36

STRAIGHT LINES

In Fig. 36.1, we see a rectangular box having six faces, which are parts of six planes. In the figure, ABCD and EFGH are parallel planes. Similarly, ADGH and BCFE are parallel planes and so are ABEH and CFGD. Two planes ABCD and CFGD intersect in the line CD. Similarly, it happens with any two adjacent planes. Also two edges, say AB and AH meet in the vertex A. It also happens with any two adjacent edges. We can see that the planes meet in lines and the edges meet in vertices.

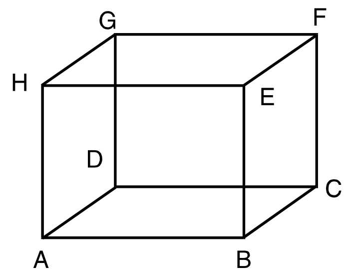

Fig. 36.1

In this lesson, we will study the equations of a line in space in symmetric form, reducing the general equation of a line into symmetric form, finding the perpendicular distance of a point from a line and finding the angle between a line and a plane. We will also establish the condition of coplanarity of two lines.

OBJECTIVES

After studying this lesson, you will be able to :

- find the equations of a line in space in symmetric form;
- convert the general equations of a line into symmetric form;
- find the perpendicular distance of a point from a line;
- find the angle between a line and a plane; and
- find the condition of coplanarity of two lines.

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of three dimensional geometry.
- Direction cosines/ratios of a line and projection of a line segment on another line.
- Condition of parallelism and perpendicularity of two lines.
- General equation of a plane.
- Equations of a plane in different forms.
- Angle between two planes.

36.1 VECTOR EQUATION OF A LINE

A line is uniquely defermined if, it passes through a given point and it has a given direction or it passes through two given points.

16.1.1 Equation of a line through a given point and parallel to a given vector : Let l be the line which passes through the point A and which is parallel to the vector $\vec{b}$. Let $\vec{a}$ be position vector of the point A and $\vec{r}$ be the position vector of an arbitrary point P on the line.

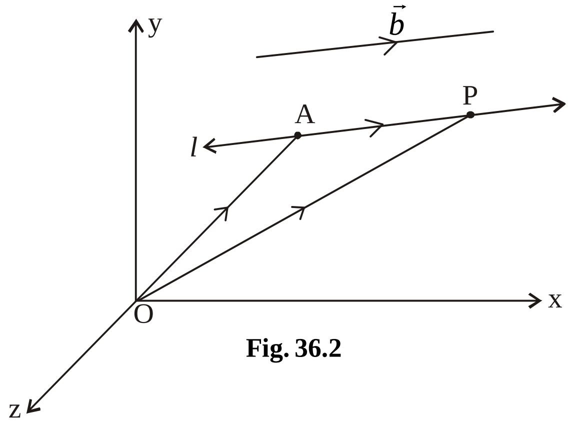

Fig. 36.2

In ΔOAP, $\quad \overrightarrow{OA}+\overrightarrow{AP} = \overrightarrow{OP}$

i.e. $\quad \overrightarrow{AP} = \overrightarrow{OP}-\overrightarrow{OA}=\vec{r}-\vec{a}$

But $\quad \overrightarrow{AP} \parallel \vec{b} \quad \therefore \quad \overrightarrow{AP}=\lambda\vec{b}$

$\therefore \quad \vec{r}-\vec{a} = \lambda\vec{b}$ ...(1)

$\Rightarrow \quad \vec{r}=\vec{a}+\lambda\vec{b}$ is the required equation of the line in vector from

36.1.2 Conversion of Vector form into Cartesian form :

Let (x_1, y_1, z_1) be the coordinates of the given point A and b_1, b_2, b_3 be the direction ratios of vector $\vec{b}$. Consider (x, y, z) as the coordinates of point P.

Then $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}, \quad \vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$

and $\quad \vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}.$

Substituting these values in equation (1) we get

$$(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k} = \lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$

$$\Rightarrow \quad \frac{x-x_1}{b_1}=\lambda, \frac{y-y_1}{b_2}=\lambda, \frac{z-z_1}{b_3}=\lambda$$

$\therefore \quad \dfrac{x-x_1}{b_1}=\dfrac{y-y_1}{b_2}=\dfrac{z-z_1}{b_3}$ is the corresponding Cartesian form of equation of the line. This is also known as symmetric form of equation of line.

36.1.3 EQUATION OF THE LINE PASSING THROUGH TWO GIVEN POINTS :

Let l be the line which passes through two points A and B. Let $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B respectively. Let $\vec{r}$ be the position vector of an arbitary point P on the line.

From the figure,

$$\overrightarrow{AP} = \vec{r} - \vec{a}$$

and $$\overrightarrow{AB} = \vec{b} - \vec{a}$$

But $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear vectors

$$\therefore \quad \overrightarrow{AP} = \lambda \overrightarrow{AB}$$

i.e. $$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a}) \qquad ...(2)$$

$$\Rightarrow \quad \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

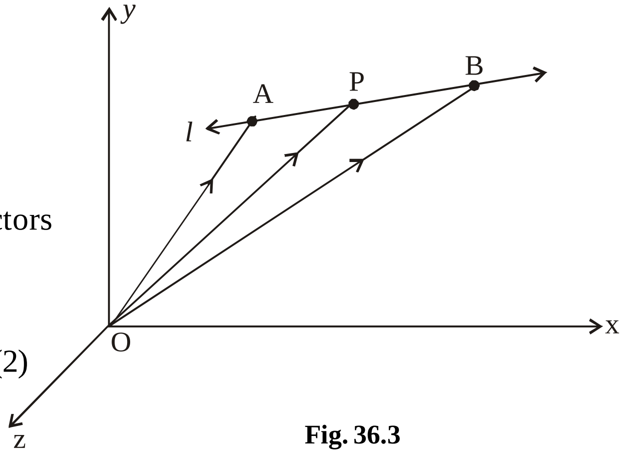

Fig. 36.3

which is the required equation in vector form.

36.1.4 CONVERSION OF VECTOR FORM INTO CARTESIAN FORM

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of point A and B respectively. Consider (x, y, z) as the coordinates of point P.

Then $$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

and $$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

Substituting these values in equation (2) we get

$$(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = \lambda(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

$$\Rightarrow \quad \frac{x - x_1}{x_2 - x_1} = \lambda, \frac{y - y_1}{y_2 - y_1} = \lambda, \frac{z - z_1}{z_2 - z_1} = \lambda$$

$\therefore \quad \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$ is the corresponding Cartesian form of equation of the line. This is known as two point form of equation of line.

Example 36.1 Find the vector equation of the line through the point (2, –3, 5) and parallel to the vector $\hat{i} + 2\hat{j} - 3\hat{k}$.

Solution : Here $$\vec{a} = 2\hat{i} - 3\hat{j} + 5\hat{k}$$

and $\qquad \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$

$\therefore \qquad \vec{r} = (2\hat{i} - 3\hat{j} + 5\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$

which is the required equation of the line.

Example 36.2 Find the vector equation of a line passing through the points (–1, 5, 2) and (4, 3, –5).

Solution : Vector equation of line in two point form is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

Here $\qquad \vec{a} = -\hat{i} + 5\hat{j} + 2\hat{k}$

and $\qquad \vec{b} = 4\hat{i} + 3\hat{j} - 5\hat{k}$

$\therefore \qquad \vec{b} - \vec{a} = 5\hat{i} - 2\hat{j} - 7\hat{k}$

Hence, the required equation is $\qquad \vec{r} = (-\hat{i} + 5\hat{j} + 2\hat{k}) + \lambda(5\hat{i} - 2\hat{j} - 7\hat{k})$

Example 36.3 Write the following equation of a line in vector form $\dfrac{x+3}{2} = \dfrac{y-2}{-3} = \dfrac{z-5}{7}$.

Solution : Comparing the given equation with $\dfrac{x - x_1}{b_1} = \dfrac{y - y_1}{b_2} = \dfrac{z - z_1}{b_3}$

We get $\qquad x_1 = -3,\ y_1 = 2,\ z_1 = 5$

$\qquad b_1 = 2,\ b_2 = -3,\ b_3 = 7$

$\therefore \qquad \vec{a} = (-3\hat{i} + 2\hat{j} + 5\hat{k})$

and $\qquad \vec{b} = (2\hat{i} - 3\hat{j} + 7\hat{k})$

Hence, $\qquad \vec{r} = (-3\hat{i} + 2\hat{j} + 5\hat{k}) + \lambda(2\hat{i} - 3\hat{j} + 7\hat{k})$ is the required equation in vector form.

Example 36.4 Find the equations of the line through the point $(1, 2, -3)$ with direction cosines

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

Solution : The equations of the line are

$$\frac{x-1}{\frac{1}{\sqrt{3}}} = \frac{y-2}{\frac{1}{\sqrt{3}}} = \frac{z+3}{-\frac{1}{\sqrt{3}}}$$

or $$\frac{x-1}{1}=\frac{y-2}{1}=\frac{z+3}{-1}$$

or $$x-1=y-2=-(z+3)$$

Example 36.5 Find the equations of a line passing through the point $(1,-3,2)$ and having direction ratios $(1,-2,3)$

Solution : The equations of the line are

$$\frac{x-1}{1}=\frac{y+3}{-2}=\frac{z-2}{3}$$

Example 36.6 Find the equations of the line passing through two points $(1,-3,2)$ and $(4,2,-3)$

Solution : The equations of the required line are

$$\frac{x-1}{4-1}=\frac{y+3}{2+3}=\frac{z-2}{-3-2} \quad \text{or} \quad \frac{x-1}{3}=\frac{y+3}{5}=\frac{z-2}{-5}$$

Example 36.7 Find the equations of the line passing through the points $(1,-5,-6)$ and parallel to the line joining the points $(0,2,3)$ and $(-1,3,7)$.

Solution : Direction ratios of the line joining the points $(0,2,3)$ and $(-1,3,7)$ are

$$-1-0,\ 3-2,\ 7-3$$

or $$-1, +1, +4$$

$\therefore$ Direction ratios of a line parallel to this line can be taken as $-1, 1, 4$.

Thus, equations of the line through the point $(1,-5,-6)$ and parallel to the given line are

$$\frac{x-1}{-1}=\frac{y+5}{1}=\frac{z+6}{4}$$

CHECK YOUR PROGRESS 36.1

1. Find the equations, in symmetric form, of the line passing through the point $(1,-2,3)$ with direction ratios $3,-4,5$.
2. Find the equations of the line, in symmetric form, passing through the points $(3,-9,4)$ and $(-9,5,-4)$.
3. Find the equations of the line, in symmetric form, passing through the points $(-7,5,3)$ and $(2,6,8)$

4. Find the equations of the line, in symmetric form, through the point (1,2,3) and parallel to the line joining the points $(-4, 7, 2)$ and $(5, -3, -2)$
5. Find the equations of a line passing through the origin and equally inclined to the co-ordinate axes.
6. Write the vector equation of the line passing through origin and (5, –2, 3).
7. Write the following equation of a line in vector form $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-3}{2}$.
8. Write the following equation of a line in Cartesian form : $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})$
9. Find the vector equation of a line passing through the point (2, –1, 4) and parallel to the vector $\hat{i}+2\hat{j}-\hat{k}$.

36.2 REDUCTION OF THE EQUATIONS OF A LINE INTO SYMMETRIC FORM

You may recall that a line can be thought of as the intersection of two non-parallel planes. Let the equations of the two intersecting planes be

$$ax + by + cz + d = 0 \qquad \text{...(i)}$$

and $$a'x + b'y + c'z + d' = 0 \qquad \text{....(ii)}$$

Let AB be the line of intersection of the two planes. Every point on the line AB lies on both the planes. Thus, the co-ordinates of any point on the line satisfy the two equations of the planes. Hence (i) and (ii) together represent the equations of a line.

The equations $ax + by + cz = 0$ and $a'x + b'y + c'z = 0$ together represent the equations of the line through the origin parallel to the above line as the above two planes also pass through origin. The above form of the equations of a line is referred to as general (***or non-symmetric***) form of the equations of a line.

To reduce the general equations of a line given by (i) and (ii) in the symmetric form, we need the direction cosines of the line as well as the co-ordinates of a point on the line.

Let the direction cosines of the line be l, m and n. The line is perpendicular to the normal to planes given by (i) and (ii).

$\therefore \quad al + bm + cn = 0 \quad$ and $\quad a'l + b'm + c'n = 0$

By cross multiplication method, we get

$$\frac{l}{bc'-b'c}=\frac{m}{ca'-ac'}=\frac{n}{ab'-a'b}$$

Thus, the direction cosines of the line are proportional to

$$(bc'-b'c),(ca'-ac') \text{ and } (ab'-a'b).$$

The point where the line meets the XY – plane is obtained by putting z = 0 in the equations (i) and (ii), which give

$$ax + by + d = 0 \qquad \text{....(iii)}$$

$$a'x + b'y + d' = 0 \qquad \text{....(iv)}$$

Solving (iii) and (iv), we get

$$x = \frac{bd' - b'd}{ab' - a'b}, \; y = \frac{da' - d'a}{ab' - a'b}$$

$\therefore$ A point on the line is $\left(\frac{bd' - b'd}{ab' - a'b}, \frac{da' - d'a}{ab' - a'b}, 0 \right)$

$\therefore$ The equations of the line in symmetric form are

$$\frac{x - \frac{bd' - b'd}{ab' - a'b}}{bc' - b'c} = \frac{y - \frac{da' - d'a}{ab' - a'b}}{ca' - c'a} = \frac{z}{ab' - a'b}$$

Note : Instead of taking z = 0, we may take x = 0 or y = o or any other suitable value for any of the x, y, z provided the two equations so obtained have a unique solution.

Example 36.8 Convert the equations of the line given by $x - 2y + 3z = 4, 2x - 3y + 4z = 5$ into symmetric form and find its direction cosines.

Solution : Let z = 0 be the z – co-ordinate of a point on each of the planes.

$\therefore$ The equations of the planes reduce to

$$x - 2y = 4$$

$$2x - 3y = 5$$

which on solving give $x = -2$ and $y = -3$

$\therefore$ The point common to two planes is $(-2, -3, 0)$.

Let *l*, m, n be the direction cosines of the line As the line is perpendicular to normal to the planes. we have

$$l - 2m + 3n = 0$$

and $$2l - 3m + 4n = 0$$

$\therefore$ $$\frac{l}{-8+9} = \frac{m}{6-4} = \frac{n}{-3+4}$$

or $$\frac{l}{1} = \frac{m}{2} = \frac{n}{1} = \pm\frac{1}{\sqrt{6}}$$

$\therefore$ The equations of the line are

$$\frac{x+2}{\pm\frac{1}{\sqrt{6}}} = \frac{y+3}{\pm\frac{2}{\sqrt{6}}} = \frac{z}{\pm\frac{1}{\sqrt{6}}}$$

or $$\frac{x+2}{1} = \frac{y+3}{2} = \frac{z}{1}$$

and the direction cosines of the line are $\pm\frac{1}{\sqrt{6}}, \pm\frac{2}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}$ (the same sign positive or negative to be taken throughout)

CHECK YOUR PROGRESS 36.2

1. Find the equations, in symmetric form, of the line given by

   (i) $x + 5y - z = 7$ and $2x - 5y + 3z = -1$

   (ii) $x + y + z + 1 = 0$ and $4x + y - 2z + 2 = 0$

   (iii) $x - y + z + 5 = 0$ and $x - 2y - z + 2 = 0$

36.3 PERPENDICULAR DISTANCE OF A POINT FROM A LINE

Let P be the point (x_1, y_1, z_1) and AQ be the given line whose equations are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

where l, m and n are the direction cosines of the line AQ, Q is the foot of the perpendicular from P on AQ and A is the point (α, β, γ).

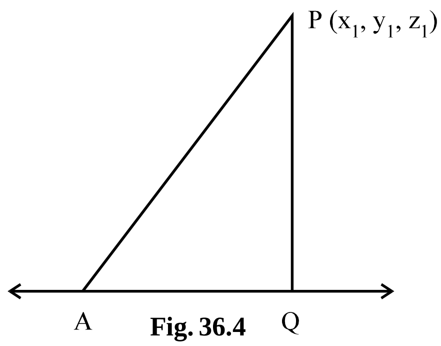

Fig. 36.4

We have $\quad PQ^2 = AP^2 - AQ^2$

Now $\quad AP^2 = (x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2$

Again AQ, the projection of AP on the line is

$(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n$

$$\therefore \quad PQ^2 = \left\{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2\right\}$$
$$- \{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n\}^2$$

which gives the length of perpendicular (PQ) from the point P to the line.

Example 36.9 Find the distance of a point (2,3,1) from the line

$$y + z - 1 = 0 = 2x - 3y - 2z + 4$$

Solution : Let z = 0 be the z-coordinate of the point common to two planes.

$\therefore$ Their equations become $y = 1$ and $2x - 3y + 4 = 0$ which give $x = -\frac{1}{2}$

$\therefore$ A point common to two planes is $\left(-\frac{1}{2}, 1, 0\right)$

Let l, m, n be the direction cosines of the given line

Then, $0l + m + n = 0$ and $2l - 3m - 2n = 0$

or $\qquad \frac{l}{1} = \frac{m}{2} = \frac{n}{-2} = \frac{1}{\pm 3}$ or $\qquad l = \pm\frac{1}{3},\ m = \pm\frac{2}{3},\ n = \mp\frac{2}{3}$

If PQ is the length of the perpendicular from (2,3,1) to the given line. Then

$$PQ^2 = \left[\left(2+\frac{1}{2}\right)^2 + (3-1)^2 + (1-0)^2\right] - \left[\frac{5}{2}\times\frac{1}{3} + \frac{2}{3}\times 2 - 1\times\frac{2}{3}\right]^2$$

$$= \left(\frac{25}{4} + 4 + 1\right) - \left(\frac{5}{6} + \frac{4}{3} - \frac{2}{3}\right)^2$$

$$= \frac{45}{4} - \frac{9}{4} = 9$$

$\therefore$ PQ = 3

Thus, the required distance is 3 units.

CHECK YOUR PROGRESS 36.3

1. Find the distance of the point from the line, for each of the following :

   (i) Point (0, 2, 3), line $\frac{x+3}{3} = \frac{y-1}{2} = \frac{z+4}{3}$

   (ii) Point $(-1, 3, 9)$, line $\frac{x-13}{5} = \frac{y+8}{-6} = \frac{z-31}{1}$

   (iii) Point (4, 1, 1), line $x + y + z = 4,\ x - 2y - z = 4$

   (iv) Point (3, 2, 1), line $x + y + z = 4,\ x - 2y - z = 4$

36.4 ANGLE BETWEEN A LINE AND A PLANE

The angle between a line and a plane is the complement of the angle between the line and normal to the plane. Let the equations of the line be

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n} \qquad .....(i)$$

and that of the plane be

$$ax + by + cz + d = 0 \qquad ....(ii)$$

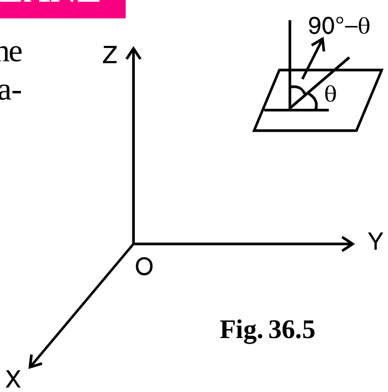

Fig. 36.5

If θ be the angle between (i) and (ii), then

$$\sin\theta = \cos(90^\circ - \theta) = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\ \sqrt{a^2 + b^2 + c^2}}$$

Example 36.10 Find the angle between the line $\frac{x-2}{3} = \frac{y+3}{3} = \frac{z-1}{1}$

and the plane $2x - 3y + 4z - 7 = 0$

Solution : Here the angle θ between the given line and given plane is given by

$$\sin\theta = \frac{2\times 3 - 3\times 3 + 4\times 1}{\sqrt{3^2 + 3^2 + 1^2}\ \sqrt{2^2 + (-3)^2 + 4^2}} = \frac{1}{\sqrt{19}\ \sqrt{29}}$$

$$= \frac{1}{\sqrt{551}}$$

or $$\theta = \sin^{-1}\left(\frac{1}{\sqrt{551}}\right)$$

CHECK YOUR PROGRESS 36.4

1. Find the angle between the following lines and the planes.

(i) Line : $\frac{x-4}{1} = \frac{y+2}{4} = \frac{z-3}{-1}$ and Plane : $3x - 4y + 5z = 5$

(ii) Line : $\frac{x-2}{2} = \frac{z-3}{3} = \frac{y+2}{1}$ and Plane : $-2x + 4y - 5z = 20$

(iii) Line : $\frac{x}{4} = \frac{y-2}{-3} = \frac{y+2}{5}$ and Plane : $x - 4y + 6z = 11$

(iv) Line : $\frac{x+2}{4} = \frac{y-3}{5} = \frac{z+4}{1}$ and Plane : $4x - 3y - z - 7 = 0$

36.5 CONDITION OF COPLANARITY OF TWO LINES

If the two lines given by

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \qquad ......(i)$$

and

$$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \qquad .....(ii)$$

intersect, they lie in the same plane.

Equation of a plane containing line (i) is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \qquad ......(iii)$$

with $$Al_1 + Bm_1 + Cn_1 = 0 \qquad \text{.....(iv)}$$

If the plane (iii) contains line (ii), the point (x_2, y_2, z_2) must lie on it.

Thus, $$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 \qquad \text{.....(v)}$$

with $$Al_2 + Bm_2 + Cn_2 = 0 \qquad \text{....(vi)}$$

Eliminating A,B and C from (iv), (v) and (vi), we have

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad \text{.....(vii)}$$

which is the necessary condition for coplanarity of lines given by (i) and (ii)

Again, eliminating A,B and C from (iii), (iv) and (vi) we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad \text{.....(viii)}$$

(viii) represents the equation of the plane containing the two intersecting lines.

We shall now show that if the condition (vii) holds, then the lines (i) and (ii) are coplanar.

Consider the plane

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad \text{....(ix)}$$

or, $$(x - x_1)(m_1 n_2 - m_2 n_1) + (y - y_1)(n_1 l_2 - n_2 l_1) + (z - z_1)(l_1 m_2 - l_2 m_1) = 0$$

A line will lie in the plane, if the normal to the plane is perpendicular to the line and any point on the line lies in the plane.

You may see that

$$l_1(m_1 n_2 - m_2 n_1) + m_1(n_1 l_2 - n_2 l_1) + n_1(l_1 m_2 - l_2 m_1) = 0$$

Hence line (i) lies in plane (ix)

By similar argument, we can say that line (ii) lies on plane (ix)

$\therefore$ The two lines are coplanar.

Thus, the condition (vii) is also sufficient for the two lines to be coplanar.

Corollary : The lines (i) and (ii) will intersect if and only if (vii) holds and lines are not parallel.

Note :

(i) Two lines in space, which are neither intersecting nor parallel, do not lie in the same plane. Such lines are called skew lines.

(ii) If the equation of one line be in symmetric form and the other in general form, we proceed as follows:

Let equations of one line be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad \text{.... (i)}$$

and that of the other line be

$$ax + by + cz + d = 0 \text{ and } a'x + b'y + c'z + d' = 0 \qquad \text{......(ii)}$$

If the two lines are coplanar, then a point on the first line should satisfy equations of the second line. A general point on line (i) is $(x_1 + lr, y_1 + mr, z_1 + nr)$.

This point lies on $ax + by + cz + d = 0$ if

$$a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr) + d = 0$$

or
$$r = -\frac{ax_1 + by_1 + cz_1 + d}{al + bm + cn}$$

Similarly, this point should lie on $a'x + b'y + c'z + d' = 0$, resulting in

$$r = -\frac{a'x_1 + b'y_1 + c'z_1 + d'}{a'l + b'm + c'n}$$

Equating the two values of r obtained above, we have the required condition as

$$\frac{ax_1 + by_1 + cz_1 + d}{al + bm + cn} = \frac{a'x_1 + b'y_1 + c'z_1 + d'}{a'l + b'm + c'n}$$

Note : In case, both the lines are in general form, convert one of them into symmetric form and then proceed as above.

Example 36.11 Prove that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ are co-planar.

Solution : For the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ ......(i)

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ .... (ii)

to be coplanar we must have

$$\begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

or $\quad 3(12+5)+3(12+35)+8(4-28)=0$

or $\quad 51+141-192=0$

or $\quad 0=0$ which is true.

$\therefore$ The two lines given by (i) and (ii) are coplanar.

Example 36.12 Prove that the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7} \text{ and } \frac{x-2}{1}=\frac{y-4}{4}=\frac{z-6}{7}$$

are coplanar. Find the equation of the plane containing these lines.

Solution : For the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7} \text{ and } \frac{x-2}{1}=\frac{y-4}{4}=\frac{z-6}{7}$$

to be coplanar, we must have

$$\begin{vmatrix} 2+1 & 4+3 & 6+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

or $\quad 3(35-28)-7(21-7)+11(12-5)=0$

or $\quad 21-98+77=0$

or $\quad 0=0.$ which is true.

$\therefore$ The given lines are coplanar.

Equation of the plane containing these lines is

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

or $\quad (x+1)(35-28)-(y+3)(21-7)+(z+5)(12-5)=0$

or $\quad 7x+7-14y-42+7z+35=0$

or $\quad 7x-14y+7z=0$

or $\quad x-2y+z=0$

CHECK YOUR PROGRESS 36.5

1. Prove that the following lines are coplanar :

   (i) $\frac{x-3}{3} = \frac{y-2}{-4} = \frac{z+1}{1}$ and $x + 2y + 3z = 0 = 2x + 4y + 3z + 3$

   (ii) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $4x - 3y + 1 = 0 = 5x - 3z + 2$

2. Show that the lines $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$

   and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ are coplanar. Find the equation of the plane containing them.

LET US SUM UP

A line is the intersection of two non-parallel planes.

- Vector equation of a line is $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\vec{a}$ is the position vector of the given point on the line and $\vec{b}$ is a vector parallel to the line.
- Its corresponding Cartesian form is

  $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$, where (x_1, y_1, z_1) are the coordinates to the given point on the line and b_1, b_2, b_3 are the direction ratios of the vector $\vec{b}$.
- $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ is another vector equation of the line where $\vec{a}$ and $\vec{b}$ are the position vectors of two distinct points on the line.
- Its corresponding Cartesian form is

  $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$, where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of two distinct given points on the line.

- The angle θ between the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane

  $ax + by + cz + d = 0$ is given by

  $$\sin\theta = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\,\sqrt{a^2 + b^2 + c^2}}$$

- The condition of coplanarity of two lines,

$$\frac{x-x_1}{l}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

is $$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

and the equation of the plane containing the lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

SUPPORTIVE WEB SITES

http://www.regentsprep.org/regents/math/algebra/ac1/eqlines.htm

http://www.purplemath.com/modules/strtlneq.htm

http://www.mathsteacher.com.au/year10/ch03_linear_graphs/02_gradient/line.htm

TERMINAL EXERCISE

1. Find the equations of the line passing through the points (1,4,7) and $(3,-2,5)$

2. Find the equations of the line passing through the point $(-1,-2,-3)$ and perpendicular to the plane $3x-4y+5z-11=0$

3. Find the direction cosines of the line which is perpendicular to the lines whose direction ratios are $1,-1,2$ and $2,1,-1$.

4. Show that the line segment joining the points (1,2,3) and (4,5,7) is parallel to the line segment joining the points $(-4,3,-6)$ and $(2,9,2)$

5. Find the angle between the lines

$$\frac{x-1}{2}=\frac{y-2}{-4}=\frac{z+5}{5} \quad \text{and} \quad \frac{x+1}{3}=\frac{y+1}{4}=\frac{z}{2}$$

6. Find the equations of the line passing through the point $(1,2,-4)$ and perpendicular to each of the two lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \quad \text{and} \quad \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

7. Convert the equations of the line $x-y+2z-5=0$, $3x+y+z-6=0$ into the symmetric form.

8. Show that the lines $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z}{-1}$ and $\frac{x-4}{3}=\frac{y-1}{-2}=\frac{z-1}{1}$ are coplanar. Find the equation of the plane containing them.

9. Find the equation of the plane containing the lines.

$$\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5} \quad \text{and} \quad \frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$$

10. Find the projection of the line segment joining the points (2,3,1) and (5,8,7) on the line $\frac{x}{2}=\frac{y+4}{3}=\frac{z+1}{6}$

11. Find the vector equation of a line which passes through the point (1, 2, –4) and is parallel to the vector $(2\hat{i}+3\hat{j}-5\hat{k})$.

12. Cartesian equation of a line is $\frac{x+5}{3}=\frac{y-4}{-5}=z$, what is its vector equation?

13. Find the vector equation of a line passing through the points (3, –2, –5) and (3, –2, 6).

14. Find the vector equation of a line which passes through the point (–2, 4, –5) and parallel to the line given by $\frac{x-3}{3}=\frac{y+4}{5}=\frac{z-8}{2}$.

ANSWERS

CHECK YOUR PROGRESS 36.1

1. $\frac{x-1}{3}=\frac{y+2}{-4}=\frac{z-3}{5}$

2. $\frac{x-3}{-6}=\frac{y+9}{7}=\frac{z-4}{-4}$

3. $\frac{x+7}{9}=\frac{y-5}{1}=\frac{z-3}{5}$

4. $\frac{x-1}{9}=\frac{y-2}{-10}=\frac{z-3}{-4}$

5. $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

6. $\vec{r}=\lambda\left(5\hat{i}-2\hat{j}+3\hat{i}\right)$

7. $\vec{r}=\left(5\hat{i}-4\hat{j}+3\hat{i}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$

8. $\frac{x-1}{1}=\frac{y-2}{-3}=\frac{z-3}{2}$

9. $\vec{r}=\left(2\hat{i}-\hat{j}+4\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right)$

CHECK YOUR PROGRESS 36.2

1. (i) $\frac{x-2}{2}=\frac{y-1}{-1}=\frac{z}{-3}$

(ii) $\frac{x+\frac{1}{3}}{1}=\frac{y+\frac{2}{3}}{-2}=\frac{z}{1}$

(iii) $\frac{x-1}{3}=\frac{y-3}{2}=\frac{z+3}{1}$

CHECK YOUR PROGRESS 36.3

1. (i) $\sqrt{21}$ units (ii) 21 units

(iii) $\sqrt{\frac{27}{14}}$ units (iv) $\sqrt{6}$ units

CHECK YOUR PROGRESS 36.4

1. (i) $\sin^{-1}\left(-\frac{3}{5}\right)$ (ii) $\sin^{-1}\left(\frac{1}{\sqrt{70}}\right)$

(iii) $\sin^{-1}\left(\frac{46}{\sqrt{2650}}\right)$ (iv) 0°.

CHECK YOUR PROGRESS 36.5

2. $x + y + z = 0$

TERMINAL EXERCISE

1. $\dfrac{x-1}{2} = \dfrac{y-4}{-6} = \dfrac{z-7}{-2}$

2. $\dfrac{x+1}{3} = \dfrac{y+2}{-4} = \dfrac{z+3}{5}$

3. $-\dfrac{1}{\sqrt{35}}, \dfrac{5}{\sqrt{35}}, \dfrac{3}{\sqrt{35}}$

5. $90°$

6. $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z+4}{6}$

7. $\dfrac{x - \frac{11}{4}}{-3} = \dfrac{y + \frac{9}{4}}{5} = \dfrac{z}{4}$

8. $2x - 5y - 16z + 13 = 0$

9. $17x - 47y - 24z + 172 = 0$

10. $\dfrac{57}{7}$ units.

11. $\vec{r} = (\hat{i} + 2\hat{y} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 5\hat{k})$

12. $\vec{r} = (-5\hat{i} + 4\hat{j}) + \lambda(3\hat{i} - 5\hat{j} + \hat{k})$

13. $\vec{r} = (3\hat{i} - 2\hat{j} - 5\hat{k}) + \lambda(11\hat{k})$

14. $\vec{r} = (-2\hat{i} + 4\hat{j} - 5\hat{k}) + \lambda(3\hat{i} + 5\hat{j} + 2\hat{k})$