(a, b)\ \forall : a = b\}\ a, b \in$A, is equivalence relation.

10. Let A = N × N, N being the set of natural numbers. Let * : A × A $\rightarrow$ A be defined as $(a, b) * (c, d) = \{ad + bc, bd)$ for all $(a, b), (c, d) \in$A. Show that

    (i) * is commutative

    (ii) * is associative

    (iii) identity element w.r.t * does not exist.

11. Let * be a binary operation on the set N of natural numbers defined by the rule $a * b = ab$ for all $a, b \in$N

    (i) Is * commutative? (ii) Is * associative?

ANSWERS

CHECK YOUR PROGRESS 23.2

1. (i) No (ii) Yes
2. (a), (b)
3. (a),
4. (a), (c),(e)
5. (a), (b)

CHECK YOUR PROGRESS 23.3

1. $\text{fog} = \dfrac{x^2}{(1-x)^2} + 2, \quad \text{gof} = \dfrac{x^2+2}{x^2+1}$

   $\text{fof} = x^4 + 4x^2 + 6, \quad \text{gog} = x$

2. (a) $\text{fog} = 4x^2 + 20x + 21, \quad \text{gof} = 2x^2 - 3$

   $\text{fof} = x^4 - 8x^2 + 12, \quad \text{gog} = 4x + 15$

   (b) $\text{fog} = 9, \ \text{gof} = 3, \ \text{fof} = x^4, \ \text{gog} = 3$

   (c) $\text{fog} = \dfrac{6-7x}{x}, \quad \text{gof} = \dfrac{2}{3x-7}, \quad \text{fof} = 9x - 28, \quad \text{gog} = x$

6. (a) $\text{fog} = \left| x^{\frac{1}{3}} \right|$ (b) $\text{goh} = \dfrac{1}{x^{\frac{1}{3}}}$ (c) $\text{foh} = \left| \dfrac{1}{x} \right|$

   (d) $\text{hog} = \dfrac{1}{x^{\frac{1}{3}}}$ (e) $\text{fogoh}(1) = 1$

CHECK YOUR PROGRESS 23.4

1. (ii) Domain is B. Range is A.

2. (a) $f^{-1}(x) = x - 3$ (b) $f^{-1}(x) = \dfrac{1-x}{3}$

   (c) Inverse does not exist. (d) $f^{-1}(x) = \dfrac{1}{x-1}$

CHECK YOUR PROGRESS 23.5

1. (i) No (ii) Yes (iii) Yes
2. 16
4. $\frac{9}{8}$
5. (0,0)
6. $\text{identity} = 0,\ a^{-1} = \frac{-a}{a+1}$

TERMINAL EXERCISE

1. (a) $fog = (4x-1)^3,\ gof = 4x^3 - 1,\ fog = x^9,\ gog = 16x - 5$

   (b) $fog = \frac{1}{(x^2-2x+3)^2},\ gof = \frac{3x^4 - 2x^2 + 1}{x^4},\ fof = x^4, gog - x^4 - 4x^3 + 4x^2$

   (c) $fog = \sqrt{x-8},\ gof = \sqrt{x-4} - 4,\ fof = \sqrt{\sqrt{x-4}-4} gog = x - 8$

   (d) $fog = x^4 + 2x^2,\ gof = x^4 - 2x^2 + 2,\ fof = x^4 - 2x^2,\ gog = x^4 + 2x^2 + 2,$

2. (a) $\left|\frac{1}{x^{1/3}}\right|$, (b) $(fog)(3) = 364,\ (gof)(3) = 289$
3. (c), (d), (e),
4. $f(x) = \sin^2 x,\ g(x) = \sqrt{x}$
8. Neither reflexive, nor symmetric, nor transitive
9. Yes, R is an equivalence relation
11. (i) Not commutative