28

311en28

DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

We are aware that population generally grows but in some cases decay also. There are many other areas where growth and decay are continuous in nature. Examples from the fields of Economics, Agriculture and Business can be cited, where growth and decay are continuous. Let us consider an example of bacteria growth. If there are 10,00,000 bacteria at present and say they are doubled in number after 10 hours, we are interested in knowing as to after how much time these bacteria will be 30,00,000 in number and so on.

Answers to the growth problem does not come from addition (repeated or otherwise), or multiplication by a fixed number. In fact Mathematics has a tool known as exponential function that helps us to find growth and decay in such cases. Exponential function is inverse of logarithmic function. We shall also study about Rolle's Theorem and Mean Value Theorems and their applications. In this lesson, we propose to work with this tool and find the rules governing their derivatives.

OBJECTIVES

After studying this lesson, you will be able to :

- define and find the derivatives of exponential and logarithmic functions;
- find the derivatives of functions expressed as a combination of algebraic, trigonometric, exponential and logarithmic functions; and
- find second order derivative of a function.
- state Rolle's Theorem and Lagrange's Mean Value Theorem; and
- test the validity of the above theorems and apply them to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Application of the following standard limits :

(i) $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$ (ii) $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(iii) $\lim\limits_{h \to 0} \frac{\left(e^h - 1\right)}{h} = 1$

- Definition of derivative and rules for finding derivatives of functions.

28.1 DERIVATIVE OF EXPONENTIAL FUNCTIONS

Let $y = e^x$ be an exponential function. .....(i)

$\therefore \quad y + \delta y = e^{(x+\delta x)}$ (Corresponding small increments) .....(ii)

From (i) and (ii), we have

$$\therefore \quad \delta y = e^{x+\delta x} - e^x$$

Dividing both sides by δx and taking the limit as $\delta x \to 0$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} e^x \frac{\left[e^{\delta x} - 1\right]}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x . 1 = e^x$$

Thus, we have $\mathbf{\frac{d}{dx}\left(e^x\right) = e^x}$.

Working rule : $\frac{d}{dx}\left(e^x\right) = e^x \cdot \frac{d}{dx}(x) = e^x$

Next, let $y = e^{ax+b}$.

Then $y + \delta y = e^{a(x+\delta x)+b}$

[δx and δy are corresponding small increments]

$$\therefore \quad \delta y = e^{a(x+\delta x)+b} - e^{ax+b}$$

$$= e^{ax+b}\left[e^{a\delta x} - 1\right]$$

$$\therefore \quad \frac{\delta y}{\delta x} = e^{ax+b} \frac{\left[e^{a\delta x} - 1\right]}{\delta x}$$

$$= a \cdot e^{ax+b} \frac{e^{a\delta x} - 1}{a\delta x}$$ [Multiply and divide by a]

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = a \cdot e^{ax+b} \cdot \lim_{\delta x \to 0} \frac{e^{a\delta x} - 1}{a\ \delta x}$$

or $$\frac{dy}{dx} = a \cdot e^{ax+b} \cdot 1 \qquad \left[\lim_{x \to 0} \frac{e^x - 1}{x} = 1\right] \quad = ae^{ax+b}$$

Working rule :

$$\frac{d}{dx}\left(e^{ax+b}\right) = e^{ax+b} \cdot \frac{d}{dx}(ax+b) = e^{ax+b} \cdot a$$

$\therefore$ $$\frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$$

Example 28.1 Find the derivative of each of the follwoing functions :

(i) e^{5x} (ii) e^{ax} (iii) $e^{-\frac{3x}{2}}$

Soution : (i) Let $y = e^{5x}$.

Then $y = e^{t}$ where $5x = t$

$\therefore$ $\frac{dy}{dt} = e^{t}$ and $5 = \frac{dt}{dx}$

We know that, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = e^{t} \cdot 5 = 5e^{5x}$

Alternatively $\frac{d}{dx}\left(e^{5x}\right) = e^{5x} \cdot \frac{d}{dx}(5x) = e^{5x} \cdot 5 = 5e^{5x}$

(ii) Let $y = e^{ax}$.

Then $y = e^{t}$ when $t = ax$

$\therefore$ $\frac{dy}{dt} = e^{t}$ and $\frac{dt}{dx} = a$

We know that, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = e^{t} \cdot a$

Thus, $\frac{dy}{dx} = a \cdot e^{ax}$

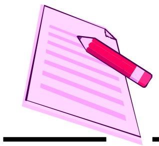
(iii) Let $y = e^{\frac{-3x}{2}}$

$\therefore$ $\frac{dy}{dt} = e^{\frac{-3}{2}x} \cdot \frac{d}{dx}\left(\frac{-3}{2}x\right)$

Thus, $\frac{dy}{dt} = \frac{-3}{2} e^{\frac{-3x}{2}}$

Example 28.2 Find the derivative of each of the following :

(i) $y = e^{x} + 2\cos x$ (ii) $y = e^{x^2} + 2\sin x - \frac{5}{3}e^{x} + 2e$



Solution : (i) $y = e^x + 2\cos x$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(e^x\right) + 2\frac{d}{dx}(\cos x) = e^x - 2\sin x$

(ii) $y = e^{x^2} + 2\sin x - \frac{5}{3}e^x + 2e$

$\therefore \quad \frac{dy}{dx} = e^{x^2}\frac{d}{dx}\left(x^2\right) + 2\cos x - \frac{5}{3}e^x + 0$ [Since e is constant]

$$= 2xe^{x^2} + 2\cos x - \frac{5}{3}e^x$$

Example 28.3 Find $\frac{dy}{dx}$, when

(i) $y = e^{x\cos x}$ (ii) $y = \frac{1}{x}e^x$ (iii) $y = e^{\frac{1-x}{1+x}}$

Solution : (i) $y = e^{x\cos x}$

$\therefore \quad \frac{dy}{dx} = e^{x\cos x}\frac{d}{dx}(x\cos x)$

$\therefore \quad \frac{dy}{dx} = e^{x\cos x}\left[x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}(x)\right]$

$$= e^{x\cos x}\left[-x\sin x + \cos x\right]$$

(ii) $y = \frac{1}{x}e^x$

$\therefore \quad \frac{dy}{dx} = e^x\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\frac{d}{dx}\left(e^x\right)$ [Using product rule]

$$= \frac{-1}{x^2}e^x + \frac{1}{x}e^x$$

$$= \frac{e^x}{x^2}[-1+x] = \frac{e^x}{x^2}[x-1]$$

(iii) $y = e^{\frac{1-x}{1+x}}$

$$\frac{dy}{dx} = e^{\frac{1-x}{1+x}}\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$= e^{\frac{1-x}{1+x}}\left[\frac{-1.(1+x)-(1-x).1}{(1+x)^2}\right]$$

$$= e^{\frac{1-x}{1+x}}\left[\frac{-2}{(1+x)^2}\right] = \frac{-2}{(1+x)^2}e^{\frac{1-x}{1+x}}$$

Example 28.4 Find the derivative of each of the following functions :

(i) $e^{\sin x} \cdot \sin e^x$ (ii) $e^{ax} \cdot \cos(bx+c)$

Solution : $y = e^{\sin x} \cdot \sin e^x$

$$\therefore \quad \frac{dy}{dx} = e^{\sin x} \cdot \frac{d}{dx}\left(\sin e^x\right) + \sin e^x \frac{d}{dx} e^{\sin x}$$

$$= e^{\sin x} \cdot \cos e^x \cdot \frac{d}{dx}\left(e^x\right) + \sin e^x \cdot e^{\sin x} \frac{d}{dx}(\sin x)$$

$$= e^{\sin x} \cdot \cos e^x \cdot e^x + \sin e^x \cdot e^{\sin x} \cdot \cos x$$

$$= e^{\sin x}[e^x \cdot \cos e^x + \sin e^x \cdot \cos x]$$

(ii) $y = e^{ax} \cos(bx+c)$

$$\therefore \quad \frac{dy}{dx} = e^{ax} \cdot \frac{d}{dx}\cos(bx+c) + \cos(bx+c)\frac{d}{dx}e^{ax}$$

$$= e^{ax} \cdot [-\sin(bx+c)]\frac{d}{dx}(bx+c) + \cos(bx+c)e^{ax}\frac{d}{dx}(ax)$$

$$= -e^{ax}\sin(bx+c) \cdot b + \cos(bx+c)e^{ax}.a$$

$$= e^{ax}[-b\sin(bx+c) + a\cos(bx+c)]$$

Example 28.5 Find $\frac{dy}{dx}$, if $y = \frac{e^{ax}}{\sin(bx+c)}$

Solution :

$$\frac{dy}{dx} = \frac{\sin(bx+c)\frac{d}{dx}e^{ax} - e^{ax}\frac{d}{dx}[\sin(bx+c)]}{\sin^2(bx+c)}$$

$$= \frac{\sin(bx+c).e^{ax}.a - e^{ax}\cos(bx+c).b}{\sin^2(bx+c)}$$

$$= \frac{e^{ax}[a\sin(bx+c) - b\cos(bx+c)]}{\sin^2(bx+c)}$$

CHECK YOUR PROGRESS 28.1

1. Find the derivative of each of the following functions :

(a) e^{5x} (b) e^{7x+4} (c) $e^{\sqrt{2}x}$ (d) $e^{\frac{-7}{2}x}$ (e) e^{x^2+2x}

2. Find $\frac{dy}{dx}$, if

(a) $y=\frac{1}{3}e^x-5e$ (b) $y=\tan x+2\sin x+3\cos x-\frac{1}{2}e^x$

(c) $y=5\sin x-2e^x$ (d) $y=e^x+e^{-x}$

3. Find the derivative of each of the following functions :

(a) $f(x)=e^{\sqrt{x+1}}$ (b) $f(x)=e^{\sqrt{\cot x}}$

(c) $f(x)=e^{x\sin^2 x}$ (d) $f(x)=e^{x\sec^2 x}$

4. Find the derivative of each of the following functions :

(a) $f(x)=(x-1)e^x$ (b) $f(x)=e^{2x}\sin^2 x$

5. Find $\frac{dy}{dx}$, if

(a) $y=\frac{e^{2x}}{\sqrt{x^2+1}}$ (b) $y=\frac{e^{2x}\cdot\cos x}{x\sin x}$

28.2 DERIVATIVE OF LOGARITHMIC FUNCTIONS

We first consider logarithmic function

Let $y=\log x$ .....(i)

$\therefore$ $y+\delta y=\log(x+\delta x)$ .....(ii)

(δx and δy are corresponding small increments in x and y)

From (i) and (ii), we get

$$\delta y=\log(x+\delta x)-\log x$$

$$=\log\frac{x+\delta x}{x}$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{1}{\delta x}\log\left[1+\frac{\delta x}{x}\right]$$

$$=\frac{1}{x}\cdot\frac{x}{\delta x}\log\left[1+\frac{\delta x}{x}\right] \quad \text{[Multiply and divide by x]}$$

$$=\frac{1}{x}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

Taking limits of both sides, as $\delta x\to 0$, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{1}{x}\lim_{\delta x\to 0}\log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \log\left\{ \lim_{\delta x \to 0} \left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \right\}$$

$$= \frac{1}{x} \log e \qquad \left[\because \lim_{\delta x \to 0} \left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}} = e\right]$$

$$= \frac{1}{x}$$

Thus, $\quad \mathbf{\dfrac{d}{dx}(\log x) = \dfrac{1}{x}}$

Next, we consider logarithmic function

$$y = \log(ax + b) \qquad \text{...(i)}$$

$\therefore \quad y + \delta y = \log[a(x + \delta x) + b] \qquad$ ...(ii)

[δx and δy are corresponding small increments]

From (i) and (ii), we get

$$\delta y = \log[a(x + \delta x) + b] - \log(ax + b)$$

$$= \log \frac{a(x + \delta x) + b}{ax + b}$$

$$= \log \frac{(ax + b) + a\delta x}{ax + b}$$

$$= \log\left[1 + \frac{a\delta x}{ax + b}\right]$$

$\therefore \quad \dfrac{\delta y}{\delta x} = \dfrac{1}{\delta x} \log\left[1 + \dfrac{a\delta x}{ax + b}\right]$

$$= \frac{a}{ax + b} \cdot \frac{ax + b}{a\delta x} \log\left[1 + \frac{a\delta x}{ax + b}\right] \quad \left[\text{Multiply and divide by } \frac{a}{ax+b}\right]$$

$$= \frac{a}{ax + b} \log\left[1 + \frac{a\delta x}{ax + b}\right]^{\frac{ax+b}{a\delta x}}$$

Taking limits on both sides as $\delta x \to 0$

$\therefore \quad \displaystyle\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{a}{ax + b} \lim_{\delta x \to 0} \log\left[1 + \frac{a\delta x}{ax + b}\right]^{\frac{ax+b}{a\delta x}}$

or $\quad \dfrac{dy}{dx} = \dfrac{a}{ax + b} \log e \qquad \left[\because \displaystyle\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e\right]$

or, $\quad \dfrac{dy}{dx} = \dfrac{a}{ax + b}$

Working rule :

$$\frac{d}{dx}\log(ax+b)=\frac{1}{ax+b}\frac{d}{dx}(ax+b)$$

$$=\frac{1}{ax+b}\times a=\frac{a}{ax+b}$$

Example 28.6 Find the derivative of each of the functions given below :

(i) $y=\log x^5$ (ii) $y=\log\sqrt{x}$ (iii) $y=(\log x)^3$

Solution : (i) $y=\log x^5=5\log x$

$\therefore$ $\frac{dy}{dx}=5\cdot\frac{1}{x}=\frac{5}{x}$

(ii) $y=\log\sqrt{x}\;=\log x^{\frac{1}{2}}$ or $y=\frac{1}{2}\log x$

$\therefore$ $\frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{x}=\frac{1}{2x}$

(iii) $y=(\log x)^3$

$\therefore$ $y=t^3,$ when $t=\log x$

$\Rightarrow$ $\frac{dy}{dt}=3t^2$ and $\frac{dt}{dx}=\frac{1}{x}$

We know that, $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=3t^2\cdot\frac{1}{x}$

$\therefore$ $\frac{dy}{dx}=3(\log x)^2\cdot\frac{1}{x}$

$\therefore$ $\frac{dy}{dx}=\frac{3}{x}(\log x)^2$

Example 28.7 Find, $\frac{dy}{dx}$ if

(i) $y=x^3\log x$ (ii) $y=e^x\log x$

Solution :

(i) $y=x^3\log x$

$\therefore$ $\frac{dy}{dx}=\log x\frac{d}{dx}(x^3)+x^3\frac{d}{dx}(\log x)$ [Using Product rule]

$$=3x^2\log x+x^3\cdot\frac{1}{x}$$

$$= x^2(3\log x + 1)$$

(ii) $$y = e^x \log x$$

$$\therefore \quad \frac{dy}{dx} = e^x \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx} e^x$$

$$= e^x \cdot \frac{1}{x} + e^x \cdot \log x$$

$$= e^x \left[\frac{1}{x} + \log x \right]$$

Example 28.8 Find the derivative of each of the following functions :

(i) $\log \tan x$ (ii) $\log [\cos (\log x)]$

Solution : (i) Let $y = \log \tan x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x)$$

$$= \frac{1}{\tan x} \cdot \sec^2 x$$

$$= \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}$$

$$= \text{cosec } x . \sec x$$

(ii) Let $y = \log [\cos (\log x)]$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos(\log x)} \cdot \frac{d}{dx}[\cos(\log x)]$$

$$= \frac{1}{\cos(\log x)} \cdot \left[-\sin \log x \frac{d}{dx}(\log x) \right]$$

$$= \frac{-\sin(\log x)}{\cos(\log x)} \cdot \frac{1}{x}$$

$$= -\frac{1}{x} \tan(\log x)$$

Example 28.9 Find $\frac{dy}{dx}$, if $y = \log(\sec x + \tan x)$

Solution : $y = \log(\sec x + \text{tax } x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{\sec x + \tan x} \cdot \left[\sec x \tan x + \sec^2 x \right]$$

$$= \frac{1}{\sec x + \tan x} \cdot \sec x \left[\sec x + \tan x\right]$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

$$= \sec x$$

Example 28.10 Find $\frac{dy}{dx}$, if

$$y = \frac{(4x^2-1)(1+x^2)^{\frac{1}{2}}}{x^3(x-7)^{\frac{3}{4}}}$$

Solution : Although, you can find the derivative directly using quotient rule (and product rule) but if you take logarithm on both sides, the product changes to addition and division changes to subtraction. This simplifies the process:

$$y = \frac{(4x^2-1)(1+x^2)^{\frac{1}{2}}}{x^3(x-7)^{\frac{3}{4}}}$$

Taking logarithm on both sides, we get

$$\therefore \quad \log y = \log\left[\frac{\left(4x^2-1\right)\left(1+x^2\right)^{\frac{1}{2}}}{x^3\left(x-7\right)^{\frac{3}{4}}}\right]$$

or $$\log y = \log\left(4x^2-1\right) + \frac{1}{2}\log\left(1+x^2\right) - 3\log x - \frac{3}{4}\log(x-7)$$

[Using log properties]

Now, taking derivative on both sides, we get

$$\frac{d}{dx}(\log y) = \frac{1}{4x^2-1}\cdot 8x + \frac{1}{2(1+x^2)}\cdot 2x - \frac{3}{x} - \frac{3}{4}\cdot\left(\frac{1}{x-7}\right)$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{8x}{4x^2-1} + \frac{x}{1+x^2} - \frac{3}{x} - \frac{3}{4\left(x-7\right)}$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{8x}{4x^2-1} + \frac{x}{1+x^2} - \frac{3}{x} - \frac{3}{4(x-7)}\right]$$

$$= \frac{(4x^2-1)\sqrt{1+x^2}}{x^3(x-7)^{\frac{3}{4}}}\left[\frac{8x}{4x^2-1} + \frac{x}{1+x^2} - \frac{3}{x} - \frac{3}{4(x-7)}\right]$$

CHECK YOUR PROGRESS 28.2

1. Find the derivative of each the functions given below:

   (a) $f(x) = 5 \sin x \; 2 \log x$ (b) $f(x) = \log \cos x$

2. Find $\frac{dy}{dx}$, if

   (a) $y = e^{x^2} \log x$ (b) $y = \frac{e^{x^2}}{\log x}$

3. Find the derivative of each of the following functions :

   (a) $y = \log(\sin \log x)$ (b) $y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

   (c) $y = \log\left[\frac{a + b\tan x}{a - b\tan x}\right]$ (d) $y = \log(\log x)$

4. Find $\frac{dy}{dx}$, if

   (a) $y = (1+x)^{\frac{1}{2}}(2-x)^{\frac{2}{3}}(x^2+5)^{\frac{1}{7}}(x+9)^{-\frac{3}{2}}$ (b) $y = \frac{\sqrt{x}(1-2x)^{\frac{3}{2}}}{(3+4x)^{\frac{5}{4}}(3-7x^2)^{\frac{1}{4}}}$

28.3 DERIVATIVE OF LOGARITHMIC FUNCTION (CONTINUED)

We know that derivative of the function x^n w.r.t. x is $n x^{n-1}$, where n is a constant. This rule is not applicable, when exponent is a variable. In such cases we take logarithm of the function and then find its derivative.

Therefore, this process is useful, when the given function is of the type $[f(x)]^{g(x)}$. For example, a^x, x^x etc.

Note : Here f(x) may be constant.

Derivative of a^x w.r.t. x

Let $y = a^x, \quad a > 0$

Taking log on both sides, we get

$$\log y = \log a^x = x \log a \qquad [\log m^n = n \log m]$$

$$\therefore \quad \frac{d}{dx}(\log y) = \frac{d}{dx}(x \log a) \quad \text{or} \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log a \times \frac{d}{dx}(x)$$

or
$$\frac{dy}{dx} = y \log a$$

$$= a^x \log a$$

Thus, $\qquad \mathbf{\frac{d}{dx} a^x = a^x \log a}, \qquad a > 0$

Example 28.11 Find the derivative of each of the following functions :

(i) $y = x^x$ $\qquad$ (ii) $y = x^{\sin x}$

Solution : (i) $\quad y = x^x$

Taking logrithms on both sides, we get

$$\log y = x \log x$$

Taking derivative on both sides, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log x \frac{d}{dx}(x) + x \frac{d}{dx}(\log x) \qquad \text{[Using product rule]}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \log x + x \cdot \frac{1}{x}$$

$$= \log x + 1$$

$\therefore \qquad \frac{dy}{dx} = y[\log x + 1]$

Thus, $\qquad \frac{dy}{dx} = x^x (\log x + 1)$

(ii) $\qquad y = x^{\sin x}$

Taking logarithm on both sides, we get

$$\log y = \sin x \log x$$

$\therefore \qquad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\sin x \log x)$

or $\qquad \frac{1}{y} \cdot \frac{dy}{dx} = \cos x . \log x + \sin x \cdot \frac{1}{x}$

or $\qquad \frac{dy}{dx} = y\left[\cos x \log x + \frac{\sin x}{x}\right]$

Thus, $\qquad \frac{dy}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right]$

Example 28.12 Find the derivative, if

$$y = (\log x)^x + \left(\sin^{-1} x\right)^{\sin x}$$

Solution : Here taking logarithm on both sides will not help us as we cannot put

$(\log x)^x + (\sin^{-1} x)^{\sin x}$ in simpler form. So we put

$$u = (\log x)^x \quad \text{and} \quad v = (\sin^{-1} x)^{\sin x}$$

Then, $y = u + v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{......(i)}$$

Now $u = (\log x)^x$

Taking log on both sides, we have

$$\log u = \log(\log x)^x$$

$$\therefore \quad \log u = x \log(\log x) \quad \left[\because \log m^n = n \log m\right]$$

Now, finding the derivative on both sides, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = 1 \cdot \log(\log x) + x \frac{1}{\log x} \cdot \frac{1}{x}$$

Thus, $$\frac{du}{dx} = u\left[\log(\log x) + \frac{1}{\log x}\right]$$

Thus, $$\frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right] \quad \text{.....(ii)}$$

Also, $v = (\sin^{-1} x)^{\sin x}$

$$\therefore \quad \log v = \sin x \log(\sin^{-1} x)$$

Taking derivative on both sides, we have

$$\frac{d}{dx}(\log v) = \frac{d}{dx}[\sin x \log(\sin^{-1} x)]$$

$$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} + \cos x \cdot \log\left(\sin^{-1} x\right)$$

or, $$\frac{dv}{dx} = v\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \cdot \log \sin^{-1} x\right]$$

$$= \left(\sin^{-1} x\right)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \log\left(\sin^{-1} x\right)\right] \quad \text{....(iii)}$$

From (i), (ii) and (iii), we have

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right] + \left(\sin^{-1} x\right)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^2}} + \cos x \log \sin^{-1} x\right]$$

Example 28.13 If $x^y = e^{x-y}$, prove that

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

Solution : It is given that $\qquad x^y = e^{x-y} \qquad$ ...(i)

Taking logarithm on both sides, we get

$$y \log x = (x-y) \log e$$

$$= (x-y)$$

or $\qquad y(1+\log x) = x \qquad [\because \quad \log e = 1]$

or $\qquad y = \frac{x}{1+\log x} \qquad$ (ii)

Taking derivative with respect to x on both sides of (ii), we get

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x\left(\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

Example 28.14 Find , $\frac{dy}{dx}$ if

$$e^x \log y = \sin^{-1} x + \sin^{-1} y$$

Solution : We are given that

$$e^x \log y = \sin^{-1} x + \sin^{-1} y$$

Taking derivative with respect to x of both sides, we get

$$e^x\left(\frac{1}{y}\frac{dy}{dx}\right) + e^x \log y = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$$

or $$\left[\frac{e^x}{y} - \frac{1}{\sqrt{1-y^2}}\right]\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - e^x \log y$$

or $$\frac{dy}{dx} = \frac{y\sqrt{1-y^2}\left[1 - e^x\sqrt{1-x^2}\log y\right]}{\left[e^x\sqrt{1-y^2} - y\right]\sqrt{1-x^2}}$$

Example 28.15 Find $\frac{dy}{dx}$, if $y = (\cos x)^{(\cos x)^{(\cos x)......\infty}}$

Solution : We are given that

$$y = (\cos x)^{(\cos x)^{(\cos x)......\infty}} = (\cos x)^y$$

Taking logarithm on both sides, we get

$$\log y = y \log \cos x$$

Differentiating (i) w.r.t.x, we get

$$\frac{1}{y}\frac{dy}{dx} = y \cdot \frac{1}{\cos x}(-\sin x) + \log(\cos x) \cdot \frac{dy}{dx}$$

or $$\left[\frac{1}{y} - \log(\cos x)\right]\frac{dy}{dx} = -y \tan x$$

or $$[1 - y\log(\cos x)]\frac{dy}{dx} = -y^2 \tan x$$

or $$\frac{dy}{dx} = \frac{-y^2 \tan x}{1 - y\log(\cos x)}$$

CHECK YOUR PROGRESS 28.3

1. Find the derivative with respect to x of each the following functions :

   (a) $y = 5^x$ (b) $y = 3^x + 4^x$ (c) $y = \sin(5^x)$

2. Find $\frac{dy}{dx}$, if

   (a) $y = x^{2x}$ (b) $y = (\cos x)^{\log x}$ (c) $y = (\log x)^{\sin x}$

   (d) $y = (\tan x)^x$ (e) $y = (1 + x^2)^{x^2}$ (f) $y = x^{(x^2 + \sin x)}$

3. Find the derivative of each of the functions given below :

   (a) $y = (\tan x)^{\cot x} + (\cot x)^x$ (b) $y = x^{\log x} + (\sin x)^{\sin^{-1} x}$

   (c) $y = x^{\tan x} + (\sin x)^{\cos x}$ (d) $y = (x)^{x^2} + (\log x)^{\log x}$

4. If $y = (\sin x)^{(\sin x)^{(\sin x)......\infty}}$, show that

   $$\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y\log(\sin x)}$$

5. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + ...\infty}}}$, show that

   $$\frac{dy}{dx} = \frac{1}{x(2x - 1)}$$

28.4 SECOND ORDER DERIVATIVES

In the previous lesson we found the derivatives of second order of trigonometric and inverse trigonometric functions by using the formulae for the derivatives of trigonometric and inverse trigonometric functions, various laws of derivatives, including chain rule, and power rule discussed earlier in lesson 21. In a similar manner, we will discuss second order derivative of exponential and logarithmic functions :

Example 28.16 Find the second order derivative of each of the following :

(i) e^x (ii) $\cos(\log x)$ (iii) x^x

Solution : (i) Let $y = e^x$

Taking derivative w.r.t. x on both sides, we get $\frac{dy}{dx} = e^x$

Taking derivative w.r.t. x on both sides, we get $\frac{d^2y}{dx^2} = \frac{d}{dx}(e^x) = e^x$

$$\therefore \qquad \frac{d^2y}{dx^2} = e^x$$

(ii) Let $y = \cos(\log x)$

Taking derivative w.r.t x on both sides, we get

$$\frac{dy}{dx} = -\sin(\log x) \cdot \frac{1}{x} = \frac{-\sin(\log x)}{x}$$

Taking derivative w.r.t. x on both sides, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{\sin(\log x)}{x}\right]$$

or

$$= -\frac{x \cdot \cos(\log x) \cdot \frac{1}{x} - \sin(\log x)}{x^2}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{\sin(\log x) - \cos(\log x)}{x^2}$$

(iii) Let $y = x^x$

Taking logarithm on both sides, we get

$$\log y = x \log x \qquad ....(i)$$

Taking derivative w.r.t. x of both sides, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x = 1 + \log x$$

or

$$\frac{dy}{dx} = y(1 + \log x) \qquad ....(ii)$$

Taking derivative w.r.t. x on both sides we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[y(1+\log x)]$$

$$= y \cdot \frac{1}{x} + (1+\log x)\frac{dy}{dx} \qquad \text{...(iii)}$$

$$= \frac{y}{x} + (1+\log x)y(1+\log x)$$

$$= \frac{y}{x} + (1+\log x)^2 y \qquad \text{(Using (ii))}$$

$$= y\left[\frac{1}{x} + (1+\log x)^2\right]$$

$$\therefore \qquad \frac{d^2y}{dx^2} = x^x\left[\frac{1}{x} + (1+\log x)^2\right]$$

Example 28.17 If $y = e^{a\cos^{-1}x}$, show that

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0 \cdot$$

Solution : We have, $y = e^{a\cos^{-1}x}$ .....(i)

$$\therefore \qquad \frac{dy}{dx} = e^{a\cos^{-1}x} \cdot \frac{-a}{\sqrt{1-x^2}}$$

$$= -\frac{ay}{\sqrt{1-x^2}} \qquad \text{Using (i)}$$

or $$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)^2\left(1-x^2\right) - a^2y^2 = 0 \qquad \text{.....(ii)}$$

Taking derivative of both sides of (ii), we get

$$\left(\frac{dy}{dx}\right)^2(-2x) + 2\left(1-x^2\right) \times \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} - a^2 \cdot 2y \cdot \frac{dy}{dx} = 0$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0 \qquad \left[\text{Dividing through out by } 2 \cdot \frac{dy}{dx}\right]$$

CHECK YOUR PROGRESS 28.4

1. Find the second order derivative of each of the following :

(a) $x^4 e^{5x}$ (b) $\tan\left(e^{5x}\right)$ (c) $\dfrac{\log x}{x}$

2. If $y = a\cos(\log x) + b\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

3. If $y = e^{\tan^{-1}x}$, prove that

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

28.5 DERIVATIVE OF PARAMETRIC FUNCTIONS

Sometimes *x* and *y* are two variables such that both are explicitly expressed in terms of a third variable, say *t*, i.e. if $x = f(t)$ and $y = g(t)$, then such functions are called parametric functions and the third variable is called the parameter.

In order to find the derivative of a function in parametric form, we use chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$$

or

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ provided } \frac{dx}{dt} \neq 0$$

Example 28.18 Find $\dfrac{dy}{dx}$, when $x = a\sin t,\ y = a\cos t$

Differentiating w.r. to 't', we get

$$\frac{dx}{dt} = a\cos t \text{ and } \frac{dy}{dt} = -a\sin t$$

Hence, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a\sin t}{a\cos t} = -\tan t$$

Example 28.19 Find $\dfrac{dy}{dx}$, if $x = 2at^2$ and $y = 2at$.

Solution : Given $x = 2at^2$ and $y = 2at$.

Differentiating w.r. to 't', we get

$$\frac{dx}{dt} = 4at \text{ and } \frac{dy}{dt} = 2a$$

Hence $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{4at} = \frac{1}{2t}$

Example 28.20 Find $\frac{dy}{dx}$, If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

Solution : Given

$$x = a(\theta - \sin\theta) \text{ and}$$

$$y = a(1 + \cos\theta)$$

Differentiating both w.r. to 'θ', we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a(-\sin\theta)$$

Hence $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)} = -\cot\theta/2$$

Example 28.21 Find $\frac{dy}{dx}$, if $x = a\cos^3 t$ and $y = a\sin^3 t$

Solution : Given $x = a\cos^3 t$ and $y = a\sin^3 t$

Differentiating both w.r. to 't', we get

$$\frac{dx}{dt} = 3a\cos^2 t\frac{d}{dt}(\cos t) = -3a\cos^2 t\sin t$$

and $$\frac{dy}{dt} = 3a\sin^2 t\frac{d}{dt}(\sin t) = 3a\sin^2 t\cos t$$

Hence $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

Example 28.22 Find $\frac{dy}{dx}$, If $x = a\frac{1-t^2}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$.

Solution : Given $x = a\frac{1-t^2}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$

Differentiating both w.r. to 't', we get

$$\frac{dx}{dt} = a\left\{\frac{(1+t^2).(0-2t)-(1-t^2)(0+2t)}{(1+t^2)^2}\right\} = \frac{-4at}{(1+t^2)^2}$$

and $$\frac{dy}{dt} = 2b\left\{\frac{(1+t^2).(1)-t.(0+2t)}{(1+t^2)^2}\right\} = \frac{2b(1-t^2)}{(1+t^2)^2}$$

Hence $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at} = \frac{-b(1-t^2)}{2at}$$

CHECK YOUR PROGRESS 28.5

Find $\frac{dy}{dx}$, when :

1. $x = 2at^3$ and $y = at^4$
2. $x = a \cos \theta$ and $y = a \sin \theta$
3. $x = 4t$ and $y = \frac{4}{t}$
4. $x = b\sin^2\theta$ and $y = a\cos^2\theta$
5. $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$
6. $x = a\sec\theta$ and $y = b\tan\theta$
7. $x = \frac{3at}{1+t^2}$ and $y = \frac{3at^2}{1+t^2}$
8. $x = \sin 2t$ and $y = \cos 2t$

28.6 SECOND ORDER DERIVATIVE OF PARAMETRIC FUNCTIONS

If two parametric functions $x = f(t)$ and $y = g(t)$ are given then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = h(t) \quad \left(\text{let here } \frac{dx}{dt} \neq 0\right)$$

Hence $$\frac{d^2y}{dx^2} = \frac{d}{dt}\big((h(t)\big) \times \frac{dt}{dx}$$

Example 28.23 Find $\frac{d^2y}{dx^2}$, if $x = at^2$ and $y = 2at$

Solution : Differentiating both w.r.t. to 't', we get

$$\frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

Differentiating both sides w.r.t. to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \times \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{t^2} \times \frac{1}{2at} = -\frac{1}{2at^3}$$

Example 28.24 Find $\frac{d^2y}{dx^2}$, if $x = a \sin^3\theta$ and $y = b \cos^3\theta$

Solution : Given $x = a \sin^3\theta$ and $y = b \cos^3\theta$

Differentiating both w.r.t. 'θ', we get

$$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3b\cos^2\theta(-\sin\theta)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3b\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\frac{b}{a}\cot\theta$$

Differentiating both sides w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\frac{d}{dx}(\cot\theta) = \frac{-b}{a}\frac{d}{d\theta}(\cot\theta) \times \frac{d\theta}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b}{a}(-\text{cosec}^2\theta) \times \frac{1}{3a\sin^2\theta\cos\theta}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{b}{3a^2}\text{cosec}^4\theta\sec\theta$$

Example 28.25 If $x = a\sin t$ and $y = b\cos t$, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$

Solution : Given $x = a\sin t$ and $y = b\cos t$

Differentiating both w.r.t. 't', we get

$$\frac{dx}{dt} = a\cos t \quad \text{and} \quad \frac{dy}{dt} = -b\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-b\sin t}{a\cos t} = \frac{-b}{a}\tan t$$

Differentiating both sides w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\frac{d}{dt}(\tan t) \times \frac{dt}{dx} = \frac{-b}{a}\sec^2 t \times \frac{1}{a\cos t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b}{a^2}\sec^3 t$$

$$\left(\frac{d^2y}{dx^2}\right) at\ t = \frac{\pi}{4} = \frac{-b}{a^2}\sec^3\frac{\pi}{4} = \frac{-b}{a^2}(\sqrt{2})^3 = \frac{-2\sqrt{2}b}{a^2}$$

CHECK YOUR PROGRESS 28.6

Find $\frac{d^2y}{dx^2}$, when

1. $x = 2at$ and $y = at^2$
2. $x = a(t + \sin t)$ and $y = a(1 - \cos t)$
3. $x = 10(\theta - \sin\theta)$ and $y = 12(1 - \cos\theta)$
4. $x = a\sin t$ and $y = b\cos 2t$
5. $x = a - \cos 2t$ and $y = b - \sin 2t$

LET US SUM UP

- (i) $\frac{d}{dx}(e^x) = e^x$ (ii) $\frac{d}{dx}(a^x) = a^x \log a$; $a > 0$
- If μ is a derivable function of x, then

  (i) $\log a \cdot \frac{dx}{dx}$; $a > 0$

  (iii) $\frac{d}{dx}(e^{ax+b}) = e^{ax+b} \cdot a = ae^{ax+b}$
- (i) $\frac{d}{dx}(\log x) = \frac{1}{x}$ (ii) $\frac{d}{dx}(\log x) = \frac{1}{x}\cdot\frac{d\mu}{dx}$, if μ is a derivable function of x.
- (iii) $\frac{d}{dx}\log(ax+b) = \frac{1}{ax+b}\cdot a = \frac{a}{ax+b}$
- If $x = f(t)$ and $y = g(t)$,

  then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$,. provided $\frac{dx}{dt} \neq 0$

- If $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = h(t)$,

  then $\frac{d^2y}{dx^2} = \frac{d}{dt}[h(t)] \times \frac{dt}{dx}$

SUPPORTIVE WEB SITES

http://www.themathpage.com/acalc/exponential.htm

http://www.math.brown.edu/utra/explog.html

http://www.freemathhelp.com/derivative-log-exponent.html

TERMINAL EXERCISE

1. Find the derivative of each of the following functions :

   (a) $(x^x)^x$ (b) $x^{(x^x)}$

2. Find $\frac{dy}{dx}$, if

   (a) $y = a^{x \log \sin x}$ (b) $y = (\sin x)^{\cos^{-1} x}$

   (c) $y = \left(1 + \frac{1}{x}\right)^{x^2}$ (d) $y = \log\left[e^x \left(\frac{x-4}{x+4}\right)^{\frac{3}{4}}\right]$

3. Find the derivative of each of the functions given below :

   (a) $f(x) = \cos x \log(x) e^{x^2} x^x$ (b) $f(x) = (\sin^{-1} x)^2 \cdot x^{\sin x} \cdot e^{2x}$

4. Find the derivative of each of the following functions :

   (a) $y = (\tan x)^{\log x} + (\cos x)^{\sin x}$ (b) $y = x^{\tan x} + (\sin x)^{\cos x}$

5. Find $\frac{dy}{dx}$, if (a) If $y = \frac{x^4 \sqrt{x+6}}{(3x+5)^2}$ (b) If $y = \frac{e^x + e^{-x}}{(e^x - e^{-x})}$

6. Find $\frac{dy}{dx}$, if (a) If $y = a^x \cdot x^a$ (b) $y = 7^{x^2+2x}$

7. Find the derivative of each of the following functions :

(a) $y = x^2 e^{2x} \cos 3x$ (b) $y = \dfrac{2^x \cot x}{\sqrt{x}}$

8. If $y = x^{x^{x^{x\ldots\ldots\infty}}}$, prove that $x\dfrac{dy}{dx} = \dfrac{y^2}{1 - y\log x}$

Find derivative of each of the following function

9. $(\sin x)^{\cos x}$

10. $(\log x)^{\log x}$

11. $\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

12. $\left(x + \dfrac{1}{x}\right)^x + x^{x + \frac{1}{x}}$

13. $x = a\left(\cos t + \log\dfrac{t}{2}\right)$ and $y = a\sin t$

14. $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

15. $x = e^t(\sin t + \cos t)$ and $y = e^t(\sin t - \cos t)$

16. $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

17. $x = a\left(t + \dfrac{1}{t}\right)$ and $y = a\left(t - \dfrac{1}{t}\right)$

18. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$

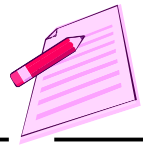
19. If $x = \dfrac{2bt}{1+t^2}$ and $y = \dfrac{a(1-t^2)}{1+t^2}$, find $\dfrac{dy}{dx}$ at $t = 2$.

20. If $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$, prove that $\dfrac{dy}{dx} = -\cot 3t$

21. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, prove that $\dfrac{dy}{dx} = \tan\left(\dfrac{3\theta}{2}\right)$

22. If $x = \cos t$ and $y = \sin t$, prove that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{3}}$ at $t = \dfrac{2\pi}{3}$

23. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$



24. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$

25. If $x = a \sin pt$ and $y = b \cos pt$, find the value of $\frac{d^2y}{dx^2}$ at $t = 0$

26. If $x = \log t$ and $y = \frac{1}{t}$, find $\frac{d^2y}{dx^2}$

27. If $x = a(1 + \cos t)$ and $y = a(t + \sin t)$, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$

28. If $x = at^2$ and $y = 2at$, find $\frac{d^2y}{dx^2}$.

ANSWERS

CHECK YOUR PROGRESS 28.1

1. (a) $5e^{5x}$ (b) $7e^{7x+4}$ (c) $\sqrt{2}e^{\sqrt{2}x}$ (d) $-\frac{7}{2}e^{-\frac{7}{2}x}$ (e) $2(x+1)e^{x^2+2x}$

2. (a) $\frac{1}{3}e^x$ (b) $\sec^2 x+2\cos x-3\sin x-\frac{1}{2}e^x$ (c) $5\cos x-2e^x$ (d) e^x-e^{-x}

3. (a) $\frac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$ (b) $e^{\sqrt{\cot x}}\left[\frac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}}\right]$

   (c) $e^{x\sin^2 x}[\sin x+2x\cos x]\sin x$ (d) $e^{x\sec^2 x}\left[\sec^2 x+2x\sec^2 x\tan x\right]$

4. (a) xe^x (b) $2e^{2x}\sin x(\sin x+\cos x)$

5. (a) $\frac{2x^2-x+2}{(x^2+1)^{3/2}}e^{2x}$ (b) $\frac{e^{2x}[(2x-1)\cot x-x\operatorname{cosec}^2 x]}{x^2}$

CHECK YOUR PROGRESS 28.2

1. (a) $5\cos x-\frac{2}{x}$ (b) $-\tan x$

2. (a) $e^{x^2}\left[2x\log x+\frac{1}{x}\right]$ (b) $\frac{2x^2\log x-1}{x(\log x)^2}.e^{x^2}$

3. (a) $\frac{\cot(\log x)}{x}$ (b) $\sec x$ (c) $\frac{2ab\sec^2 x}{a^2-b^2\tan^2 x}$ (d) $\frac{1}{x\log x}$

4. (a) $(1+x)^{\frac{1}{2}}(2-x)^{\frac{2}{3}}(x^2+5)^{\frac{1}{7}}(x+9)^{-\frac{3}{2}}\times\left[\frac{1}{2(1+x)}-\frac{2}{3(2-x)}+\frac{2x}{7(x^2-5)}-\frac{3}{2(x+9)}\right]$

   (b) $\frac{\sqrt{x}(1-2x)^{\frac{3}{2}}}{(3+4x)^{\frac{5}{4}}(3-7x^2)^{\frac{1}{4}}}\left[\frac{1}{2x}-\frac{3}{1-2x}-\frac{5}{3+4x}+\frac{7x}{2(3-7x^2)}\right]$

CHECK YOUR PROGRESS 28.3

1. (a) $5^x\log 5$ (b) $3^x\log 3+4^x\log 4$ (c) $\cos 5^x 5^x\log 5$

2. (a) $2x^{2x}(1+\log x)$ (b) $(\cos x)^{\log x}\left[\frac{\log\cos x}{x}-\tan x\log x\right]$

   (c) $(\log x)^{\sin x}\left[\cos x\log(\log x)+\frac{\sin x}{x\log x}\right]$

(d) $(\tan x)^x \left[\log \tan x + \dfrac{x}{\sin x \cos x}\right]$ (e) $(1+x)^{x^2}\left[2x\log(1+x^2) + 2\dfrac{x^3}{1+x^2}\right]$

(f) $x^{(x^2+\sin x)}\left[\dfrac{x^2+\sin x}{x} + (2x+\cos x)\log x\right]$

3. (a) $\operatorname{cosec}^2 x(1-\log\tan x)(\tan x)^{\cot x} + \left(\log\cot x - x\operatorname{cosec}^2 x\tan x\right)(\cot x)^x$

(b) $2x^{(\log x-1)}\log x + (\sin x)^{\sin^{-1}x}\left[\cot x\sin^{-1}x + \dfrac{\log\sin x}{\sqrt{1-x^2}}\right]$

(c) $x^{\tan x}\left(\dfrac{\tan x}{x} + \sec^2 x\log x\right) + (\sin x)^{\cos x}[\cos x\cot x - \sin x\log\sin x]$

(d) $(x)^{x^2}\cdot x(1+2\log x) + (\log x)^{\log x}\left[\dfrac{1+\log(\log x)}{x}\right]$

CHECK YOUR PROGRESS 28.4

1. (a) $e^{5x}\left(25x^4 + 40x^3 + 12x^2\right)$ (b) $25e^{5x}\sec^2\left(e^{5x}\right)\left\{1 + 2e^{5x}\tan e^{5x}\right\}$

(c) $\dfrac{2\log x - 3}{x^3}$

CHECK YOUR PROGRESS 28.5

1. $\dfrac{2t}{3}$ 2. $-\cot\theta$ 3. $-\dfrac{1}{t^2}$ 4. $-\dfrac{a}{b}$ 5. $\dfrac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$

6. $\dfrac{b}{a}\operatorname{cosec}\theta$ 7. $\dfrac{2t}{1-t^2}$ 8. $-\tan 2t$

CHECK YOUR PROGRESS 28.6

1. $\dfrac{1}{2a}$ 2. $\dfrac{\sec^4 t/2}{4a}$ 3. $\dfrac{-3}{100}\operatorname{cosec}^4 \theta/2$ 4. $\dfrac{-4b}{a^2}$ 5. $\operatorname{cosec}^3 2t$

TERMINAL EXERCISE

1. (a) $(x^x)^x[x + 2x\log x]$ (b) $x^{(x)^x}[x^{x-1} + \log x(\log x + 1)x^x]$

2. (a) $a^{x\log\sin x}[\log\sin x + x\cot x]\log a$

(b) $(\sin x)^{\cos^{-1}x}\left[\cos^{-1}x\cot x - \dfrac{\log\sin x}{\sqrt{1-x^2}}\right]$

(c) $\left(1+\frac{1}{x}\right)^{x^2}\left[2x\log\left(x+\frac{1}{x}\right)-\frac{1}{1+\frac{1}{x}}\right]$ (d) $1+\frac{3}{4(x-4)}-\frac{3}{4(x+4)}$

3. (a) $\cos x\log(x)e^{x^2}\cdot x^x\left[-\tan x+\frac{1}{x\log x}+2x+1+\log x\right]$

(b) $(\sin^{-1}x)^2.x^{\sin x}e^{2x}\left[\frac{2}{\sqrt{1-x^2}\sin^{-1}x}+\cos x\log x+\frac{\sin x}{x}+2\right]$

4. (a) $(\tan x)^{\log x}\left[2\operatorname{cosec}2x\log x+\frac{1}{x}\log\tan x\right]$

$+(\cos x)^{\sin x}[-\sin x\tan x+\cos x\log(\cos x)]$

(b) $x^{\tan x}\left[\frac{\tan x}{x}+\sec^2 x\log x\right]+(\sin x)^{\cos x}[\cot x\cos x-\sin x\log\sin x)]$

5. (a) $\frac{x^4\sqrt{x+6}}{(3x+5)^2}\left[\frac{4}{x}+\frac{1}{2(x+6)}-\frac{6}{(3x+5)}\right]$ (b) $\frac{-4e^{2x}}{(e^{2x}-1)^2}$

6. (a) $a^x.x^{a-1}[a+x\log_e a]$ (b) $7^{x^2+2x}(2x+2)\log_e 7$

7. (a) $x^2e^{2x}\cos 3x\left\{\frac{2}{x}+2-3\tan 3x\right\}$ (b) $\frac{2^x\cot x}{\sqrt{x}}\left[\log 2-2\operatorname{cosec}2x-\frac{1}{2x}\right]$

9. $(\sin x)^{\cos x}[-\sin x\log\sin x+\cos x.\cot x]$

10. $(\log x)^{\log x}\left[\frac{\log(\log x)+1}{x}\right]$

11. $\frac{(x-1)(x-2)}{(x-3)(x-4)}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}\right]$

12. $\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]+x^{x+\frac{1}{x}}\left[\frac{x^2-1}{x^2}\log x+\frac{x^2+1}{x^2}\right]$

13. $\tan t$ 14. $\tan\theta$ 15. $\tan t$ 16. $\frac{-y\log x}{x\log y}$ 17. $\frac{x}{y}$ 18. $-\sqrt{3}$

19. $\frac{4a}{3b}$ 20. $\frac{\sec^3\theta}{a\theta}$ 21. $\frac{1}{a}$ 22. $\frac{-b}{a^2}$ 23. $\frac{1}{t}$ 24. $\frac{-1}{a}$

25. $\frac{-1}{2at^3}$ 26. $\frac{-1}{t}$ 27. -2 28. $\frac{1}{t}$