2

ALGEBRA OF COMPLEX NUMBERS

2.1 INTRODUCTION

Let us consider a quadratic equations $x^2 - 5xi - 6 = 0$. On solving this equation, we find that the roots are $2i$ and $3i$. Suppose the roots of a quadratic equation are i and $-i$. Now to find the corresponding quadratic equation we need to find the sum of the roots and the product of the roots. That is, we need to perform addition and multiplication of complex numbers. In the present lesson we would like to verify whether the four fundamental operations can be performed in Complex numbers or not, if yes how? Which properties hold good in the operations?

2.2 OBJECTIVES

After studying this lesson, you will be able to:

- state the condition for equality of two complex numbers.
- add two or more complex numbers.
- subtract a complex number from another complex number.
- multiply two given complex numbers.
- calculate the multiplicative inverse of a complex number.
- divide a complex number by another complex number.
- state the various properties of addition and multiplication

of complex numbers and recognise the use of these properties.

2.3 PREVIOUS KNOWLEDGE

(a) Four fundamental operations in real numbers.

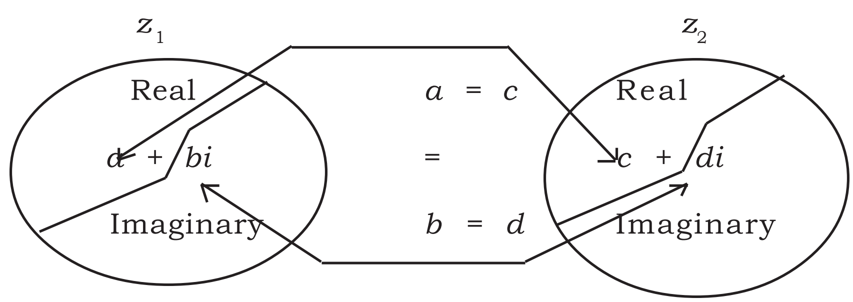
(b) Properties of operations in real numbers

(c) Properties of modulus and conjugate of complex numbers.

(d) Solving problems related to real numbers.

2.4 EQUALITY OF COMPLEX NUMBERS

Two complex numbers are said to be equal if and only if their real parts and imaginary parts are respectively equal, e.g. $a + bi = c + di$ if and only if $a = c$ and $b = d$.



Example A: For what value of x and y

(i) $5x + 6yi$ and $30 + 30i$ are equal.

(ii) $3(x + yi)$ and $-3(3 - 2i)$ are equal.

Solution: (i) We are given that :

$5x + 6yi = 30 + 30i$ (Reapt part = Real part and Imaginary part = Imaginary part)

$\Rightarrow$ $5x = 30$ and $6y = 30$

$\Rightarrow$ $x = 6$ and $y = 5$

(ii) Here

$3(x + yi) = -3(3 - 2i)$

i.e. $3x + 3yi = -9 + 6i$

$\Rightarrow$ $3x = -9$ and $3y = 6$

$$\Rightarrow \quad x = -3 \text{ and } y = 2$$

Check-point 1:

If a and b are real numbers, under what condition(s) will $a + bi = b + ai$

{ ***Ans:*** $\boldsymbol{a = b}$ }

INTEXT QUESTIONS 2.1

1. Find the value of x and y if

   (a) $2x + 3yi = 4 - 9i$

   (b) $(\frac{3}{2}x - 2) + 5\sqrt{2}\,yi = \sqrt{2}$

   (c) $(\frac{1}{2}x - \frac{1}{3}y) + \frac{2}{3}yi = -2 + 4i$

   (d) $(x - yi) + 7 - 2i = 9 - i$

   (e) $10 - 3xi = 7y + 9i$

   (f) $(x - 2y) + (y - x)i = 3 - 7i$

2.5 ADDTION OF COMPLEX NUMBERS

When $2x + y$ and $x + 3y$ are added we add $x's$ and $y's$ to respective value

i.e. $(2x + y) + (x + 3y)$

$= (2 + 1)x + (1 + 3)y$

$= 3x + 4y$

In complex numbers, addition will imply adding real part to the real and imaginary part to the imaginary.

In general, $z_1 + z_2$ where $z_1 = a + bi$ and $z_2 = c + di$ will be $(a + bi) + (c + di)$, i.e., $(a + c) + (b + d)i$

Example B: Simplify

(i) $(3 + 2i) + (4 - 3i)$

(ii) $(2 + 5i) + (-3 - 7i) + (1 - i)$

Solution (i) $(3 + 2i) + (4 - 3i) = (3 + 4) + (2 - 3)i$

$= 7 + (-1)i$

or, $(3 + 2i) + (4 - 3i) = 7 - i$

(ii) $(2 + 5i)+(-3-7i)+(1-i) = (2-3+1) + (5-7-1)i$

$= (3-3) + (5-8)i$

$= 0 - 3i$

or, $(2+5i) + (-3-7i) + (1-i) = -3i$

Check-point 2:

1. Fill in the blank:

   The sum of a complex number and its conjugate is always a ______________________________.

2. Simplify :

   $(2 + 5i) + (1 - 4i)$

2.6 SUBTRACTION OF COMPLEX NUMBERS

Let $z_1 = a + bi$ and $z_2 = c + di$ then

$z_1 - z_2 = (a + bi) - (c + di)$

$\Rightarrow$ $z_1 + (-z_2) = (a - c) + (b - d)i$

Example C: Subtract $3 + 2i$ from $5 + 6i$

Solution $(5 + 6i) - (3 + 2i) = (5 - 3) + (6 - 2)i$

$= 2 + 4i$

Example D: What should be added to i to obtain 5?

Solution: The given complex number is $0 + i$

Let z = (a+bi) be added to 0+i to obtain (5+0i)

Then $(0 + i) + (a + bi) = 5 + 0i$

or, $0 + a = 5$ and $1 + b = 0$

or, $a = 5$ and $b = -1$

Alternatively $i = 0 + i$ and $5 = 5 + 0i$. The number is

$$5 + 0i - (0 + i)$$

$$= 5 - i$$

Hence 5–i is the complex number which when added to i, gives 5.

Check-point 3:

Show that $-(\bar{z}) = \overline{(-z)}$ for all complex number z.

Properties:

1. **Closure:** Sum of two complex numbers will be a complex number.

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

Now, $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ which is again a complex number as $(a_1 + a_2)$ and $(b_1 + b_2)$ are real numbers. This proves the closure property of complex number.

e.g. $(1 + i) + (2 + 3i) = (1 + 2) + (1 + 3)i = 3 + 4i$

Similarly, the difference of two complex numbers will be a complex number

e.g. $(2 + 4i) - (1 - 4i) = (2 - 1) + (4 - 4)i = 1 + 8i$

2. **Commutative :**

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

Now, $z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$

$= (a_1 + a_2) + (b_1 + b_2)i$

$= (a_2+a_1) + (b_2+b_1)i$

$= (a_2+b_2 i) + (a_1+b_1 i)$

$= z_2 + z_1$

i.e., $z_1 + z_2 = z_2 + z_1$

Hence, additional of complex numbers is commutative.

Let $z_1 = 8 + 7i$ and $z_2 = 9 - 3i$ then

$z_1 + z_2 = (8+7i) + (9-3i)$ and $z_2 + z_1 = (9-3i) + (8+7i)$

$= (8+9) + (7-3)i$ and $= (9+8) + (-3+7)i$

or, $z_1 + z_2 = 17 + 4i$ and $z_2 + z_1 = 17 + 4i$

we get, $z_1 + z_2 = z_2 + z_1$

Now, $z_1 - z_2 = (a_1 + b_1 \mathrm{i}) - (a_2 + b_2 i)$

$= (a_1 - a_2) + (b_1 - b_2)i$

and $z_2 - z_1 = (a_2 + b_2 i) - (a_1 + b_1 \mathrm{i})$

$= (a_2 + b_2 i) - (b_2 - b_1)i$

$= -(a_1 - a_2) - (b_1 - b_2)i$

$= -(a_1 + b_1 i) + (a_2 + b_2 i)$

$= z_2 + z_1$

$\therefore \quad z_1 - z_2 \neq z_2 - z_1$

Hence, subtraction of complex numbers is not commutative.

e.g. $z_1 - z_2 = (8+7i) - (9-3i)$ and $z_2 - z_1 = (9-3i) - (8+7i)$

$= (8-9) + (7+3)i$ and $= (9-8) + (-3-7)i$

or, $z_1 - z_2 = -1 + 10i$ and $z_2 - z_1 = 1-10i$

$\therefore \quad z_1 - z_2 \neq z_2 - z_1$

Associative:

Let $z_1 = a_1 + b_1 i, \quad z_2 = a_2 + b_2 i, \quad z_3 = a_3 + b_3 i$

Now $z_1 + (z_2 + z_3)$

$= (a_1 + b_1 i) + \{ (a_2 + b_2 i) + (a_3 + b_3 i) \}$

$= (a_1 + b_1 i) + \{ (a_2 + a_3) + (b_2 + b_3) i \}$

$= \{ (a_1 + (a_2 + a_3) \} + \{ b_1 + (b_2 + b_3) \} i$

$= \{ (a_1 + a_2) + a_3 \} + \{ (b_1 + b_2) + b_3 \} i$

$= \{ (a_1 + a_2) + (b_1 + b_2) i \} + (a_3 + b_3)$

$= \{ (a_1 + b_1 i) + (a_2 + b_2 i) \} + (a_3 + b_3 i)$

$= (z_1 + z_2) + z_3$

Hence, the associativity property holds good in the case of addition of complex numbers.

e.g. let $z_1 = 2 + 3i$, $z_2 = 3i$ and $z_3 = 1 - 2i$. Then

$$z_1 + (z_2 + z_3) = (2 + 3i) + \{(3i) + (1 - 2i)\}$$

$$= (2 + 3i) + (1 + i)$$

$$= (3 + 4i)$$

$$(z_1 + z_2) + z_3 = \{(2 + 3i) + (3i)\} + (1 - 2i)$$

$$= (2 + 6i) + (1 - 2i)$$

$$= (3 + 4i)$$

We get $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

The equality of two sums is the consequence of the associative property of addition of complex numbers.

Like commutativity, you can observe that associativity also does not hold good in the case of subtraction.

Identity:

Let $z_2 = x + yi$ be the additive identity of $z_1 = 2+3i$ then

$z_1 + z_2 = z_1$

i.e. $(2 + 3i) + (x + yi) = 2 + 3i$

or $(2 + x) + (3 + y)i = 2 + 3i$

or $(2 + x) = 2$ and $3 + y = 3$

or $x = 0$ and $y = 0$

i.e. $z_2 = x + yi = 0 + 0i$ is the additive identity.

i.e. if $z = a + bi$ is any complex number, then

$(a + bi) + (0 + 0i) = a + bi$

i.e. 0 is the additive identity.

$$z_1 - z_2 = (2 + 3i) - (0 + 0i)$$
$$= (2 - 0) + (3 - 0)i$$
$$= 2 + 3i = z_1$$

∴ $z_2 = 0 + 0i$ is the identity in substraction also

i.e. $(a + bi) - (0 + 0i) = a + bi$

Inverse:

Let $z = a + bi$, $z_1 = a_1 + b_1 i$. Then

z_1 is said to be the additive inverse of z if $z + z_1 = 0 + 0i$

ie., $z + z_1 = 0$

Let $z_1 = 4 + 5i$ and $z_2 = x + yi$ be the additive inverse of z_1

Then, $z_1 + z_2 = 0$

or $(4 + 5i) + (x + yi) = 0 + 0i$

or $(4 + x) + (5 + y)i = 0 + 0i$

or $4 + x = 0$ and $5 + y = 0$

or $x = -4$ and $y = -5$

Thus, $z_2 = -4 - 5i$ is the additive inverse of $z_1 = 4 + 5i$

consider $z_1 - z_2 = 0$

or $(4 + 5i) - (x + yi) = 0 + 0i$

or $(4 - x) + (5 - y)i = 0 + 0i$

or $4 - x = 0$ and $5 - y = 0$

or $x = 4$ and $y = 5$

i.e. $z_1 - z_2 = 0$ gives $z_2 = 4 + 5i$

Thus in subtraction the number itself is the inverse.

i.e. $(a + bi) - (a + bi) = 0 + 0i$ or 0

INTEXT QUESTIONS 2.2

1. Simplify

(a) $(\sqrt{2} + \sqrt{5i}) + (\sqrt{5} - \sqrt{2i})$ (b) $(i + 4) + (2 - 3i)$

(c) $\frac{2+i}{3} + \frac{2-i}{6}$ (d) $(3 + 4i) - (5i - 2)$

(e) $(1 + i) - (1 - 6i)$ (f) $(\sqrt{2} - \sqrt{3i}) + (\sqrt{2}) -(- 2 - 7i)$

2. If $z_1 = (5 + i)$ and $z_2 = (6 + 2i)$, find :

(a) $z_1 + z_2$, (b) $z_2 + z_1$, (c) Is $z_1 + z_2 = z_2 + z_1$?

(d) $z_1 - z_2$, (e) $z_2 - z_1$, (f) Is $z_1 - z_2 = z_2 - z_1$?

3. If $z_1 = (1 + i)$, $z_2 = (1 - i)$ and $z_3 = (2 + 3i)$, find :

(a) $z_1 + (z_2 + z_3)$, (b) $(z_1 + z_2) + z_3$,

(c) Is $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$? (d) $z_1 - (z_2 - z_3)$,

(e) $(z_1 - z_2) - z_3$, (f) Is $z_1 - (z_2 - z_3) = (z_1 - z_2) - z_3$?

4. Find the additive inverse of the following:

(a) $12 - 7i$ (b) $4 - 3i$

5. What should be added to $(-15 + 4i)$ to obtain $(3 - 2i,)$

6. Show that $\left\{\overline{(3+7i) - (5+2i)}\right\} = \overline{(3+7i)} - \overline{(5+2i)}$

2.7 MULTIPLICATION OF COMPLEX NUMBERS

Two complex numbers can be multiplied by following the usual laws of addition and multiplication as is done in the case of real numbers.

Let $z_1 = (a + bi)$ and $z_2 = (c+di)$ then,

$z_1.z_2 = (a + bi).(c + di)$

or $ac + adi + bci + bdi^2$

or $(ac - bd) + (ad + bc)i$

> If $(a + bi)$ and $(c + di)$ are two complex numbers, their product is defined as the complex number$(ac - bd)+(ad+bc)i$

Example E: Simplify

(i) $(1 + 2i)\ (1 - 3i)$ (ii) $\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right)$ (iii) $(3 - 2i)^2$

Solution

(i) $(1 + 2i)(1 - 3i) = \{1-(-6)\} + (-3 + 2)i$

$= 7-i$

(ii) $\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right) = \{3-(-1)\} + (-\sqrt{3}+\sqrt{3})i$

$= 4+ 0i$ or 4

(iii) $(3 - 2i)^2 = (3 - 2i)\ (3 - 2i)$

$= (9 - 4) + (- 6 - 6)i$

$= 5 - 12i$

2.8 DIVISION OF COMPLEX NUMBERS

Dividing a complex number $a + bi$ by a non–zero complex number $c + di$ means finding the complex number which is equal to :

$$\frac{a+bi}{c+di}(c+di \neq 0)$$

$$\frac{a+bi}{c+di} = a+bi\frac{1}{(c+di)}\cdot\frac{(c-di)}{(c-di)}$$

$$= \quad (a+bi)\left|\frac{(c-di)}{c^2+d^2}\right|$$

$$= \quad (a+bi)\left(\frac{c}{c^2+d^2}-\frac{di}{c^2+d^2}\right)$$

Thus $$\frac{a+bi}{c+di} = (a+bi)\left(\frac{c}{c^2+d^2}-\frac{d}{c^2+d^2}i\right)$$

$$= \quad \frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}$$

Example F: Divide 3+i by 4–2i

Solution $$\frac{3+i}{4-2i}$$

Multiplying $\frac{3+i}{4-2i}$ and dividing the numerator and denominator by 4+2i

$$= \quad (3+i) \left(\frac{3}{20}+\frac{2}{20}i\right)$$

$$= \quad (3+i) \left(\frac{1}{15}+\frac{1}{10}i\right)$$

$$= \quad \left(\frac{3}{5}+\frac{1}{10}\right) + \left(\frac{3}{10}+\frac{1}{5}\right)i$$

$$= \quad \left(\frac{5}{10}\right) + \left(\frac{5}{10}\right)i$$

Thus $$\frac{3+i}{4-2i}=\frac{1}{2}+\frac{1}{2}i$$

Check-point 4:

Find to $$\frac{1+2i}{2+3i}$$

Properties

Closure: From the definitions of product and division of complex numbers we can say closure property holds in these two operations.

Commutative:

Let $z_1 = (a + bi)$ and $z_2 = (c + di)$ then

$$z_1.z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

$$z_2.z_1 = (c + di)\ (a + bi) = (ca - db) + (cb + da)i$$

$$= (ac - bd) + (ad + bc)i$$

Thus $z_1.z_2 = (ac - bd) + (ad + bc)i = z_2.z_1$

Hence commutativity holds good in multiplication.

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = (a+bi)\left(\frac{c}{c^2+d^2} - \frac{d}{c^2+d^2}i\right)$$

$$= \left(\frac{ac}{c^2+d^2} + \frac{bd}{c^2+d^2}\right) + \left(\frac{bc}{c^2+d^2} - \frac{ad}{c^2+d^2}\right)i$$

$$\frac{z_2}{z_1} = \left(\frac{c+di}{a+bi}\right) = (c+di)\left(\frac{a}{c^2+b^2} - \frac{b}{c^2+b^2}i\right)$$

$$= \left(\frac{ac}{a^2+b^2} + \frac{bd}{a^2+b^2}\right) + \left(\frac{ad}{a^2+b^2} + \frac{bc}{a^2+b^2}i\right)$$

We get $\frac{z_1}{z_2} \neq \frac{z_2}{z_1}$

Commutativity does not hold good in division.

Let us check it with certain example:

Let $z_1 = (1 + i)$ and $z_2 = (1 - 2i)$

$$z_1 \,.\, z_2 = (1 + i)\ (1 - 2i)$$

$$= (1 + 2) + (-2 + 1)i$$

$$= 3 - i$$

$$z_2 . z_1 = (1 - 2i)(1 + i)$$

$$= (1 + 2) + (1 - 2)i$$

$$= 3 - i$$

$$\therefore \quad z_1 . z_2 = 3 - i = z_2 . z_1$$

Now $\dfrac{z_1}{z_2} = \dfrac{(1+i)}{(1-2i)}$

$$= (1+i)\left(\frac{1}{5}+\frac{2}{5}i\right)$$

$$= \left(\frac{1}{5}-\frac{2}{5}\right)+\left(\frac{1}{5}+\frac{2}{5}\right)i$$

$$= \frac{-1}{5}+\frac{3}{5}i$$

and $\dfrac{z_2}{z_1} = \dfrac{(1-2i)}{(1+i)}$

$$= (1 - 2i)\left(\frac{1}{2}\frac{1}{2}i\right)$$

$$= \left(\frac{1}{2}-\frac{2}{2}\right)+\left(\frac{-1}{2}\frac{-2}{2}\right)i = \frac{-1}{2}-\frac{3}{2}i$$

Gives $\dfrac{z_1}{z_2} \neq \dfrac{z_2}{z_1}$

Associativity:

Let $z_1 = (a + bi)$, $z_2 = c + di$ and $z_3 = (e + fi)$, then

$$z_1(z_2.z_3) = (a + bi)\{(c + di)(e + fi)\}$$

$$= (a + bi)\{(ce - df) + (cf + de)i\}$$

$$= a(ce - df) - b(cf + de) + \{a(cf + de) + b(ce - df)\}i$$

$$= (ace - adf - bcf - bde) + (acf + ade + bce - bdf)i$$

$(z_1.z_2)z_3 = \{(a + bi)(c + di)\}(e + fi)$

$= \{(ac - bd) + (bc + ad)i\}(e + fi)$

$= \{(ac - bd)e - (bc + ad)f\} + \{(ac - bd)f + (bc + ad)e\}i$

$= (ace - bde - bcf - adf) + (acf - bdf + bce + ade)i$

$\therefore \quad z_1(z_2.z_3) = (z_1.z_2)z_3$

The equality of the two products shows that **product of comlex numbers is associative.** Like commutativity it can be observed that **associativity does not hold good in division of complex numbers.**

Let us check it with certain examples :

If $z_1 = (1 + i)$, $z_2 = (2 + i)$ and $z_3 = (3 + i)$ then

$z_1(z_2.z_3) = (1 + i)\{(2 + i)(3 + i)\}$

$= (1 + i)\{(6 - 1) + (3 + 2)i\}$

$= (1 + i)(5 + 5i)$

$= (5 - 5) + (5 + 5)i$

$= 0 + 10i = 10i$

and $(z_1.z_2)z_3 = \{(1 + i)(2 + i)\}(3 + i)$

$= \{(2 - 1) + (1 + 2)i\}(3 + i)$

$= (1 + 3i)(3 + i)$

$= (3 - 3) + (1 + 9)i$

$= 0 + 10i = 10i$

$\therefore \quad z_1(z_2.z_3) = (z_1.z_2)z_3$

Identity

Let $z_1 = a + bi$ and $z_2 = x + yi$ be the identity

Then $z_1.z_2 = z_1$

i.e. $(a + bi)(x + yi) = a + bi$

or $(ax - by) + (ay + bx)i = a + bi$

or $\quad ax - by = a$ and $ay + bx = b$

or $\quad x = 1$ and $y = 0$

i.e. $\quad z_2 = x + yi =$ **1 + 0*i* is the multiplicative identiy.**

Let us check if $\frac{z_1}{z_2} = z_1$

i.e., $$\frac{(a+bi)}{1+0i} = (a+bi)\left(\frac{1}{1} - \frac{0}{1}\right)$$

$$= a + bi$$

Thus (1+0i) is the division identity.

The complex number $1 = 1 + 0i$ is the identity for multiplication and division.

e.g. if $z = 2 + 3i$ then

$$(2+3i)\left(\frac{1}{1} - \frac{0}{1}i\right)$$

$$z.\ (1 + 0i) = (2 + 3i)\ (1 + 0i)$$

$$= (2 - 0) + (3 + 0)i$$

$$= 2 + 3i$$

and $\quad \frac{z}{1+0i} =$

$$= 2 + 3i$$

Inverse:

The multiplicative inverse of a non–zero complex number $z = a + bi$ is dfined as the $x + yi$ such that

$$(a + bi)\ (x + yi) = 1 \text{ or } 1 + 0i$$

$$x + yi = \frac{1+0i}{a+bi}$$

$$= \frac{1+0i}{a+bi} \times \frac{a-bi}{a-bi}$$

or $x + yi = \frac{a - bi}{a^2 + b^2}$ or $\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$

i.e. $x = \frac{a}{a^2+b^2}$ and y =

Thus the multiplicative inverse of $(a + bi)$ is

$$\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right)$$

Let us check if $\frac{z_1}{z_2} = 1+0i$

or $\frac{(a+bi)}{(x+yi)} = 1+ 0i$

or $a + bi = (x + yi)\ (1+0i)$

or $a + bi = x + yi$

or $a = x$ and $b = y \Rightarrow \quad z_2 = x + yi = a + bi$

Thus the division inverse of $z_1 = a + bi$ is $z_2 = a + bi$

e.g. the multiplicative inverse of $z_1 = 2 + 4i$ is

$\left(\frac{2}{20} - \frac{4}{20}i\right)$ or $\left(\frac{1}{10} - \frac{1}{5}i\right)$ and

the division inverse of $z_1 = 2 + 4i$ is the number itself.

Distributive property of multiplication over addition.

Let $z_1 = a_1 + b_1 i, \quad z_2 = a_2 + b_2 i, \quad z_3 = a_3 + b_3 i$

Then, $z_1\ (z_2 + z_3) = z_1 z_2 + z_1 z_3$

INTEXT QUESTIONS 2.3

1. Simplify:

(a) $(1+2i)\ (\sqrt{2} - i)$ (b) $(\sqrt{2} + i)^2$

(c) $(3+0i) \quad (0-5i)$ (d) $(-4+5i)(5-3i)(-2+i)$

(e) $(3+i)(1-i)(-1+i)$ (f) $(2+3i) \div (1-2i)$

(g) $(1+2i) \div (1+i)$ (h) $(1+0i) \div (3+7i)$

2. Let $z_1 = (2+i)$, $z_2 = (-2+i)$ and $z_3 = (2-i)$ then find

(a) $(z_1.z_2) \quad z_3$ (b) $z_1 \quad (z_2 \quad z_3)$

(c) Is $(z_1.z_2) \quad z_3 = z_1 \quad (z_2 \, . \, z_3)$?

3. Compute the multiplicative inverse for the following complex numbers

(a) $\sqrt{3} - 4i$ (b) $3 + 7i$

2.9 FURTHER PROPERTIES OF CONJUGATES OF COMPLEX NUMBERS

$\overline{\left(\frac{z_1}{z_2}\right)}, \frac{\overline{z_1}}{\overline{z_2}}$

Example G: If $z_1 = a+bi$ and $z_2 = c+di$ then

(i) find $\overline{z_1 + z_2}$, $\overline{z_1} + \overline{z_2}$ and state your observation

(ii) find $\overline{z_1 - z_2}$, $\overline{z_1} - \overline{z_2}$ and state your observation

(iii) find $\overline{z_1 z_2}$, $\overline{z_1} . \overline{z_2}$ and state your observation

(iv) find and state your observation

Solution

(i) $\overline{z_1 + z_2} = \overline{(a+bi)+(c+di)}$

$= \overline{(a+c)+(b+d)i}$

$= (a+c) - (b+d)i$

$\overline{z_1} + \overline{z_1} = \overline{(a+bi)} + \overline{(c+di)}$

$= (a-bi) + (c-di)$

$= (a+c) - (b+d)i$

we get $\overline{z_1 + z_2} = (a+c) - (b+d)\ \mathrm{i} = \overline{z_1} + \overline{z_1}$

∴ Conjugate of the sum of two complex numbers is equal to the sum of their conjugates.

(ii) $\overline{z_1 - z_2} = \overline{(a+bi)-(c-di)}$

$= \overline{(a-c)+(b-d)i}$

$= (a-c) - (b-d)i$

$\overline{z_1} - \overline{z_2} = \overline{(a+bi)} - \overline{(c+di)}$

$= (a-bi) - (c-di)$

$= (a-c) - (b-d)i$

we get $\overline{z_1 - z_2} = (a-c) - (b-d)i = \overline{z_1} - \overline{z_2}$

∴ Conjugate of the difference of two complex numbers is equal to the difference of their conjugates.

(iii) $\overline{z_1 \cdot z_2} = \overline{(a+bi)(c+di)}$

$= \overline{(ac-bd)+(ad+bc)i}$

$= (ac-bd) - (ad+bc)i$

$\overline{z_1} \cdot \overline{z_2} = \overline{(a+bi)}\overline{(c+di)}$

$= (a-bi)\ (c-di)$

$= (ac-bd) - (ad+bc)i$

we get $\overline{z_1 \cdot z_2} = (ac-bd) - (ad+bc)i = \overline{z_1} \cdot \overline{z_2}$

∴ Conjugate of the product of two complex numbers is equal to the product of their conjugate.

(iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{a+bi}{c+di}\right)}$

$$= \overline{\left[\frac{(a+bi)(c-di)}{(c+di)(c-di)}\right]}$$

$$= \overline{\left[\left(\frac{ac}{c^2+d^2}+\frac{bd}{c^2+d^2}\right)+\left(\frac{bc}{c^2+d^2}-\frac{ad}{c^2+d^2}\right)i\right]}$$

$$= \left(\frac{ac}{c^2+d^2}+\frac{bd}{c^2+d^2}\right)-\left(\frac{bc}{c^2+d^2}-\frac{ad}{c^2+d^2}\right)$$

i.e., $$\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{ac}{c^2+d^2}+\frac{bd}{c^2+d^2}\right)+\left(\frac{ad}{c^2+d^2}-\frac{bc}{c^2+d^2}\right)i$$

$$\frac{\overline{z_1}}{\overline{z_2}} = \frac{\overline{(a+bi)}}{\overline{((c+di)}}$$

$$= \frac{a-bi}{c-di}$$

$$= (a-bi)\left(\frac{c}{c_2+d_2}+\frac{di}{c_2+d_2}\right)$$

$$= \left(\frac{ac}{c^2+d^2}+\frac{bd}{c^2+d^2}\right)+\left(\frac{ad}{c^2+d^2}-\frac{bc}{c^2+d^2}\right)i$$

Thus we get $$\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{ac}{c^2+d^2}+\frac{bd}{c^2+d^2}\right)+\left(\frac{ad}{c^2+d^2}-\frac{bc}{c^2+d^2}\right)i$$

$$= \frac{\overline{z_1}}{\overline{z_2}}$$

$\therefore$ Conjugate of the quotient of two complex numbers is equal to the quotient of their conjugates.

Check-point 5:

Verify the above four results for $z_1 = 1+2i$ and $z_2 = 2+3i$.

2.10 FURTHER PROPERTIES ON MODULUS OF COMPLEX NUMBERS.

(i) $|\mathbf{z}| = \mathbf{0} \iff \mathbf{z} = \mathbf{0}$

Let $z = a + bi$

$\therefore \quad |z| = 0$

$\iff \sqrt{a^2+b^2} = 0$

$\iff a = 0$ and $b = 0$

$\iff z = a + bi = 0$

$\mathbf{z_1 = z_2 \Rightarrow |z_1| = |z_2|}$

Let $z_1 = a + bi$, $z_2 = c + di$

$z_1 = z_2$ gives $a + bi = c + di$

or, $a = c$ and $b = d$

$\iff \sqrt{a^2+b^2} = \sqrt{c^2+d^2}$

$\iff |z_1| = |z_2|$

Note: You may note that if $|z_1| = |z_2|$, it does not always imply that $z_1 = z_2$

e.g. if $z_1 = 1 + 2i$ and $z_2 = -1 - 2i$

then, $|z_1| = \sqrt{1^2+2^2}$ and $|z_2| = \sqrt{(-1)^2+(-2)^2}$

$= \sqrt{5}$

But $z_1 \neq z_2$

(ii) $|\mathbf{z_1+z_2}| \leq |\mathbf{z_1}| + |\mathbf{z_2}|$

Let $z_1 = 1$ or $1 + 0i$ and $z_2 = 0$ or $0 + 0$

then, $|z_1| = 1, |z_2| = 0$ and $|z_1| + |z_2| = 1 + 0 = 1$

$$|z_1 + z_2| = |1 + 0i| = \sqrt{1+0} = 1$$

$$\therefore \quad |z_1 + z_2| = |z_1| + |z_2| \quad \ldots\ldots\ldots \quad \text{(i)}$$

Let $z_1 = 4 + 3i$ and z_2 $3 + 4i$,

Then $z_1 + z_2 = 7 + 7i$

$$|z_1| = \sqrt{16+9} = \sqrt{25} = 5 \text{ and } |z_2| = \sqrt{9+16} = 5$$

$$|z_1 + z_2| = \sqrt{49+49} = \sqrt{98}$$

$$\therefore \quad |z_1 + z_2| < |z_1| + |z_2| \quad \ldots\ldots\ldots \quad \text{(ii)}$$

(i) and (ii) gives the required result.

WHAT YOU HAVE LEARNT

- $(a + bi) = (c + di)$ if and only if $a = c$ and $b = d$
- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $(a + bi)\ (c + di) = (ac - bd) + (ad + bc)i$
- $(a+bi) \div (c+di) = \left(\dfrac{ac}{c^2+d^2} + \dfrac{bd}{c^2+d^2}\right) + \left(\dfrac{bc}{c^2+d^2} - \dfrac{ad}{c^2+d^2}\right)i$

Operation	+	—	×	÷
Closure	√	√	√	√
Commutative	√	×	√	×
Associative	√	×	√	×
Identity	√	√	√	√
Inverse	√	√	√	√

- $-\left(\overline{z}\right) = \left(\overline{-z}\right)$

- $\overline{(z_1+z_2)} = \overline{z_1} + \overline{z_2}$
- $\overline{(z_1-z_2)} = \overline{z_1} - \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
- $z + \overline{z} = 2 \operatorname{Re}(z)$
- $z_1 = z_2 \Rightarrow \quad = |z_2|$
- $|z_1+z_2| \le |z_1| + |z_2|$

TERMINAL QUESTIONS

1. Find the values of x and y if

   $11(x + yi) - (x - yi) = 11$

2. Express in the form a+bi

   (a) $(7 + 2i) + (8 - 4i)$

   (b) $(3 + i) + (4 - i) + (8 - 2i)$

   (c) $\left(\sqrt{2}+\sqrt{6}i\right)+\left(\sqrt{6}+\sqrt{2}i\right)$

3. Verify if the following statements are true or false

   (a) $(7 + 2i) + (9 + 3i) = (9 + 3i) + (7 + 2i)$

   (b) $(-4 + 6i)(-4 - 6i) = (-4 -6i)(-4 + 6i)$

4. What should be added to $(1 + i)$ to obtain $(1 + 6i)$

5. Show that $\left\{\overline{(a+bi)-(c+di)}\right\} = \left(\overline{a+bi}\right)-\left(\overline{c+di}\right)$

6. Solve the following

   (a) $(4 + i)(1 - i)(-1 + i)$

(b) $\left(\frac{1}{7}+i\right)\left(\quad - i\right)\left(\quad - 2i\right)$

7. Write the additive and multiplicative inverse of

(a) $7-9i$ (b) $11-2i$

(c) $+4i$ (d) $6-11i$

ANSWERS TO CHECK POINTS

CHECK-POINT 1: a=b

CHECK-POINT 2: (1)Purely real number (2) 3+i

CHECK-POINT 4: $\left(\frac{8}{13}+\frac{1}{13}i\right)$

ANSWERS TO INTERXT QUESTIONS

2.1

1. (a) $(2, -3)$, (b) $\left(\sqrt{2}+\sqrt{5}\right)$ $(\sqrt{2}+2),0)$ (c) $(0, 6)$

(d) $(2, -1)$ (e) $\left(-3, \frac{10}{7}\right)$ (f) $(11, 4)$

2.2

1. (a) $)+\left(\sqrt{5}+\sqrt{2}\right)i$ (b) $6-2i$ (c) $1+\frac{1}{6}i$

(d) $5-i$ (e) $7i$ (f) $2(\sqrt{2}+i)+(7-\sqrt{3})i$

2. (a) $11+3i$ (b) $11+3i$ (c) Yes

(d) $-1-i$ (e) $1+i$ (f) No

3. (a) $4+3i$ (b) $4+3i$ (c) Yes

(d) $2+5i$ (e) $-2-i$ (f) No

4. (a) $-12+7i$ (b) $-4+3i$

5. $18-6i$

2.3

1. (a) $(\sqrt{2}+2)+(2\sqrt{2}-1)i$ (b) $1 + 2\sqrt{2}\,i$ (c) $-15i$

   (d) $-27 - 79i$ (e) $-2 + 6i$ (f) $\frac{-4}{5}+\frac{7}{5}i$

   (g) $\frac{3}{2}+\frac{1}{2}i$ (h) $\frac{3}{58}-\frac{7}{58}i$

2. (a) $-10 + 5i$ (b) $-10 + 5i$ (c) Yes

3. (a) $\frac{\sqrt{3}}{19}+\frac{4}{19}i$ (b) $\frac{3}{58}-\frac{7}{58}i$

ANSWERS TO TERMINAL QUESTIONS

1. $x = \frac{11}{10}$, $y = 0$

2. (a) $15 - 2i$ (b) $15 - 2i$ (c) $+\left(\sqrt{2}+\sqrt{6}\right)i$

3. (a) True (b) True

4. $5i$

6. (a) $-2 + 8i$ (b) $\frac{251}{343}-\frac{99}{49}i$

7. (a) $-7 + 9i$, $\frac{7}{130}+\frac{9}{130}i$,

   (b) $-11+2i$, $\frac{11}{125}+\frac{2}{125}i$,

   (c) $-\sqrt{3}- 4i$, $\frac{\sqrt{3}}{19}-\frac{4}{19}i$,

   (d) $-6+11i$, $\frac{6}{157}-\frac{11}{157}i$