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311en37

LINEAR PROGRAMMING

37.1 INTRODUTION TO LINEAR PROGRAMMING PROBLEMS

A toy dealer goes to the wholesale market with Rs. 1500.00 to purchase toys for selling. In the market there are various types of toys available. From quality point of view, he finds that the toy of type ‘A’ and type ‘B’ are suitable. The cost price of type ‘A’ toy is Rs. 300 each and that of type ‘B’ is Rs. 250 each. He knows that the type ‘A’ toy can be sold for Rs. 325 each, while the type ‘B’ toy can be sold for Rs. 265 each. Within the amount available to him he would like to make maximum profit. His problem is to find out how many type ‘A’ and type ‘B’ toys should be purchased so to get the maximum profit.

He can prepare the following table taking into account all possible combinations of type ‘A’ and type ‘B’ toys subject to the limitation on the investment.

‘A’ type	‘B’ type	Investment	Amount after sale (including the unutilised amount if any)	Profit on the investment
0	6	1500.00	1590.00	90.00
1	4	1300.00	1585.00	85.00
2	3	1350.00	1595.00	95.00
3	2	1400.00	1605.00	105.00
4	1	1450.00	1615.00	115.00
5	0	1500.00	1625.00	125.00

Now, the decision leading to maximum profit is clear. Five type A toys should be purchased.

The above problem was easy to handle because the choice was limited to two types, and the number of items to be purchased was small. Here, all possible combinations were thought of and the corresponding gain calculated. But one must make sure that he has taken all possibilities into account.

A situation faced by a retailer of radio sets similar to the one given above is described below.

A retailer of radio sets wishes to buy a number of transistor radio sets from the wholesaler. There are two types (type A and type B) of radio sets which he can buy. Type A costs Rs.360 each and type *B* costs Rs. 240 each. The retailer can invest up to Rs. 5760. By selling the radio sets, he can make a profit of Rs. 50 on each set of type A and of Rs. 40 on each set of type B. How many of each type should he buy to maximize his total profit?

Here we have to maximize the profit. Sometimes we come across a problem in which the costs are to be minimized. Consider the following problem :

Two tailors A and B earn Rs.150 and Rs.200 per day respectively. A can stitch 6 shirts and 4 pants per day, while B can stitch 4 shirts and 7 pants per day. How many days shall each work if they want to produce at least 60 shirts and 72 pants at a minimum labour cost?

In this problem we have to minimise the labour cost.

These types of problems of maximisation and minimisation are called optimisation problems.

The technique followed by mathematicians to solve such problems is called **'Linear Programming'.**

OBJECTIVES

After studying this lesson, you will be able to :

- undertstand the terminology used in linear programming;
- convert different type of problems into a linear programming problem;
- use graphical mehtod to find solution of the linear programming problems

EXPECTED BACKGROUND KNOWLEDGE

- good idea of converting a mathematical information into a in equality
- to be able to solve system of on equalities using graphical method.

37.2 DEFINITIONS OF VARIOUS TERMS INVOLVED IN LINEAR PROGRAMMING

A close examination of the examples cited in the introduction points out one basic property that all these problems have in common, i.e., in each example, we were concerned with maximising or minimising some quantity.

In first two examples, we wanted to maximise the return on the investment. In third example, we wanted to minimise the labour cost. In linear programming terminology the maximization or minimization of a quantity is referred to as the objective of the problem.

37.2.1 OBJECTIVE FUNCTION

In a linear programming problem. z, the linear function of the variables which is to be optimized is called **objective function.**

Here, a linear form means a mathematical expression of the type

$$a_1x_1 + a_2\, x_2 + ... + a_n x_n\,,$$

where $a_1,\ a_{2,\,.....,}\ a_n$ are **constants** and $x_1,\ x_{2,\ .....,}\ x_n$ are **variables.**

In linear programming problems, the products, services, projects etc. that are competing with each other for sharing the given limited resources are called the variables or decision variables.

37.2.2 CONSTRAINTS

The limitations on resources (like cash in hand, production capacity, man power, time, machines, etc.) which are to be allocated among various competing variables are in the form of linear equations or inequations (inequalities) and are called **constraints** or **restrictions.**

37.2.3 NON-NEGATIVE RESTRICTIONS

All decision variables must assume non-negative values, as negative values of physical quantities is an impossible situation.

37.3 FORMULATION OF A LINEAR PROGRAMMING PROBLEM

The formulation of a linear programming problem as a mathematical model involves the following key steps.

Step 1 : Identify the decision variables to be determined and express them in terms of algebraic symbols such as $x_{1,} x_{2,} x_{3,}$ ....

Step 2 : Identify all the limitations in the given problem and then express them as linear equations or inequalities in terms of above defined decision variables.

Step 3 : Identify the objective which is to be optimised (maximised or minimised) and express it as a linear function of the above defined decision variables.

Example 37.1 A retailer wishes to buy a number of transistor radio sets of types *A* and *B*. Type *A* cost Rs.360 each and type *B* cost Rs. 240 each. The retailer knows that he cannot sell more than 20 sets, so he does not want to buy more than 20 sets and he cannot afford to pay more than Rs.5760. His expectation is that he would get a profit of Rs.50 for each set of type *A* and Rs.40 for each set of type *B*. Form a mathematical model to find how many of each type should be purchased in order to make his total profit as large as possible?

Solution : Suppose the retailer purchases x_1 sets of type *A* and x_2 sets of type *B*. Since the number of sets of each type is non-negative, so we have

$$x_1 \geq 0, \qquad \cdots (1)$$

$$x_2 \geq 0, \qquad \cdots (2)$$

Also the cost of x_1 sets of type *A* and x_2 sets of type *B* is 360 x_1 + 240 x_2 and it should be equal to or less than Rs.5760, that is,

$$360\, x_1 + 240\, x_2 \leq 5760$$

or $$3x_1 + 2x_2 \leq 48 \qquad \cdots (3)$$

Further, the number of sets of both types should not exceed 20, so

$$x_1 + x_2 \le 20 \qquad \cdots(4)$$

Since the total profit consists of profit derived from selling the x_1 type *A* sets and x_2 type *B* sets, therefore, the retailer earns a profit of Rs.50 x_1 on type *A* sets and Rs.40 x_2 on type *B* sets. So the total profit is given by :

$$z = 50x_1 + 40x_2 \qquad \cdots(5)$$

Hence, the mathematical formulation of the given linear programming problem is as follows :

Find $x_{1,}$ x_2 which

Maximise z = $50x_1 + 40x_2$ **(Objective function) subject to the conditions**

$$\left.\begin{array}{l} 3x_1 + 2x_2 \le 48 \\ x_1 + x_2 \le 20 \\ x_1 \ge 0, x_2 \ge 0 \end{array}\right\} \textbf{Constraints}$$

Example 37.2 A soft drink company has two bottling plants, one located at *P* and the other at *Q*. Each plant produ ces three different soft drinks *A*, *B*, and *C*. The capacities of the two plants in terms of number of bottles per day, are as follows :

Plants → / Products ↓	P	Q
A	3000	1000
B	1000	1000
C	2000	6000

A market survey indicates that during the month of May, there will be a demand for 24000 bottles of *A*, 16000 bottles of *B* and 48000 bottles of *C*. The operating cost per day of running plants *P* and *Q* are respectively Rs.6000 and Rs.4000. How many days should the firm run each plant in the month of May so that the production cost is minimised while still meeting the market demand.

Solution : Suppose that the firm runs the plant *P* for x_1 days and plant Q for x_2 days in the month of May in order to meet the market demand.

The per day operating cost of plant *P* is Rs.6000. Therefore, for x_1 days the operating cost will be Rs.6000 x_1.

The per day operating cost of plant *Q* is Rs.4000. Therefore, for x_2 days the operating cost will be Rs.4000 x_2.

Thus the total operating cost of two plants is given by :

$$z = 6000x_1 + 4000x_2 \qquad \cdots (1)$$

Plant P produces 3000 bottles of soft drink A per day. Therefore, in x_1 days plant P will produce 3000 x_1 bottles of soft drink A.

Plant Q produces 1000 bottles of soft drink A per day.

Therefore, in x_2 days plant Q will produce 1000 x_2 bottles of soft drink A.

Total production of soft drink A in the supposed period is 3000 x_1 + 1000 x_2

But there will be a demand for 24000 bottles of this soft drink, so the total production of this soft drink must be greater than or equal to this demand.

$$\therefore \quad 3000x_1 + 1000x_2 \geq 24000$$

$$\text{or} \quad 3x_1 + x_2 \geq 24 \qquad \cdots (2)$$

Similarly, for the other two soft drinks, we have the constraints

$$1000x_1 + 1000x_2 \geq 16000$$

$$\text{or} \quad x_1 + x_2 \geq 16 \qquad \cdots (3)$$

and

$$2000x_1 + 6000x_2 \geq 48000$$

$$\text{or} \quad x_1 + 3x_2 \geq 24 \qquad \cdots (4)$$

x_1 and x_2 are non-negative being the number of days, so

$$x_1 \geq 0, \; x_2 \geq 0 \qquad \cdots (5)$$

Thus our problem is to find x_1 and x_2 which

Minimize $z = 6000x_1 + 4000x_2$ ***(objective function)***

subject to the conditions

$$\left.\begin{aligned} 3x_1 + x_2 &\geq 24 \\ x_1 + x_2 &\geq 16 \\ x_1 + 3x_2 &\geq 24 \end{aligned}\right\} \quad \textit{(constraints)}$$

and $x_1 \geq 0, \; x_2 \geq 0$

Example 37.3 A firm manufactures two types of products *A* and *B* and sells them at a profit of Rs.2 on type *A* and Rs.3 on type *B*. Each product is processed on two machines *G* and *H*. Type A requires one minute of processing time on *G* and 2 minutes on H, type *B* requires one minute on *G* and one minute on *H*. The machine *G* is available for not more than 6 hours and 40 minutes while machine *H* is available for 10 hours during one working day. Formulate the problem as a linear programming problem so as to maximise profit.

Solution : Let x_1 be the number of products of type *A* and x_2 be the number of products of type *B*.

The given information in the problem can systematically be arranged in the form of following table :

Machine	Processing time of the products (in minute)		Available time (in minute)
	Type *A* (x_1 units)	Type *B* (x_2 units)	
G	1	1	400
H	2	1	600
Profit per unit	Rs.2	Rs.3	

Since the profit on type *A* is Rs.2 per product, so the profit on selling x_1 units of type *A* will be $2x_1$. Similarly, the profit on selling x_2 units of type *B* will be $3x_2$. Therefore, total profit on selling x_1 units of type *A* and x_2 units of type *B* is given by

$$z = 2x_1 + 3x_2 \qquad \text{(objective function)} \qquad \cdots(1)$$

Since machine G takes 1 minute time on type *A* and 1 minute time on type *B*, therefore, the total number of minutes required on machine *G* is given by

$$x_1 + x_2$$

But the machine *G* is not available for more than 6 hours and 40 minutes (i.e., 400 minutes). Therefore,

$$x_1 + x_2 \le 400 \qquad \cdots(2)$$

Similarly, the total number of minutes required on machine *H* is given by

$$2x_1 + x_2$$

Also, the machine *H* is available for 10 hours (i.e., 600 minutes). Therefore,

$$2x_1 + x_2 \le 600 \qquad \cdots(3)$$

Since, it is not possible to produce negative quantities, so

$$x_1 \geq 0,\ x_2 \geq 0 \qquad \cdots (4)$$

Thus, the problem is to find x_1 and x_2 which

Maximize $\quad z = 2x_1 + 3x_2 \qquad$ (objective function)

subject to the conditions

$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 600$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Example 37.4 A furniture manufacturer makes two types of sofas – sofa of type *A* and sofa of type *B*. For simplicity, divide the production process into three distinct operations, say carpentary, finishing and upholstery. The amount of labour required for each operation varies. Manufacture of a sofa of type *A* requires 6 hours of carpentary, 1 hour of finishing and 2 hours of upholstery. Manufacture of a sofa of type *B* requires 3 hours of carpentary, 1 hour of finishing and 6 hours of upholstery. Owing to limited availability of skilled labour as well as of tools and equipment, the factory has available each day 96 man hours of carpentary, 18 man hours for finishing and 72 man hours for upholstery. The profit per sofa of type *A* is Rs.80 and the profit per sofa of type *B* is Rs. 70. How many sofas of type *A* and type *B* should be produced each day in order to maximise the profit? Formulate the problems as linear programming problem.

Solution : The different operations and the availability of man hours for each operation can be put in the following tabular form :

Operations	Sofa of type *A*	Sofa of type *B*	Available labour
Carpentary	6 hours	3 hours	96 man hours
Finishing	1 hour	1 hour	18 man hours
Upholstery	2 hours	6 hours	72 man hours
Profit	Rs. 80	Rs. 70	

Let x_1 be the number of sofas of type *A* and x_2 be the number of sofas of type *B*.

Each row of the chart gives one restriction. The first row says that the amount of carpentary required is 6 hours for each sofa of type *A* and 3 hours for each sofa of type *B*. Further, only 96 man hours of carpentary are available per day. We can compute the total number of man hours of carpentary required per day to produce x_1 sofas of type *A* and x_2 sofas of type *B* as follows:

Number of man - hours per day of carpentary

= { (Number of hours carpentary per sofa of type *A*) $\times$ (Number of sofas of type A) }

+ { (Number of hours carpentary per sofa of type B)$\times$

(Number of sofas of type B) }

$= 6x_1 + 3x_2$

The requirement that at most 96 man hours of carpentary per day means

$$6x_1 + 3x_2 \leq 96$$

or $\quad \mathbf{2x_1 + x_2 \leq 32} \qquad \cdots$ **(1)**

Similarly, second and third row of the chart give the restrictions on finishing and upholstery respectively as

$\mathbf{x_1 + x_2 \leq 18} \qquad \cdots$ **(2)**

and

$$2x_1 + 6x_2 \leq 72$$

or $\quad \mathbf{x_1 + 3x_2 \leq 36} \qquad \cdots$ **(3)**

Since, the number of the sofas cannot be negative, therefore

$\mathbf{x_1 \geq 0, x_2 \geq 0} \qquad \cdots$ **(4)**

Now, the profit comes from two sources, that is, sofas of type A and sofas of type B. Therefore,

Profit = (Profit from sofas of type A) + (Profit from sofas of type B)

= { (Profit per sofa of type A) $\times$ (Number of sofas of type A) }

+ { (Profit per sofa of type B) $\times$ (Number of sofas of type B) }

$z = 80x_1 + 70x_2$ **(objective function)** $\cdots$ **(5)**

Thus, the problem is to find x_1 and x_2 which

Maximize $z = 80\,x_1 + 70\,x_2$ (objective function)

subject to the constraints

$$\left.\begin{aligned} &2x_1 + x_2 \leq 32 \\ &x_1 + x_2 \leq 18 \\ &x_1 + 3x_2 \leq 36 \\ &x_1 \geq 0,\ x_2 \geq 0 \end{aligned}\right\} \quad \textbf{(Constraints)}$$

CHECK YOUR PROGRESS 37.1

1. A company is producing two products *A* and *B*. Each product is processed on two machines *G* and *H*. Type *A* requires 3 hours of processing time on *G* and 4 hours on *H*; type *B* requires 4 hours of processing time time on G and 5 hours on *H*. The available time is 18 hours and 21 hours for operations on *G* and *H* respectively. The products *A* and *B* can be sold at the profit of Rs. 3 and Rs. 8 per unit respectively. Formulate the problem as a linear programming problem.

2. A furniture dealer deals in only two items, tables and chairs. He has Rs. 5000 to invest and a space to store at most 60 pieces. A table costs him Rs. 250 and a chair Rs. 50. He can sell a table at a profit of Rs. 50 and a chair at a profit of Rs. 15. Assuming, he can sell all the items that he buys, how should he invest his money in order that may maximize his profit? Formulate a linear programming problem.

3. A dairy has its two plants one located at *P* and the other at *Q*. Each plant produces two types of products *A* and *B* in 1 kg packets. The capacity of two plants in number of packets per day are as follows:

Products ↓ \ Plants →	P	Q
A	2000	1500
B	4000	6000

   A market survey indicates that during the month of April, there will be a demand for 20000 packets of *A* and 16000 packets of *B*. The operating cost per day of running plants *P* and *Q* are respectively Rs.4000 and Rs.7500. How many days should the firm run each plant in the month of April so that the production cost is minimized while still meeting the market demand? Formulate a Linear programming problem.

4. A factory manufactures two articles *A* and *B*. To manufacture the article *A*, a certain machine has to be worked for 1 hour and 30 minutes and in addition a craftsman has to work for 2 hours. To manufacture the article *B*, the machine has to be worked for 2 hours and 30 minutes and in addition the craftsman has to work for 1 hour and 30 minutes. In a week the factory can avail of 80 hours of machine time and 70 hours of craftsman's time. The profit on each article A is Rs.5 and that on each article *B* is Rs.4. If all the articles produced can be sold away, find how many of each kind should be produced to earn the maximum profit per week. Formulate the problem as a linear programming problem.

37.4 GEOMETRIC APPORACH OF LINEAR PROGRAMMING PROBLEM

Let us consider a simple problem in two variables *x* and *y*. Find *x* and *y* which satisfy the following equations

$$x + y = 4$$
$$3x + 4y = 14$$

Solving these equations, we get $x = 2$ and $y = 2$. What happens when the number of equations and variables are more?

Can we find a unique solution for such system of equations?

However, a unique solution for a set of simultaneous equations in n-variables can be obtained if there are exactly n-relations. What will happen when the number of relations is greater than or less then n ?

A unique solution will not exist, but a number of trial solutions can be found. Again, if the number of relations are greater than or less than the number of variables involved and the relation are in the form of inequalities.

Can we find a solution for such a system?

Whenever the analysis of a problem leads to minimising or maximising a linear expression in which the variable must obey a collection of linear inequalities, a solution may be obtained using linear programming techniques. One way to solve linear programming problems that involve only two variables is geometric approach called **graphical solution of the linear programming problem.**

37.5 SOLUTION OF LINEAR PROGRAMMING PROBLEMS

In the previous section we have seen the problems in which the number of relations are not equal to the number of variables and many of the relations are in the form of inequation (i.e., $\leq$ or $\geq$) to maximise (or minimise) a linear function of the variables subject to such conditions.

Now the question is how one can find a solution for such problems?

To answer this questions, let us consider the system of equations and inequations (or inequalities).

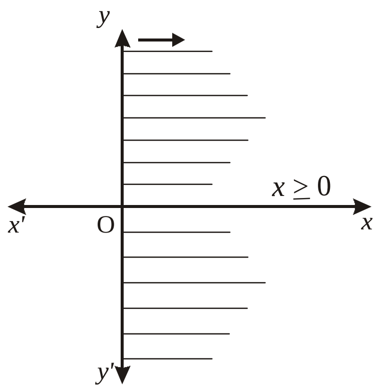

Fig. 37.1

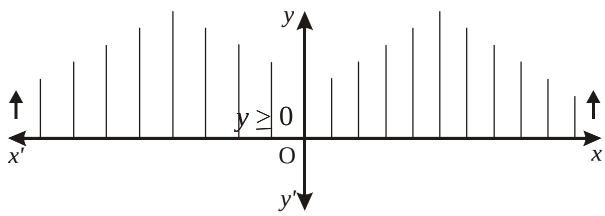

Fig. 37.2

We know that $x \geq 0$ represents a region lying towards the right of y - axis including the y - axis.

Similarly, the region represented by $y \geq 0$, lies above the x - axis including the x-axis.

The question arises: what region will be represented by $x \geq 0$ and $y \geq 0$ simultaneously.

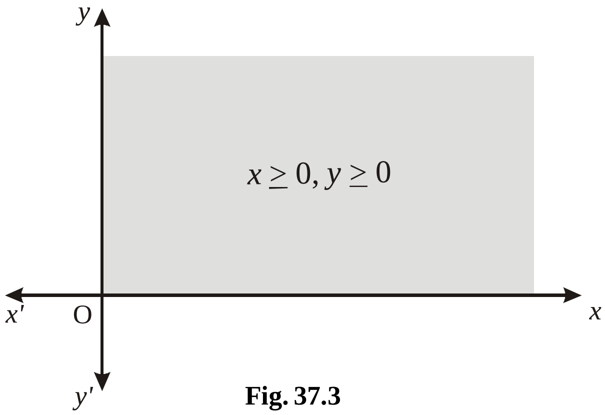

Fig. 37.3

Obviously, the region given by $x \geq 0$, $y \geq 0$ will consist of those points which are common to both $x \geq 0$ and $y \geq 0$. It is the first quadrant of the plane.

Next, we consider the graph of the equation $x + 2y \leq 8$. For this, first we draw the line $x + 2y = 8$ and then find the region satisfying $x + 2y \leq 8$.

Usually we choose x = 0 and calculate the corresponding value of y and choose y = 0 and calculate the corresponding value of x to obtain two sets of values (This method fails, if the line is parallel to either of the axes or passes through the origin. In that case, we choose any arbitrary value for x and choose y so as to satisfy the equation).

Plotting the points (0,4) and (8,0) and joining them by a straight line, we obtain the graph of the line as given in the Fig. 37.4 below.

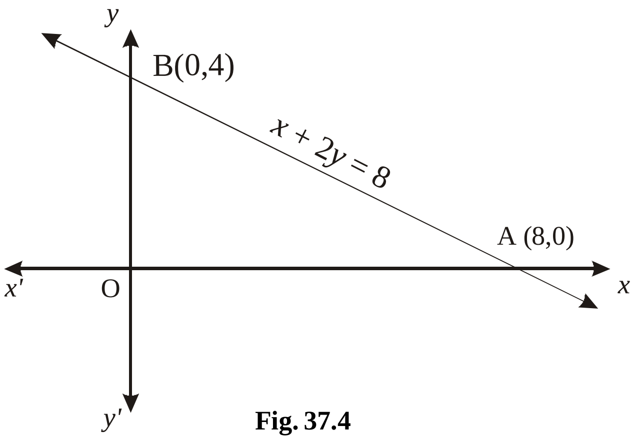

Fig. 37.4

We have already seen that $x \geq 0$ and $y \geq 0$ represents the first quadrant. The graph given by $x + 2y < 8$ lies towards that side of the line

$x + 2y = 8$ in which the origin is situated because any point in this region will satisfy the inequality. Hence the shaded region in the Fig. 37.5 represents $x \geq 0$, $y \geq 0$ and $x + 2y \leq 8$ simultaneously.

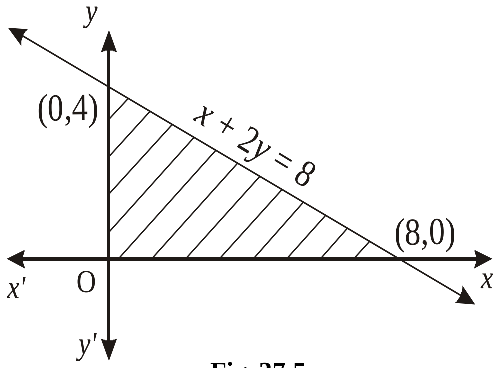

Fig. 37.5

Similarly, if we have to consider the regions bounded by $x \geq 0$, $y \geq 0$ and $x + 2y \geq 8$, then it will lie in the first quadrant and on that side of the line $x + 2y = 8$ in which the origin is not located. The graph is shown by the shaded region, in Fig. 37.6

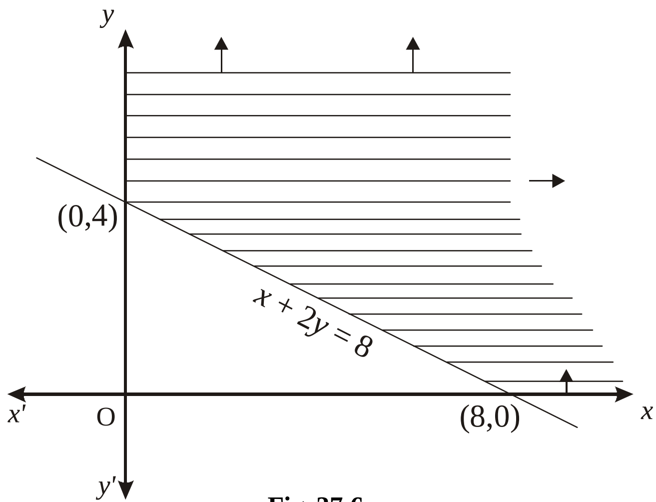

Fig. 37.6

The shaded region in which all the given constraints are satisfied is called the **feasible region.**

37.5.1 Feasible Solution

A set of values of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called a **feasible solution** of the problem.

37.5.2 Optimal Solution

A feasible solution of a linear programming problem which optimises its objective functions is called the **optimal solution** of the problem.

> ***Note :*** *If none of the feasible solutions maximise (or minimise) the objective function, or if there are no feasible solutions, then the linear programming problem has* ***no solution.***

In order to find a graphical solution of the linear programming problem, following steps be employed.

Step 1 : Formulate the linear programming problem.

Step 2 : Graph the constraints (inequalities), by the method discussed above.

Step 3 : Identify the feasible region which satisfies all the constraints simultaneously. For less than or equal to' constraints the region is generally below the lines and 'for greater than or equal to' constraints, the region is above the lines.

Step 4 : Locate the solution points on the feasible region. These points always occur at the vertex of the feasible region.

Step 5 : Evaluate the objective function at each of the vertex (corner point)

Step 6 : Identify the optimum value of the objective function.

Example 37.5 Minimise the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : The objective function to be minimised is

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0,\ x_2 \geq 0$$

First of all we draw the graphs of these inequalities, which is as follows :

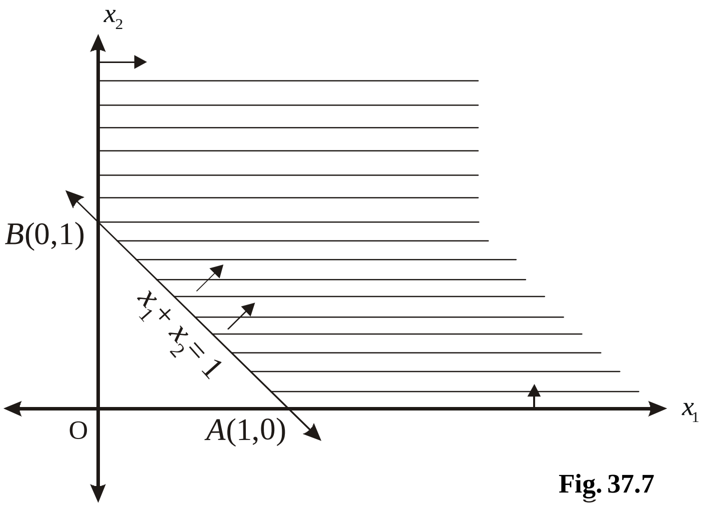

Fig. 37.7

As we have discussed earlier that the region satisfied by $x_1 \geq 0$ and $x_2 \geq 0$ is the first quadrant and the region satisfied by the line $x_1 + x_2 \geq 1$ along with $x_1 \geq 0,\ x_2 \geq 0$ will be on that side of the line $x_1 + x_2 = 1$ in which the origin is not located. Hence, the shaded region is our feasible solution because every point in this region satisfies all the constraints. Now, we have to find optimal solution. The vertex of the feasible region are A (1,0) and B (0,1).

The value of z at $A = 1$

The value of z at $B = 2$

Take any other point in the feasible region say (1,1), (2,0), (0,2) etc. We see that the value of z is minimum at A (1,0).

Example 37.6 Minimise the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : The objective function to be minimised is

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

First of all we draw the graphs of these inequalities (as discussed earlier) which is as follows :

The shaded region is the feasible region. Every point in the region satisfies all the mathematical inequalities and hence the feasible solution.

Now, we have to find the optimal solution.

The value of z at $B\ (1.5, 0)$ is 1.5

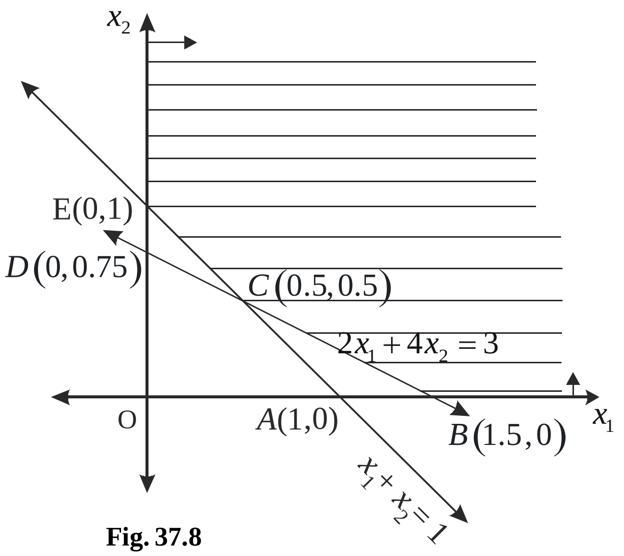

Fig. 37.8

The value of z at $C\ (0.5, 0.5)$ is 1.5

The value of z at E (0,1) is 2

If we take any point on the line $2x_1 + 4x_2 = 3$ between B and C we will get $\frac{3}{2}$ and elsewhere in the feasible region greater than $\frac{3}{2}$. Of course, the reason any feasible point (between B and C) on $2x_1 + 4x_2 = 3$ minimizes the objective function (equation) $z = x_1 + 2x_2$ is that the two lines are parallel (both have slope $-\frac{1}{2}$). Thus this linear programming problem has infinitely many solutions and two of them occur at the vertices.

Example 37.7 Maximise

$$z = 0.25x_1 + 0.45x_2$$

subject to the constraints

$$x_1 + 2x_2 \le 300$$
$$3x_1 + 2x_2 \le 480$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Solution : The objective function is to maximise

$$z = 0.25x_1 + 0.45x_2$$

subject to the constraints

$$x_1 + 2x_2 \le 300$$
$$3x_1 + 2x_2 \le 480$$
$$x_1 \ge 0,\ x_2 \ge 0$$

First of all we draw the graphs of these inequalities, which is as follows :

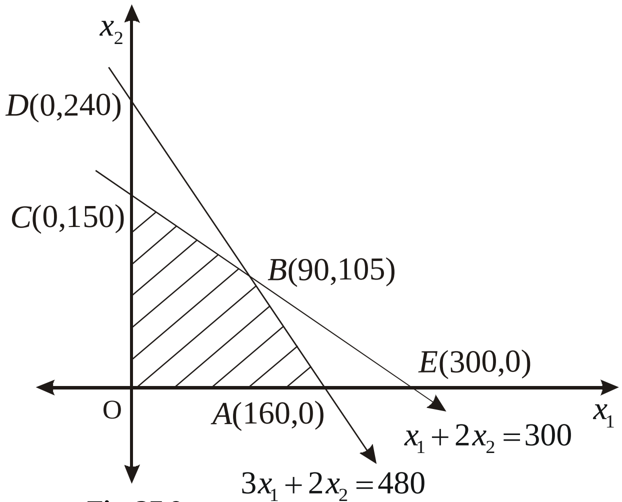

Fig. 37.9

The shaded region $OABC$ is the feasible region. Every point in the region satisfies all the mathematical inequations and hence the feasible solutions.

Now, we have to find the optimal solution.

The value of z at A (160, 0) is 40.00

The value of z at B (90, 105) is 69.75.

The value of z at C (0, 150) is 67.50

The value of z at O (0, 0) is 0.

If we take any other value from the feasible region say (60, 120), (80, 80) etc. we see that still the maximum value is 69.75 obtained at the vertex B (90, 105) of the feasible region.

Note : *For any linear programming problem that has a solution, the following general rule is true.*

If a linear programming problem has a solution it is located at a vertex of the feasible region. If a linear programming problem has multiple solutions, at least one of them is located at a vertex of the feasible region. In either case, the value of the objective function is unique.

Example 37.8 In a small scale industry a manufacturer produces two types of book cases. The first type of book case requires 3 hours on machine A and 2 hours on machines B for completion, whereas the second type of book case requires 3 hours on machine A and 3 hours on machine B. The machine A can run at the most for 18 hours while the machine B for at the most 14 hours per day. He earns a profit of Rs. 30 on each book case of the first type and Rs. 40 on each book case of the second type.

How many book cases of each type should he make each day so as to have a maximum porfit?

Solution : Let x_1 be the number of first type book cases and x_2 be the number of second type book cases that the manufacturer will produce each day.

Since x_1 and x_2 are the number of book cases so

$$x_1 \geq 0,\ x_2 \geq 0 \qquad \cdots \textbf{(1)}$$

Since the first type of book case requires 3 hours on machine A, therefore, x_1 book cases of first type will require $3x_1$ hours on machine A. second type of book case also requires 3 hours on machine A, therefore, x_2 book cases of second type will require $3x_2$ hours on machine A. But the working capacity of machine A is at most 18 hours per day, so we have

$$3x_1 + 3x_2 \leq 18$$

or $$x_1 + x_2 \leq 6 \qquad \cdots \textbf{(2)}$$

Similarly, on the machine B, first type of book case takes 2 hours and second type of book case takes 3 hours for completion and the machine has the working capacity of 14 hours per day, so we have

$\mathbf{2x_1 + 3x_2 \leq 14}$ $\quad\cdots$ **(3)**

Profit per day is given by

$z = 30x_1 + 40x_2$ $\quad\cdots$ **(4)**

Now, we have to determine x_1 and x_2 such that

Maximize $z = 30x_1 + 40x_2$ (objective function) subject to the conditions

$$\left.\begin{aligned} x_1 + x_2 &\leq 6 \\ 2x_1 + 3x_2 &\leq 14 \\ x_1 \geq 0,\ x_2 &\geq 0 \end{aligned}\right\} \quad \textbf{constraints}$$

We use the graphical method to find the solution of the problem. First of all we draw the graphs of these inequalities, which is as follows :

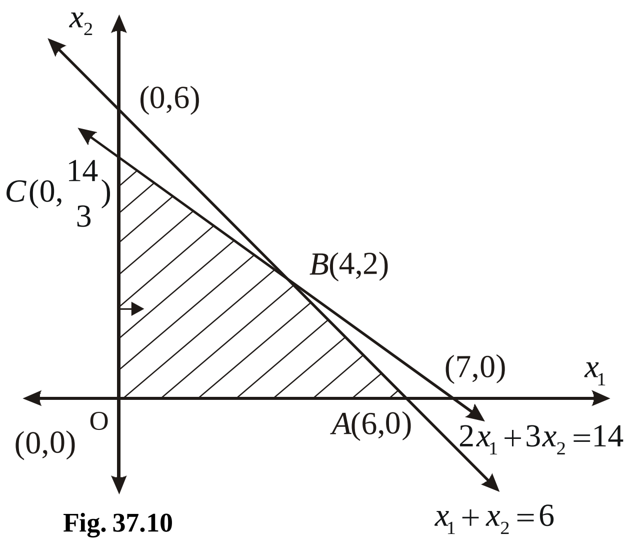

Fig. 37.10

The shaded region OABC is the feasible region. Every point in the region satisfies all the mathematical inequations and hence known as feasible solution.

We know that the optimal solution will be obtained at the vertices O (0, 0), A (6, 0). B (4,2). Since the co-ordinates of C are not integers so we don't consider this point. Co-ordinates of B are calculated as the intersection of the two lines.

Now the profit at O is zero.

Profit at A = 30 × 6 + 40 × 0

= 180

Profit at B = 30 × 4 + 40 × 2

$= 120 + 80$

$= 200$

Thus the small scale manufacturer gains the maximum profit of Rs.200 if he prepares 4 first type book cases and 2 second type book cases.

Example 37.9 Maximize the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0$$

Solutions : First we graph the constraints

$$x_1 + x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0$$

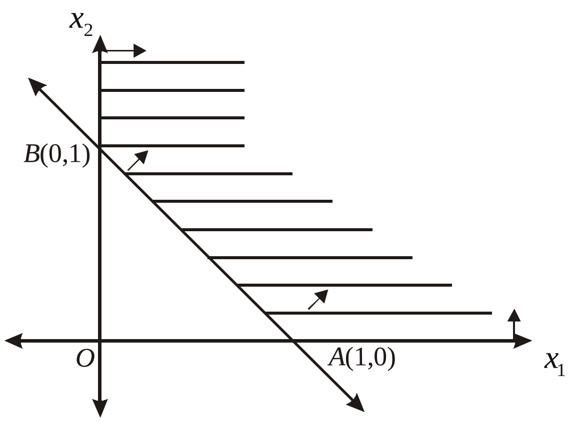

Fig. 37.11

The shaded portion is the set of feasible solution.

Now, we have to maximize the objective function.

The value of z at $A(1, 0)$ is 1.

The value of z at $B(0, 1)$ is 2.

If we take the value of z at any other point from the feasible region, say (1, 1) or (2, 3) or (5, 4) etc, then we notice that every time we can find another point which gives the larger value than the previous one. Hence, there is no feasible point that will make z largest. Since there is no feasible point that makes z largest, we conclude that this linear programming problem has no solution.

Example 37.10 Solve the following problem graphically.

Minimize $z = 2x_1 - 10x_2$

subject to the constraints

$$x_1 - x_2 \geq 0$$
$$x_1 - 5x_2 \leq -5$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : First we graph the constraints

$$x_1 - x_2 \geq 0$$
$$x_1 - 5x_2 \leq -5$$
$$x_1 \geq 0,\ x_2 \geq 0$$

$$x_2 - x_1 \le 0,$$

or $$5x_2 - x_1 \ge 5$$

$$x_1 - x_2 = 0$$

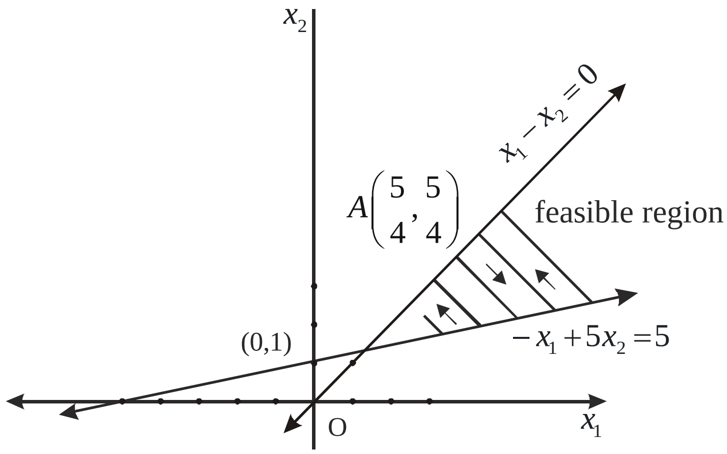

Fig. 37.12

The shaded region is the feasible region.

Here, we see that the feasible region is unbounded from one side.

But it is clear from Fig. 37.26 that the objective function attains its minimum value at the point A which is the point of intersection of the two lines $x_1 - x_2 = 0$ and $-x_1 + 5x_2 = 5$.

Solving these we get $x_1 = x_2 = \frac{5}{4}$

Hence, z is minimum when $x_1 = \frac{5}{4}$, $x_2 = \frac{5}{4}$, and its minimum value is

$$2 \times \frac{5}{4} - 10 \times \frac{5}{4} = -10.$$

Note : *If we want to find max. z with these constraints then it is not possible in this case because the feasible region is unbounded from one side.*

CHECK YOUR PROGRESS 37.2

Solve the following problems graphically

1. **Maximize** $z = 3x_1 + 4x_2$

   subject to the conditions

2. **Maximize** $\neq = 2x_1 + 3x_2$

   subject to the conditions

$x_1 + x_2 \le 40$
$x_1 + 2x_2 \le 60$
$x_1 \ge 0,\ x_2 \ge 0$

$x_1 + x_2 \le 400$
$2x_1 + x_2 \le 600$
$x_1 \ge 0, x_2 \ge 0$

3. **Minimize** $z = 60x_1 + 40x_2$

subject to the conditions

$3x_1 + x_2 \ge 24$
$x_1 + x_2 \ge 16$
$x_1 + 3x_2 \ge 24$
$x_1 \ge 0,\ x_2 \ge 0$

4. **Maximize** $z = 20x_1 + 30x_2$

subject to the conditions

$x_1 + x_2 \le 12,$
$5x_1 + 2x_2 \le 50$
$x_1 + 3x_2 \le 30,$
$x_1 \ge 0,\ x_2 \ge 0$

5. **Maximize** $z = 50x_1 + 15x_2$

subject to the conditions

$5x_1 + x_2 \le 100$
$x_1 + x_2 \le 60$
$x_1 \ge 0, x_2 \ge 0$

6. **Minimize** $z = 4000x_1 + 7500x_2$

subject to the conditions

$4x_1 + 3x_2 \ge 40$
$2x_1 + 3x_2 \ge 8$
$x_1 \ge 0, x_2 \ge 0$

LET US SUM UP

- Linear programming is a technique followed by mathematicians to solve the optimisation problems.
- A set of values of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called a **feasible solution**.
- A feasible solution of a linear programming problem which optimises its objective function is called the **Optimal solution** of the problem.
- The optimal solution of a linear programming problem is located at a vertex of the set of feasible region.
- If a linear programming problem has multiple solutions, at least one of them is located at a vertex of the set of feasible region. But in all the cases the value of the objective function remains the same.

SUPPORTIVE WEB SITES

http://people.brunel.ac.uk/~mastjjb/jeb/or/morelp.html
http://en.wikipedia.org/wiki/Simplex_algorithm
http://www.youtube.com/watch?v=XbGM4LjM52k

TERMINAL EXERCISE

1. A dealer has ₹ 1500 only for a purchase of rice and wheat. A bag of rice costs ₹ 1500 and a bag of wheat costs ₹ 1200. He has a storage capacity of ten bags only and the dealer gets a profit of ₹ 100 and ₹ 80 per bag of rice and wheat respectively. Formulate the problem as a linear programming problem to get the maximum profit.

2. A business man has ₹ 600000 at his disposal and wants to purchase cows and buffaloes to take up a business. The cost price of a cow is ₹ 20,000 and that of a buffalo is ₹ 60000. The man can store fodder for the live stock to the extent of 40 quintals per week. A cow gives 10 litres of milk and buffalo gives 20 litres of milk per day. Profit per litre of milk of cow is ₹ 5 and per litre of the milk of a buffalo is ₹ 7. If the consumption of fodder per cow is 1 quintal and per buffalo is 2 quintals a week, formulate the problem as a linear programming problem to find the number of live stock of each kind the man has to purchase so as to get maximum profit (assuming that he can sell all the quantity of milk, he gets from the livestock)

3. A factory manufactures two types of soaps each with the help of two machines *A* and *B*. *A* is operated for two minutes and *B* for 3 minutes to manufacture the first type, while the second type is manufactured by operating A for 3 minutes and B for 5 minutes. Each machine can be used for at most 8 hours on any day. The two types of soaps are sold at a profit of 25 paise and 50 paise each respectively. How many soaps of each type should the factory produce in a day so as to maximize the profit (assuming that the manufacturer can sell all the soaps he can manufacture). Formulate the problem as a linear programming problem.

4. Determine two non-negative rational numbers such that their sum is maximum provided that their difference exceeds four and three times the first number plus the second should be less than or equal to 9. Formulate the problem as a linear programming problem.

5. Vitamins *A* and *B* are found in two different foods *E* and *F*. One unit of food *E* contains 2 units of vitamin *A* and 3 units of vitamin *B*. One unit of food *F* contains 4 units of vitamin *A* and 2 units of vitamin *B*. One unit of food *E* and *F* costs Rs.5 and Rs.2.50 respectively. The minimum daily requirements for a person of vitamin *A* and *B* is 40 units and 50 units respectively. Assuming that anything in excess of daily minimum requirement of vitamin *A* and *B* is not harmful, find out the optimal mixture of food *E* and *F* at the minimum cost which meets the daily minimum requirement of vitamin *A* and *B*. Formulate this as a linear programming problem.

6. A machine producing either product *A* or *B* can produce *A* by using 2 units of chemicals and 1 unit of a compound and can produce *B* by using 1 unit of chemicals and 2 units of the compound. Only 800 units of chemicals and 1000 units of the compound are available. The profits available per unit of *A* and *B* are respectively Rs. 30 and Rs.20. Find the optimum allocation of units between *A* and *B* to maximise the total profit. Find the maximum profit.

7. Solve the following Linear programming problem graphically.

(a) Maximize $z = 25x_1 + 20x_2$

subject to the constraints

$$3x_1 + 6x_2 \le 50$$
$$x_1 + 2x_2 \le 10$$
$$x_1 \ge 0,\ x_2 \ge 0$$

(b) Maximize $z = 9x_1 + 10x_2$

subject to the constraints

$$11x_1 + 9x_2 \le 9900$$
$$7x_1 + 12x_2 \le 8400$$
$$3x_1 + 8x_2 \le 4800$$
$$x_1 \ge 0,\ x_2 \ge 0$$

(c) Maximise $z = 22x_1 + 18x_2$

subject to the constraints

$$x_1 + x_2 \le 20,$$
$$3x_1 + 2x_2 \le 48$$
$$x_1 \ge 0,\ x_2 \ge 0$$

ANSWERS

CHECK YOUR PROGRESS 37.1

1. Maximize $z = 3x_1 + 8x_2$

   subject to the constraints

   $$3x_1 + 4x_2 \le 18$$
   $$4x_1 + 5x_2 \le 21$$
   $$x_1 \ge 0,\ x_2 \ge 0.$$

2. Maximize $z = 50x_1 + 15x_2$

   subject to the constraints

   $$5x_1 + x_2 \le 100$$
   $$x_1 + x_2 \le 60$$
   $$x_1 \ge 0,\ x_2 \ge 0.$$

3. Minimize $z = 4000x_1 + 7500x_2$

   subject to the constraints

   $$4x_1 + 3x_2 \ge 40$$
   $$2x_1 + 3x_2 \ge 8$$
   $$x_1 \ge 0,\ x_2 \ge 0$$

4. Maximize $z = 5x_1 + 4x_2$

   subject to the constraints

   $$1.5x_1 + 2.5x_2 \le 80$$
   $$2x_1 + 1.5\ x_2 \le 70$$
   $$x_1 \ge 0,\ x_2 \ge 0$$

CHECK YOUR PROGRESS 37.2

1.

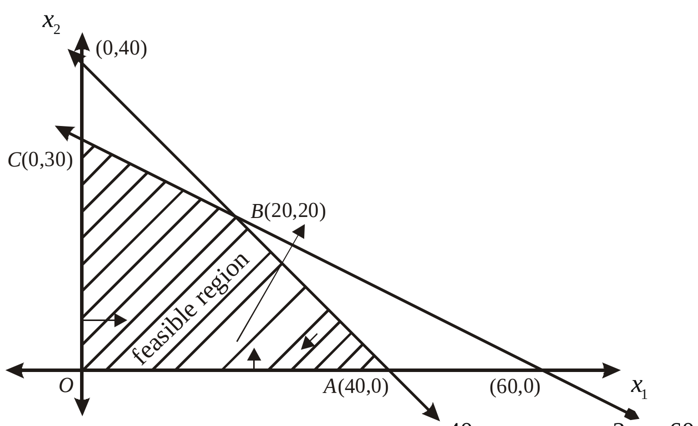

Fig. 37.13

Maximum $z = 140$ at $B(20,20)$

2.

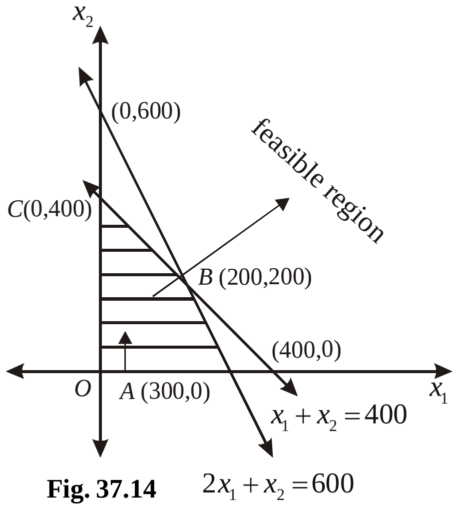

Fig. 37.14

Maximize $z = 1200$ at $C(0,400)$

3.

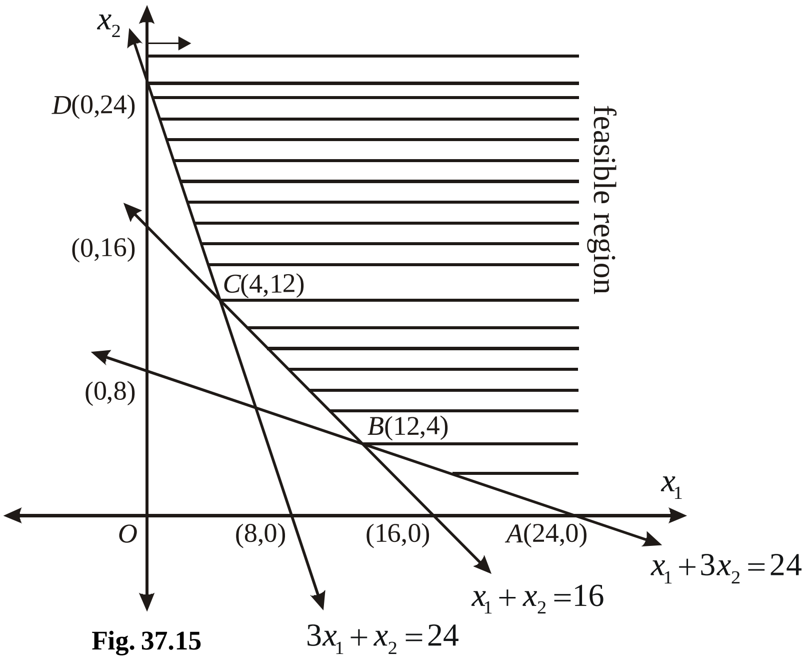

Fig. 37.15

Minimize $z = 720$ *at* $C(4,12)$, $x_1 = 4$, $x_2 = 12$

4.

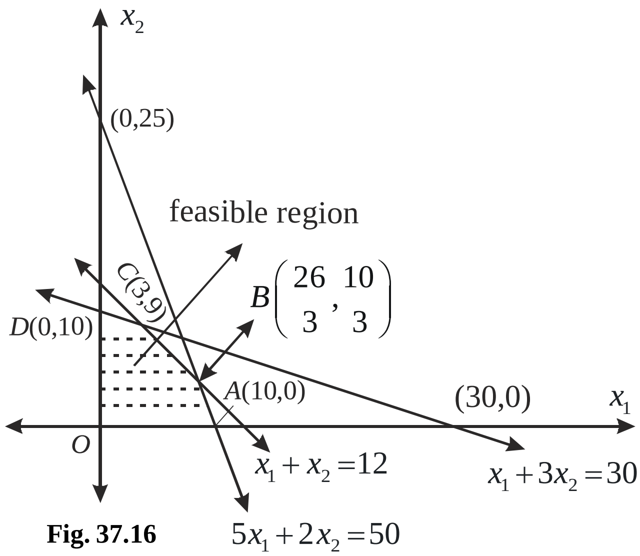

Fig. 37.16

Maximum $z = 330$ *at* $C(3,9)$, $x_1 = 3$, $x_2 = 9$

5.

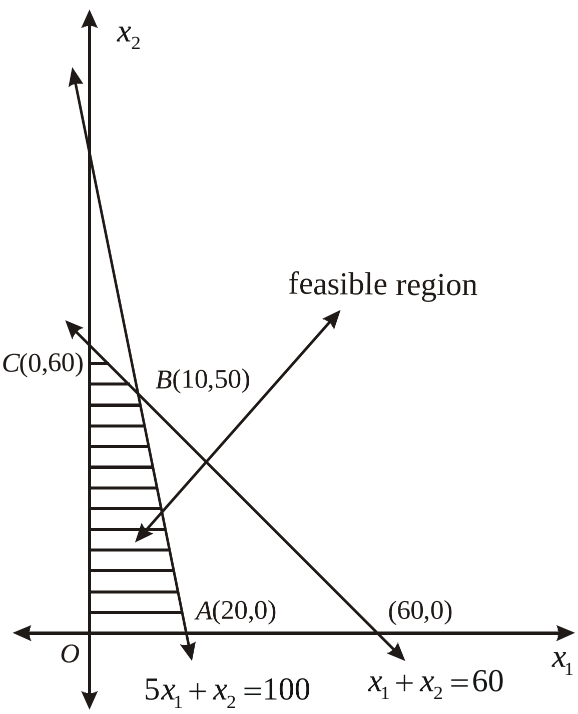

Fig. 37.17

Maximum $z = 1250$ *at* $B(10,50)$, $x_1 = 10$, $x_2 = 50$

6.

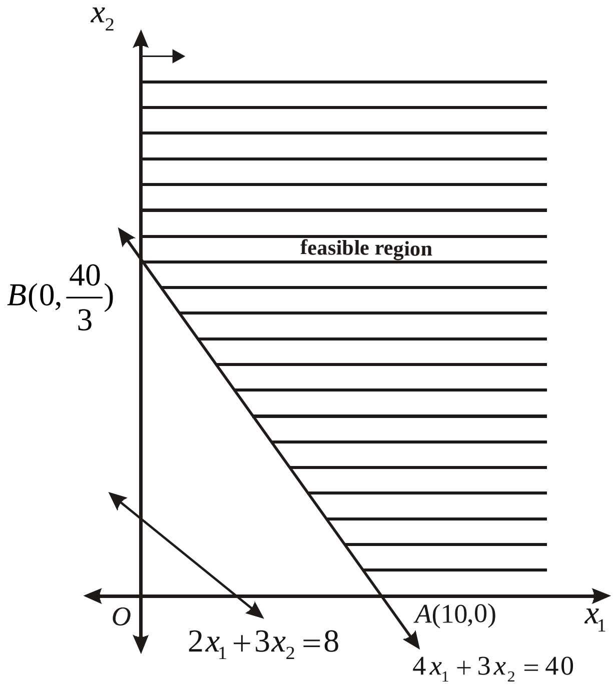

Fig. 37.18

Maximum z $=$ 40,000 at A(10,0), $x_1 = 10,\ x_2 = 0$

TERMINAL EXERCISE

1. Maximize $z = 100x_1 + 80x_2$

   subject to the conditions

   $5x_1 + 4x_2 \le 50$
   $x_1 + x_2 \le 10$
   $x_1 \ge 0,\ x_2 \ge 0$

2. Maximize $z = 150x_1 + 980x_2$

   subject to the conditions

   $x_1 + 3x_2 \le 30$
   $7x_1 + 14x_2 \le 40$
   $x_1 \ge 0,\ x_2 \ge 0$

3. Maximize $z = 25x_1 + 50x_2$

   subject to the conditions

   $2x_1 + 3x_2 \le 480$
   $3x_1 + 5x_2 \le 480$
   $x_1 \ge 0,\ x_2 \ge 0$

4. Maximize $z = x_1 + x_2$

   subject to the conditions

   $x_1 - x_2 \ge 4$
   $3x_1 + x_2 \le 9$
   $x_1 \ge 0,\ x_2 \ge 0$

5. Minimize $z = 5x_1 + 2.5x_2$

   subject to the conditions

$x_1 + 2x_2 \geq 20$
$3x_1 + 2x_2 \geq 50$
$x_1 \geq 0,\ x_2 \geq 0$

6. Maximize $z = 30x_1 + 20x_2$
subject to the constraints

$2x_1 + x_2 \leq 800$
$x_1 + 2x_2 \leq 1000$
$x_1 \geq 0,\ x_2 \geq 0$

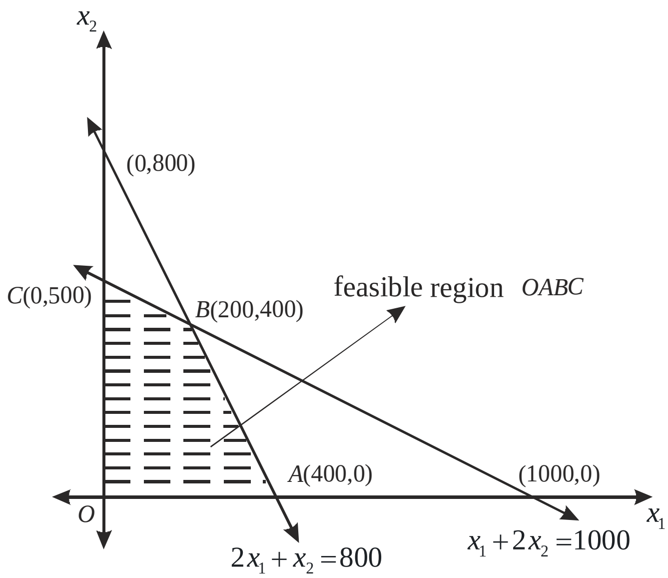

Fig. 37.19

Maximum z = 14000 at
B (200,400). $14000 = z$

8. (a)

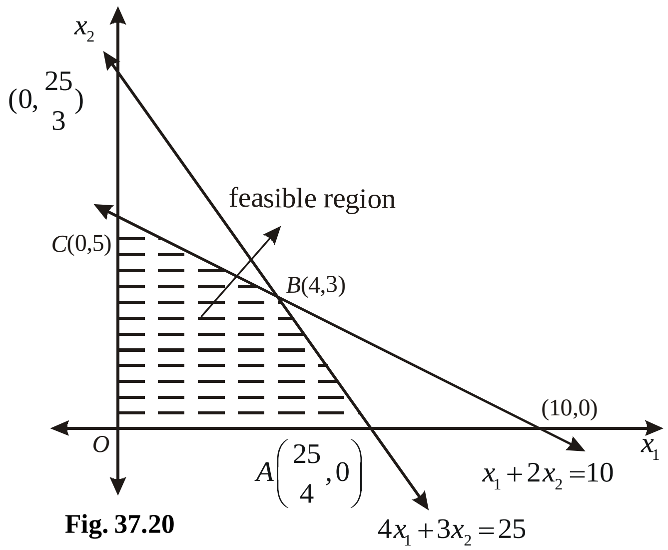

Fig. 37.20

Maximum $z = 160$ at $B(4,3), x_1 = 4 x_2 = 3$

(b)

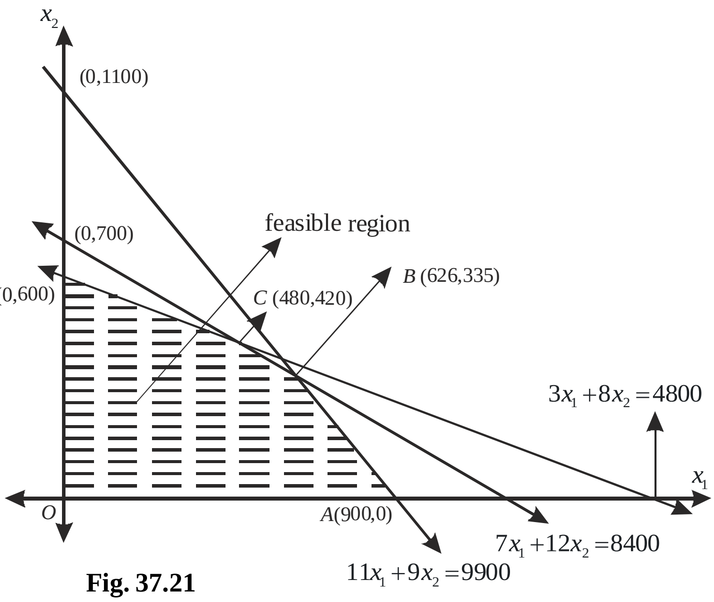

Fig. 37.21

A (900,0) D(0,600) B (626, 335), O(0, 0) and C (480, 420)

Maximum $z = 8984$ *at* $B(626, 335)$ $x_1 = 626$, $x_2 = 335$

(c)

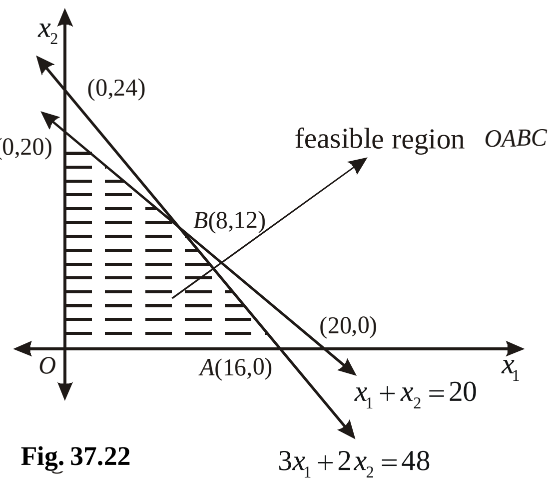

Fig. 37.22

Maximum $z = 392 \, at \, B(8,12) \, x_1 = 8 x_2 = 12$