3

MODULUS AND ARGAND DIAGRAM

3.1 INTRODUCTION

Consider the representation of following real numbers on real line.

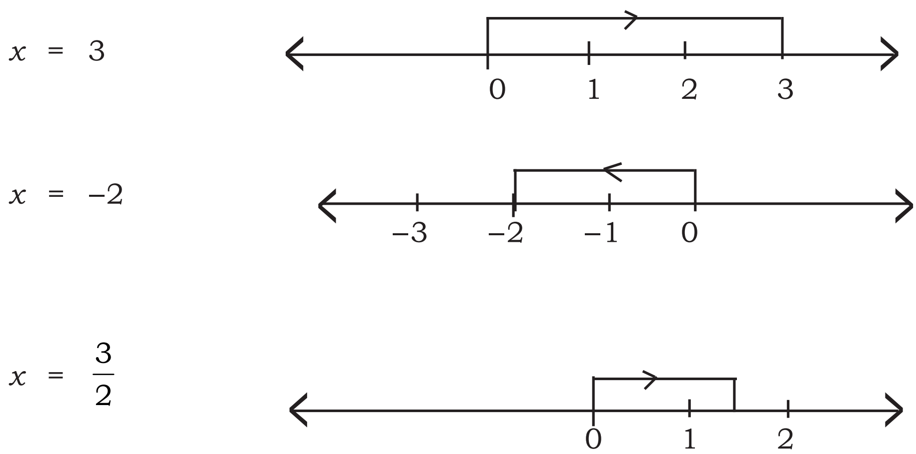


Fig. 3.1

Now let us consider the complex number $3 + 2i$. Here 3 and 2 are real numbers. Can we represent this complex number on real line? Surely no.

This number is a composition of real and imaginary parts. So, one number line is not sufficient here. How can we represent a complex number geometrically by a point ?

We will learn geometric representation of complex numbers and related properties in this lesson.

3.2 OBJECTIVES

After studying this lesson, you will be able to:

- represent complex numbers on argand plane.
- identify the complex numbers $x + iy$ corresponding to a given point $P\ (x\ ,y)$ in the agrand plane.
- recognise that there is a unique complex number associated with every point in the argand plane.
- State and represent diagramatically the following properties of a complex number:

(i) $= 0 \Leftrightarrow \quad z = 0$

(ii) $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$

(iii) $|z_1 + z_2| \le |z_1| = |z_2|$

3.3 PREVIOUS KNWOLEDGE

(a) Representation of real numbers on number a line

(b) Complex numbers

(c) Algebra of complex numbers.

3.4 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

You have already learnt that

1. Complex number $a + ib$ can be determined by an ordered pair (a,b)
2. An ordered pair (a,b) is represented by a point on coordinate plane.

So, ***complex number a+ib can be represented by a point (a,b) on coordinate plane.***

This plane is called ***Argand Plane.***

The diagram is called ***Argand Diagram***

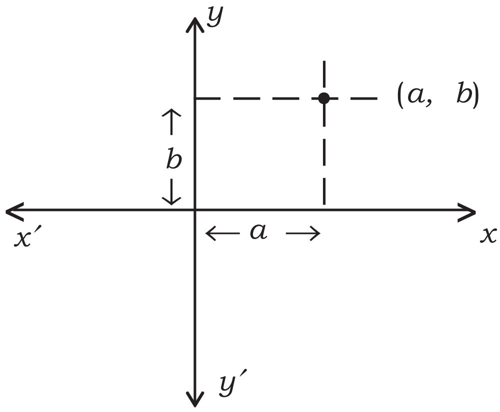


Fig. 3.2

Example A

Complex number a + 0i is denoted by P (a, 0)

1. Point P(a,0) lies on the x–axis
2. a is the real part of complex number a + ib
3. x axis is called real axis as the real part is represented on the x–axis.

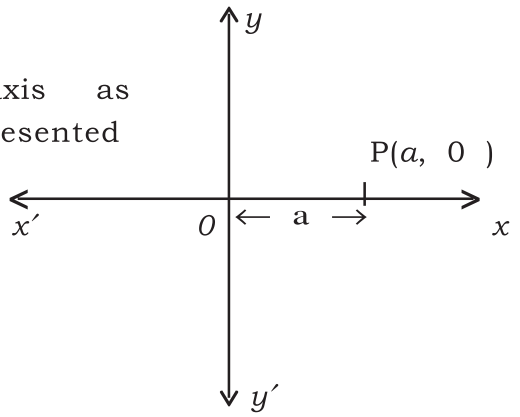


Fig. 3.3

Example B

Complex number 0 +bi is represented by Q(0,b)

1. Point Q(0, b) lies on y–axis
2. b is the imaginary part of complex number 0 + bi
3. y–axis is called the imaginary axis as the imaginary part is represented on y–axis.

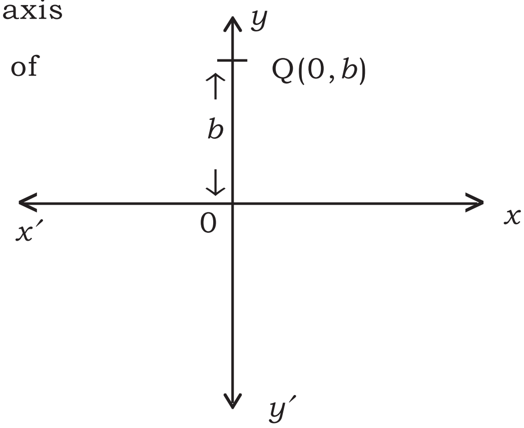


Fig. 3.4

Example C

Represent $2 + 3i$ and $3 + 2i$ on the same argand plane.

Solution:

1. $2 + 3i$ is represented by $A(2, 3)$
2. $3 + 2i$ is represented by $B(3, 2)$
   Point A and B are different

Representation of $a + bi$ is not same as of $b + ai$, if $a \neq b$

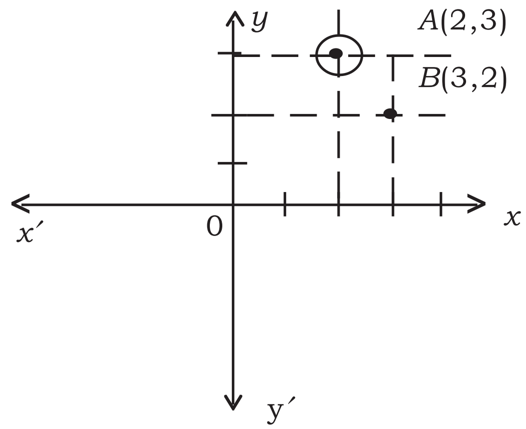


Fig. 3.5

Example D

Represent $2 + 3i$ and $-2 - 3i$ on the same argand plane.

Solution

1. $2 + 3i$ is point $P(2,3)$
2. $-2 - 3i$ is point $Q(-2,-3)$

Points P and Q are different and lie in the *I* quadrant and *III* quadrant respectively.

Representation of $a + bi$ is not same as of $-a - bi$

$\mathbf{z \neq -z}$

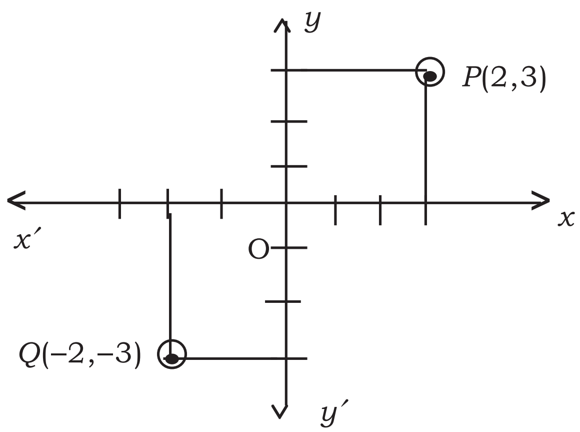


Fig. 3.6

Example E:

Represent $2 + 3i$ and $2 - 3i$ on the same argand plane

Solution

1. $2 + 3i$ is point $R(2,3)$
2. $2 - 3i$ is point $S(2,-3)$
3. Point R and S are different

Representation of $a + bi$ is not same as of $a - bi$

$\mathbf{z \neq \bar{z}}$

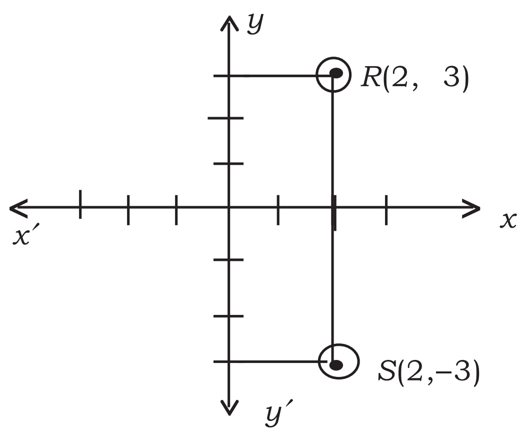


Fig. 3.7

Example F:

Represent $2 + 3i$, $-2 -3i$, $2 - 3i$ on the same argand plane

Solution

1. $2 + 3i$ is point $P(2,3)$
2. $-2 - 3i$ is point $Q(-2,-3)$
3. $2 - 3i$ is point $R(2,-3)$

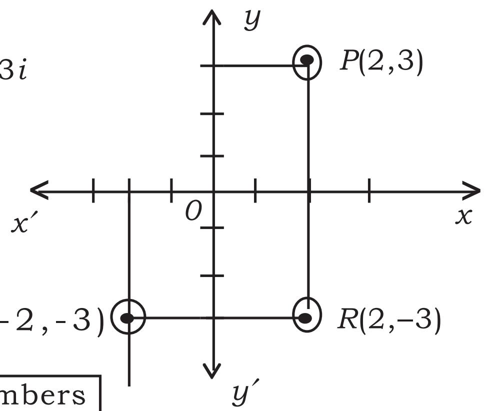


Fig. 3.8

Representation of Complex numbers
$z = a + bi$
$z = a - bi$
and $-z = -a - bi$
are all different
$\mathbf{z} \neq \quad \neq -\mathbf{z}$

Checkpoint 1:

Tick mark the correct answer:

1. The point representing the complex number $3 + 5i$ on argand plane is
   - (i) same as the point representing $3 - 5i$
   - (ii) same as the point representing $-3 - 5i$
   - (iii) same as the point representing 5 + 3i
   - (iv) none of the above
2. Complex number $a - bi$ is represented on an argand plane by the point.
   - (i) (a, b)
   - (ii) $(a, -b)$
   - (iii) $(-a, -b)$
   - (iv) $(-a, b)$

   { **Ans : 1–(iv), 2–(ii)** }

3.5 MODULUS OF A COMPLEX NUMBER

Any complex number $a + ib$ can be represented by a point in a plane.

How can we find the distance of this point from the origin?

Consider $P(a, b)$ a point in the plane representing $a + ib$. If we look at Fig.3.9, we find that

OM = a

MP = b

What is the distance of P from the origin?

Certainly, it is OP.

How do you find OP?

We may note that PM and OM are perpendicular to each other.

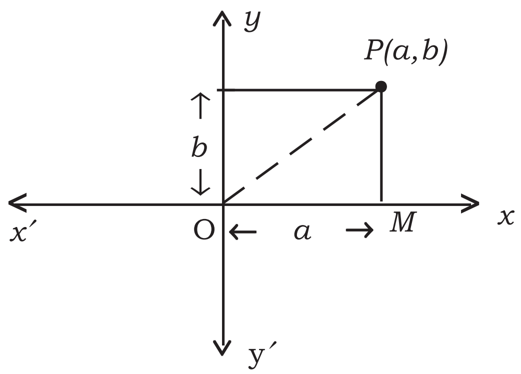


Fig. 3.9

$\therefore$ OP =

$$= \sqrt{a^2 + b^2}$$

OP is called the **modulus of complex number** or **absolute value of the complex number,** $a + ib$.

$\therefore$ Modulus of any complex number z such that

$$z = a + bi, \; a \in R, \; b \in R$$

is denoted by $|z|$ and is given $\sqrt{a^2 + b^2}$

$$\therefore \quad |z| = a + ib = \sqrt{a^2 + b^2}$$

Example G:

Find the modulus of the complex numbers shown in an argand plane (Fig. 3.10)

Solution

(i) P(4, 3) represents the complex number

$$z = 4 + 3i$$

$$\therefore \; OP = |z| = \sqrt{4^2 + 3^2}$$

$$= \sqrt{25}$$

or $|z| = 5$

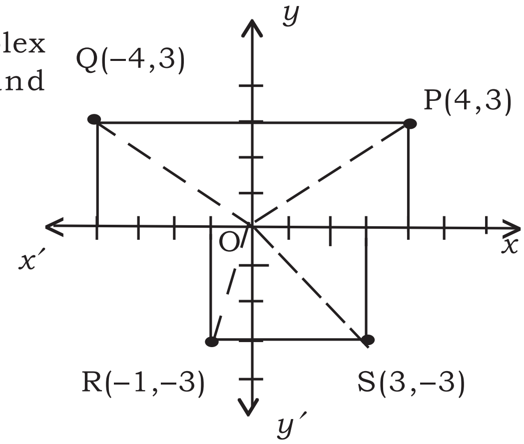


Fig. 3.10

(ii) Q(-4,3) represents z = -4 + 2i

$$OQ = |z| = \sqrt{(-4)^2 + 2^2}$$

$$= \sqrt{16+4}$$

$$= \sqrt{20}$$

(iii) R(−1, −3) represents z = −1 − 3i

$$OR = |z| = \sqrt{(-1)^2 + (-3)^2}$$

$$= \sqrt{1+9}$$

$$= \sqrt{10}$$

(iv) S(3, −3) represents z = 3 − 3i

$$OS = |z| = \sqrt{(3)^2 + (-3)^2}$$

$$= \sqrt{9+9}$$

$$= \sqrt{18}$$

$\sqrt{3^2+4^2}$

Example H:

Find the modulus of z and $\bar{z}$

if z = 3 + 4i

Solution

z = 3 + 4i

then = 3 − 4i

∴ OP = z =

$$= \sqrt{9+16}$$

$$= \sqrt{25} = 5$$

and $OQ = |\bar{z}| = \sqrt{3^2 + (-4)^2}$

$$= \sqrt{9+16}$$

$$= \sqrt{25} = 5$$

$$\boxed{|z| = |\bar{z}|}$$

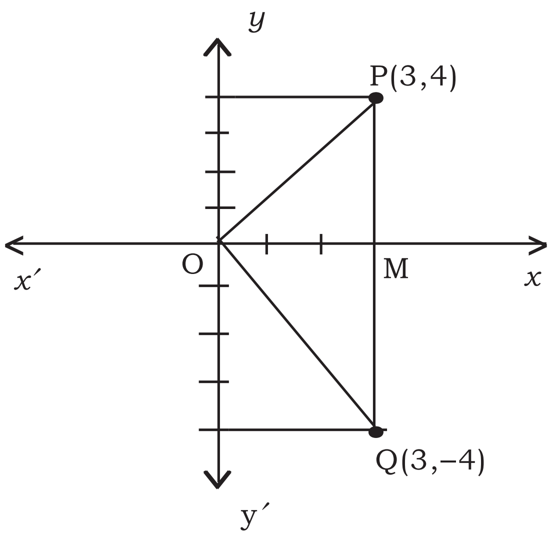


Fig. 3.11

Example I:

Find the modulus of z and $-z$ if $z = 5 + 2i$

Solution: $z = 5 + 2i$

then $-z = -(5 + 2i) = -5 - 2i$

$\therefore$ OP $= |z| = \sqrt{5^2 + 2^2}$

$= \sqrt{25 + 4}$

$= \sqrt{29}$

OQ $= |-z| = \sqrt{(5)^2 + (-2)^2}$

$= \sqrt{25 + 4}$

$= \sqrt{29}$

$$|z| = |-z|$$

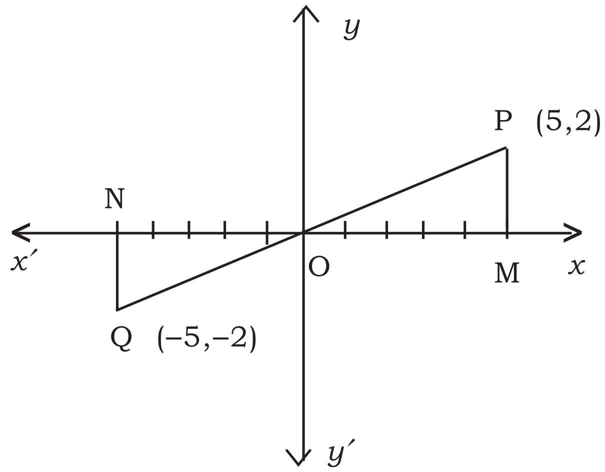


Fig. 3.12

Example J:

Find the modulus of z, $-z$ and $\bar{z}$ if $z = 1 + 2i$

Solution $z = 1 + 2i$

$-z = -1 - 2i$

$\bar{z} = 1 - 2i$

OP $= |z| = \sqrt{1^2 + 2^2} = \sqrt{5}$

OQ $= |-z| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$

OR $= |\bar{z}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

$$|z| = |-z| = |\bar{z}|$$

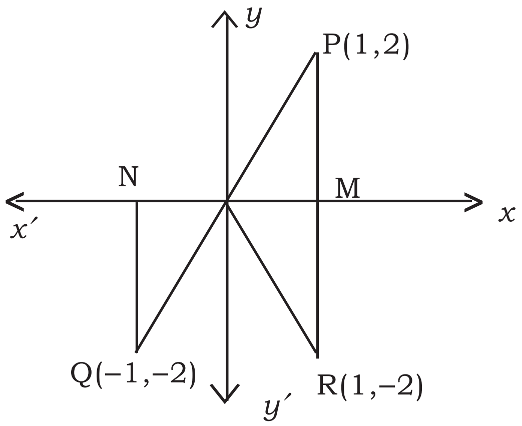


Fig. 3.13

Check-point 2:

Choose the appropriate answer:

1. Modulus of the complex number $a + ib$ is

   (i) $\sqrt{a^2 + b^2}$

   (ii) $\sqrt{a^2 - b^2}$

   (iii) $\sqrt{-a^2 + b^2}$

   (iv) $\sqrt{-a^2 - b^2}$

2. Modulus of the complex number $a + ib$ is

   (i) equal to the modulus of $-a - ib$

   (ii) not equal to modulus of –a –ib

Q.3 $|z| = |-z| = |\bar{z}|$ is

   (i) always true

   (ii) never true

   (iii) sometimes true

INTEXT QUESTIONS 3.1

Q.1 Represent the following complex numbers on Argand Plane

(a) (i) $2 + 0i$
(ii) $-3 + 0i$
(iii) $0 - 0i$
(iv) $3 - 0i$

(b) (i) $0 + 2i$
(ii) $0 - 3i$
(iii) $4i$
(iv) $-5i$

(c) (i) $2 + 5i$ and $5 + 2i$

(ii) $3 - 4i$ and $-4 + 3i$

(iii) $-7 + 2i$ and $2 - 7i$

(iv) $-2 - 9i$ and $-9 -2i$

(d) (i) $1 + i$ and $-1 - i$

(ii) $6 + 5i$ and $-6 - 5i$

(iii) $-3 + 4i$ and $3 - 4i$

(iv) $4 - i$ and $-4 + i$

(e) (i) $1 + i$ and $1 - i$

(ii) $-3 + 4i$ and $-3 - 4i$

(iii) $6 - 7i$ and $6 + 7i$

(iv) $-5 - i$ and $-5 + i$

2. (a) Find the modulus of the following complex numbers

(i) $1 + i$

(ii) $-3 -5i$

(iii) $2 - 3i$

(iv) $5 - 8i$

(v) $-6 + 6i$

(b) For the following complex numbers, verify that $|z| = |-z|$

(i) $5 + 9i$

(ii) $-6 + 8i$

(iii) $-3 -7i$

(iv) $i + 9$

(c) For the following complex numbers, verify that $|z| = |-z|$

(i) $-3 - 9i$

(ii) $14 + i$

(iii) $11 - 2i$

(iv) $-7 + 9i$

(d) For the following complex numbers, verify that $|z| = |-z| = |\bar{z}|$

(i) $2 - 3i$

(ii) $5 + 4i$

(iii) $-6 - i$

(iv) $7 - 2i$

3. Write the complex numbers corresponding to the points shown in the argand plane

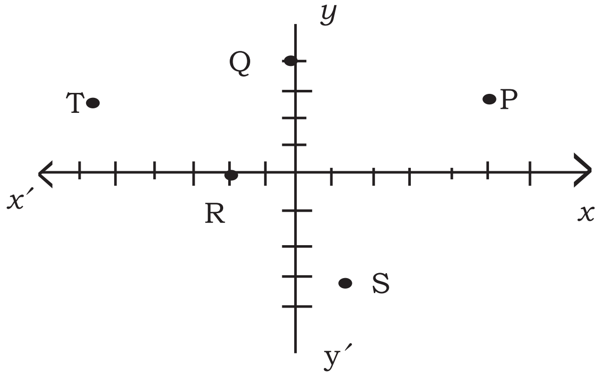


Fig. 3.14

4. Find the modulus of the following complex numbers

(a) $i + \sqrt{3}\, i$ (b) $-2i$ (c) -3 (d) $5 - 2i$

(e) $i^2 + i^3$ (f) $\dfrac{1+i}{1-i}$ (g) $(1 + i)(2 - i)$

3.6 DIAGRAMATIC REPRESENTATION OF THE PROPERTIES OF COMPLEX NUMBERS

We have learnt the representation of complex number in the argand plane. Now we will state and represent the properties of complex numbers diagramatically in the complex plane.

(i) $= 0 \Leftrightarrow z = 0$

consider $z = 0 + 0i$

Let P(0, 0) be the point on argand plane to represent z.

From Fig. 3.15 we may observe that the origin O and point P coincide.

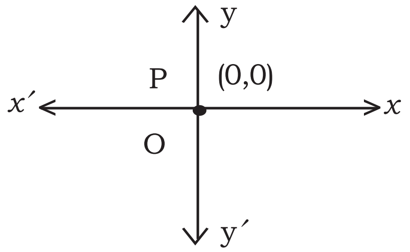


Fig. 3.15

$\therefore \quad |OP| = 0 \qquad |OP| = |z|$

(By definition of modulus of complex number)

$\Rightarrow \quad |z| = 0$

(ii) (a) $z_1 = z_2 \Rightarrow |z_1| = |z_2|$

Let P(a,b) and Q(a,b) be two points representing z_1 and z_2 on complex plane such that they coincide with each other.

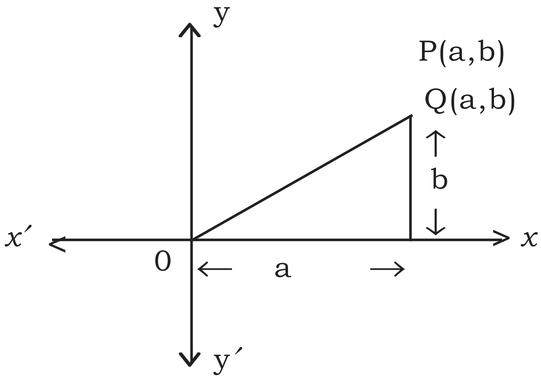


Fig. 3.16

then OP $= \sqrt{a^2+b^2}$

and OQ $= \sqrt{a^2+b^2}$

since OP = OQ

$\Rightarrow \quad |z_1| = |z_2|$

(b) But $|z_1| = |z_2|$ does not always imply $z_1 = z_2$

Let $z_1 = a + ib, \quad a \in R, \ b \in R$

$z_2 = a - ib, \quad a \in R, \ b \in R$

Let P(a, b) and Q(a, $-b$) represent z_1 and z_2 respectively on the argand plane.

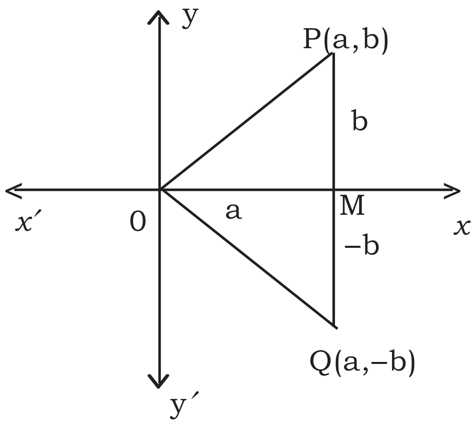


Fig. 3.17

Then, $|z_1| = \text{OP} = \sqrt{a^2+b^2}$

$|z_2| = \text{OQ} = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$

$\therefore \quad |z_1| = |z_2|$, but P and Q are two different points on the complex plane.

(iii) $|z_1 + z_2| \leq |z_1| + |z_2|$

Let $z_1 = a + ib,\ a \in R,\ b \in R$

and $z_2 = c + id\ c \in R,\ d \in R$

Then $z_1 + z_2 = (a + c) + i(b + d)$

Let points P, Q, R represent the numbers z_1, z_2 and $z_1 + z_2$ respectively in the argand plane.

Join OP, OQ and OR

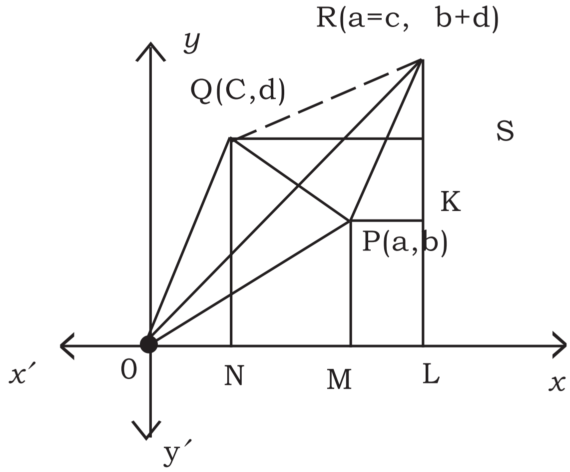


Fig. 3.18

Then OP = $|z_1|$

OQ = $|z_2|$

and OR = $|z_1 + z_2|$

Draw PM ⊥ x (ii) axis

QN ⊥ x (ii) axis

RL ⊥ x (iii) axis

Join QR and QP

Draw QS ⊥ RL and PK ⊥ RL

In Δ QON,	In Δ ROL	In Δ POM
ON = c	OL = a+c	OM = a
and QN = d	RL = b+d	PM = b

Also PK = ML

= OL − OM

= a + c − a = c

RK = RL − KL

= b + d − b = d

In Δ QON and Δ RPK

ON = PK = oc

QN = RK = d

and /QNO = /RKP = 90°

$\therefore \ \Delta$ QON $\cong \Delta$ RPK

$\Rightarrow$ OQ = PR and OQ||PR

$\Rightarrow$ OPRQ is a parallelogram

and OR is diagonal of the parallelogram.

Therefore we can say that the sum of two complex numbers is represented by the diagonal of a parallelogram.

We also know that

In Δ OPR

OR $\leq$ OP + PR

or OR $\leq$ OP + OQ ($\because$ OQ = PR)

$\Rightarrow \quad \leq |z_1| + |z_2|$

Check-point 3:

Choose the appropriate answer.

1. For a non–zero complex number $a + ib$
   (i) the modulus is always zero.
   (ii) the modulus is always non–zero.

2. For complex numbers z_1 and z_2 such that $z_1 = z_2$
   (i) $|z_1| = |z_2|$
   (ii) $|z_1| \neq |z_2|$
   (iii) $|z_1| \leq |z_2|$
   (iv) $|z_1| \geq |z_2|$

Q.3 For complex numbers z_1 , z_2 and $z_1 + z_2$
   (i) $|z_1 + z_2| = |z_1| + |z_2|$
   (ii) $|z_1 + z_2| \geq |z_1| + |z_2|$
   (iii) $|z_1 + z_2| \leq |z_1| + |z_2|$
   (iv) $|z_1 + z_2| \neq |z_1| + |z_2|$

Example K:

Draw diagram to represent $z_1 + z_2$

If $z_1 = 2 + 3i$ and $z_2 = 1 + i$

Also verify that $|z_1 + z_2| \le |z_1| + |z_2|$

Solution: Let $z_1 = 2 + 3i$

$z_2 = 1 + i$

Then $z_1 + z_2 = 3 + 4i$

Let A (2, 3) be z_1

B(1, 1) be z_2

Then C(3, 4) be $z_1 + z_2$

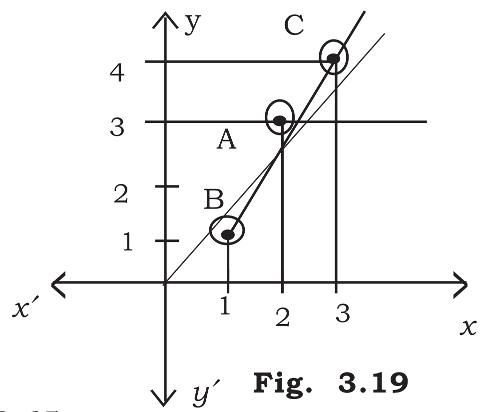


Fig. 3.19

Verification

$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$ = 3.65 aprox.

$|z_2| = \sqrt{1^2 + i^2} = \sqrt{1+1} = \sqrt{2}$ = 1.41 aprox.

$|z_1 + z_2| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Now, $|z_1| + |z_2| = 3.60 + 1.41 = 5.01$

$\therefore \quad |z_1 + z_2| < |z_1| + |z_2|$

Example L:

Represent diagramatically $|z_1 - z_2| \ge |z_2| - |z_1|$ on complex plane

Solution

For the above inequality, consider the Fig. 3.20

Let P and Q represent z_1 and z_2 respectively.

Then Q′ represent $-z_2$

R represent $z_1 + (-z_2)$

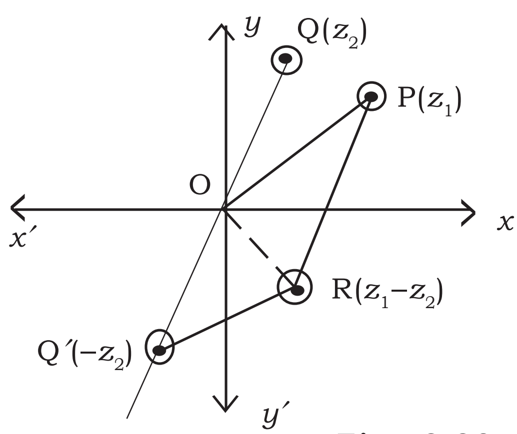


Fig. 3.20

Complete the parallelogram. OPRQ′ we see that OR is diagonal of this parallelogram and OR = $|z_1 - z_2|$

Also OP || Q′R and OP = Q′R

OQ′ || PR and OQ′ = PR

∴ In Δ OPR

PR ≤ OP + OR (Why?)

$\Rightarrow \quad |z_2| \leq |z_1| + |z_1 - z_2|$

or $\quad |z_2| - |z_1| \leq |z_1 - z_2|$

or $\quad |z_1 - z_2| \geq |z_2| - |z_1|$

INTEXT QUESTIONS 3.2

1. Draw a diagram to represent the addition $z_1 + z_2$ of following complex numbers:

   (a) $z_1 = 1 + i \qquad z_2 = 2 + 5i$

   (b) $z_1 = -2 + 3i \qquad z_2 = -1 - 4i$

   (c) $z_1 = 4 - i \qquad z_2 = 5 + 2i$

   Also veryfy that $|z_1 + z_2| \leq |z_1| + |z_2|$

2. Represent diagramatically

   $|z_1 - z_2| \leq |z_1| + |z_2|$

3. Draw a diagram to represent $z_1 - z_2$ for the following complex numbers:

   (a) $z_1 = 3 - 2i \qquad z_2 = 1 + i$

   (b) $z_1 = 4 + 3i \qquad z_2 = -4 + 3i$

   (c) $z_1 = -2 - 5i \qquad z_2 = -3 + 7i$

   (d) $z_1 = 2 + i \qquad z_2 = 3 + i$

   In each of the above verify that

   $|z_1 - z_2| \geq |z_1| - |z_2|$

3.7 POLAR FORM OF COMPLEX NUMBER

A point (a, b) in the plane, is completely determined by

(i) its distance from the origin

(ii) the angle θ, which it makes with the positive x–axis.

Let P(a, b) represent the complex number $z = a + ib$, $a \in R$, $b \in R$, and OP makes angle θ with positive x-axis.

Let OP = r

It right Δ OMP

OM=a

MP=b

∴ r cos θ = a

$r \sin \theta = b$

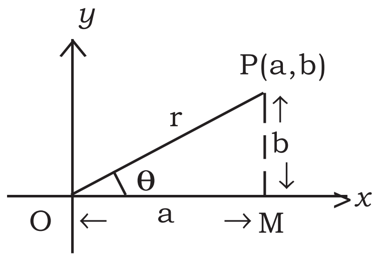


Fig. 3.21

Then $z = a + ib$ can be written as

$z = r$ (Cos θ + i sin θ) is polar form

Where $r = \sqrt{a^2 + b^2}$ is modulus

and $\tan\theta = \dfrac{b}{a}$

or $\theta = \tan^{-1}$ is argument

Here θ is the principle argument.

Example M:

Express 1 + i in polar form

Solution

Here $a = 1$, $b = 1$

∴ $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1} \dfrac{1}{1} = \tan^{-1} (1) =$

∴ In the form

1 + i can be written as $\left(\text{Cos } \dfrac{\pi}{4} + i \sin \right)$

Example N:

Express – i in polar form.

Solution

Here r $= \sqrt{(\sqrt{3})^2 + (-1)^2}$

$= \sqrt{3+1}$

$= \sqrt{4} = 2$

tan $\theta = \frac{-1}{\sqrt{3}}$

we know that

tan 30° $= \frac{1}{\sqrt{3}}$

and tan $(-\theta) = -\tan\theta$

$\Rightarrow$ tan (–30°) = –tan (30°) $= -\frac{1}{\sqrt{3}}$

$\therefore$ $\theta = -30°$

$\therefore$ Polar form is 2 {cos (–30°) + i sin (–30°)}

Example O:

Express – 5 – 5i in the polar form

Solution

Here, r $= \sqrt{25+25} = \sqrt{50} + 5\sqrt{2}$

tan $\theta = \frac{-5}{-5} = 1$

$\Rightarrow$ $\theta = 45°$

But the point (–5, –5) lies in the III quadrant.

$\therefore$ we consider the property

tan (180° + θ) = tan θ

$\Rightarrow$ tan (180° + 45°) = tan 45°

or, tan (225°) = tan (45°)

$\therefore$ Polar form is

(cos 225° + i sin 225°)

Example P:

(a) Express $-1 + \sqrt{3}\,i$ in polar form

(b) is polar representation unique?

Solution

Here r = $\sqrt{(-1)^2 + (\sqrt{3})^2} + \sqrt{1+3} = \sqrt{4} = 2$

(a) $\tan\theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$

$\Rightarrow$ $\theta = -60°$ $(\because \tan(-\theta) = -\tan\theta)$

So, the polar form is

$2\{\cos(-60°) + i\sin(-60°)\}$

(b) We may note that the point (–1,   ) representing the complex number $-1 + \sqrt{3}\,i$, i.e.,

point $(-1, \sqrt{3})$ lies in the II quadrant

$\therefore$ We consider

$\tan(180° - \theta) = -\tan\theta$

$\tan(180° - 60°) = -\tan 60°$

or $\tan 120° = \tan(-60°)$

Polar form can be written as

$4(\cos 120° + i\sin 120°)$

$\therefore$ We can say that polar representation is not unique. as the argument θ is not unique.

Check-point 4:

1. Choose the appropriate answer.

   $z = a + ib$ can be expressed in polar form as

   (i) $r(\cos\theta - i\sin\theta)$

   (ii) $r\cos\theta + i\sin\theta$

   (iii) $r(\cos\theta + i\sin\theta)$

2. Fill in the blank

In polar representation of $z = a + ib$

$\sqrt{a^2+b^2}$ is ___________ and $\tan^{-1}\frac{b}{a}$ is _________.

INTEXT QUESTIONS 3.3

1 Express the following in the polar form

(a) (i) $4 + 4i$

(ii) $+ i$

(b) (i) $1 - i$

(ii) $1 - \sqrt{3}\,i$

2. Write at least two polar representations for the following complex numbers

(a) (i) $1 - \sqrt{3}\,i$

(ii) $2 - 2i$

(iii) $-1 - i$

(iv) $6 + 6i$

3. Write each of the following complex numbers in the form $a + bi$

(a) $5(\cos 30° + i \sin 30°)$

(b) $11(\cos 120° + i \sin 120°)$

(c) $\cos 75° + i \sin 75°$

(d) $3\left\{(\cos(-225°) + i \sin(-225°)\right\}$

3.7 POLAR REPRESENTATION OF DIVISION

Let $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$
$z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$

then $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\}$$

$$= \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

Thus,we can see that

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$$

and its argument $= \Theta_1 - \Theta_2$

Also, we can observe that

$r_1 = |z_1|$

and $r_2 = |z_2|$

Thus, we can write

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Geometrical Representation of Division in *C*

Let $z_1 = r_1(\cos\Theta_1 + i\sin\Theta_1)$

and $z_2 = r_2(\cos\Theta_2 + \sin\Theta_2)$

be represented by points P and Q respectively

Let us take a point I(1,0) on plane.

Construct

$\Delta OPR \sim \Delta OQI$

$\therefore$ In ΔOPR and ΔOQI

$$\frac{OQ}{OI} =$$

$\Rightarrow$ OR = $\times$ OI

But $OP = r_1$, $OQ = r_2$, $OI = 1$

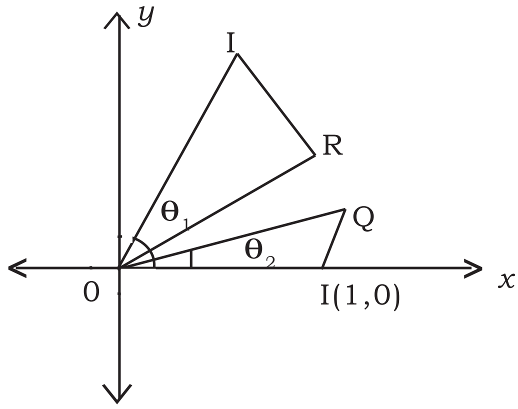


Fig. 3.22

$\therefore$ $OR = \dfrac{r_1}{r_2}$ .... (i)

Also $\angle ROX = \angle POX - \angle POR = \theta_1 - \theta_2$

$= \angle POX - \angle QOX$

$\therefore$ R is point in plane with modulus

$\dfrac{r_1}{r_2}$ and argument $\theta_1 - \theta_2$

$\therefore$ Point R represents the complex number $\dfrac{z_1}{z_2}$

$$OR = \left|\frac{z_1}{z_2}\right| \text{ .... (ii)}$$

from (i) and (ii) we get

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Check-point:

1. Tick mark the right answer.

(a) Modulus of complex number $\dfrac{z_1}{z_2}$ is

(i) $r_1 - r_2$

(ii) $\frac{r_1}{r_2}$

(iii) $r_1 - r_2$

Where $|z_1| = r_1$ and $|z_2| = r_2$

(b) Argument of $\frac{z_1}{z_2}$ is

(i) $\theta_1 - \theta_2$

θ_1

θ_2

(iii) $\theta_1 + \theta_2$

Where ar $(z_1) = \theta_1$ and ar $(z_2) = \theta_2$

{ **Ans: 1(a) (ii), (b)(i)** }

Example Q:

Find the modulus of the complex number

(ii)

$\frac{2+i}{3-i}$

$\frac{+i}{-i}$

Solution

Let z =

$\therefore$ = $\left|\frac{2+i}{3-i}\right|$

But we know that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\left|\frac{2+i}{3-i}\right| = \frac{|2+i|}{|3-i|}$$

$\therefore$ $|2+i| = \sqrt{4+1} = \sqrt{5}$

and $|3+i| = \sqrt{9+i} = \sqrt{10}$

$$\therefore \qquad |z| = \left|\frac{\sqrt{5}}{\sqrt{10}}\right| = \frac{1}{\sqrt{2}}$$

Example R:

(a) Write the polar representation for $z_1 \cdot z_2$

(b) Represent geometrically

$$|z_1 z_2| = |z_1| \; |z_2|$$

Solution

(a) Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ Here $z_1 = r_1$, $\arg(z_1) = \theta_1$

$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ Here $z_2 = r_2$, $\arg(z_2) = \theta_2$

then $z_1 . z_2 = r_1 \, r_2 (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2)$

$= r_1 \, r_2 (\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 +$
$i \sin \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2)$

$= r_1 \, r_2 \{ (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) +$
$i (\sin \theta_2 \cos \theta_1 + \sin \theta_1 \cos \theta_2)$

$= r_1 \, r_2 (\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2) \}$

Thus we get $|z_1 z_2| = r_1 \, r_2 = |z_1| \; |z_2|$

and $\arg (z_1 \; z_2) = \theta_1 + \theta_2$

Geometrical Representation

Let P (r_1, θ_1) and Q$(r_2 \; \theta_2)$ represent z_1 and z_2 respectively. Take I (1,0) on a plane and construct

Δ OQR ~ ΔOIP

$$\Rightarrow \quad \frac{OR}{OQ} = \frac{OP}{OI}$$

$$\Rightarrow \quad OR = \frac{OQ \times OP}{OI} =$$

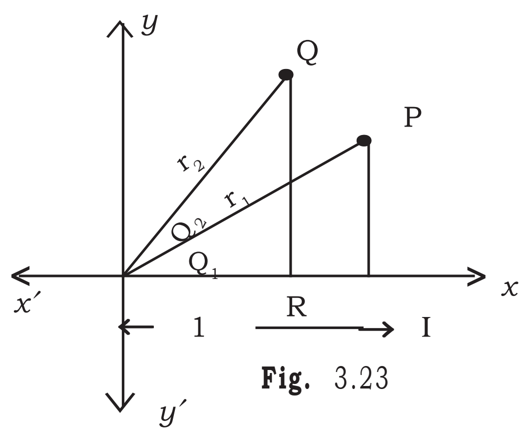


Fig. 3.23

$\Rightarrow \quad OR = r_1 \; r_2$

Also $\angle ROX = \angle ROQ + \angle QOX$

$= \angle POI + \angle QOX = \theta_1 + \theta_2$

Example S:

Find the modulus of the complex number

$(1 + i)\,(2 + 3i)$

Solution: Let $z = (1 + i)\,(2 + 3i)$

then $|z| = |(1+i)(2+3i)|$

$= |(1+i)(2+3i)|$ ($\because$ $=$ $|z_1|$ $|z_2|$)

But $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$|2+3i| = \sqrt{2^2+3^2} = \sqrt{2+9} = \sqrt{13}$

$\therefore \quad |z| = \sqrt{2}.\sqrt{13} = \sqrt{26}$

INTEXT QUESTIONS 3.4

1. Find the modulus of the following complex numbers:

(a) (i) $\dfrac{1+i}{3-i}$

(ii)

(iii)

(b) (i) $(5 - i)\,(2 + i)$

(ii) $(-i)\,(i + 3)$

(iii) $(6 + 2i)\,(5 + 4i)$

(c) (i) (ii) $\dfrac{i+\sqrt{2}}{3-\sqrt{5}i}$

(iii) $(i+i^2)\,(2i-3)$ (iv) $(4-3i)\,(i^2-2i^3+4)$

3.8 WHAT YOU HAVE LEARNT

- Every complex number $z = a + ib$ can be written as (a, b) and hence can be represented in complex coordinate plane.

  The plane is called the **argand plane.**

  The diagram is called the **argand diagram**.

  The horizontal axis is called the **real axis.**

  The vertical axis is called the **imaginary axis.**

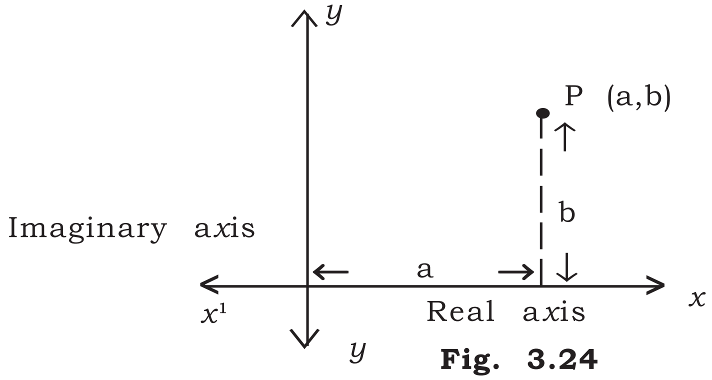


Fig. 3.24

- Every complex number has a unique representation in the argand plane and every point on complex plane can be associated to the unique complex number.

- Modulus of $z = a + ib$, $a \in R$, $b \in R$ is

$$|z| = \sqrt{a^2 + b^2}$$

- $|z| = 0 \Leftrightarrow z = 0$

- $|z_1 + z_2| = 0 \Rightarrow |z_1| = |z_2|$

  but $|z_1| = |z_2|$ does not always imply that $z_1 = z_2$

- $$|z_1 + z_2| \leq |z_1| + |z_2|$$

  Geometrically, it means that the sum of two complex numbers is represented by the diagonal of the parallelogram.

- $$|z_1 - z_2| \geq |z_2| - |z_1|$$

- For $z = a + ib$, $a \in$ R, $b \in R$

polar representation is

$z = r(\cos\theta + i\sin\theta)$

Where $r = \sqrt{a^2+b^2}$ is called the **modulus**

$\tan\theta = \frac{b}{a}$ is called the **argument**

- $= \frac{|z_1|}{|z_2|}$
- $|z_1\ z_2| = \quad |z_2|$

TERMINAL QUESTIONS

1. Represent the following complex numbers on the argand plane

   $2 + 9i$, $-11 - 5i$, $5i$, -2, $3 - \frac{1}{3}i$, i^2, $i^2 + 4i$

2. Write the complex numbers corresponding to the following points on the argand plane

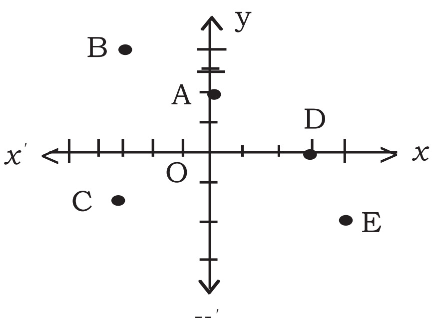


Fig. 3.25

3. Find the modulus of the following complex numbers

   (a) $2 + i$

   (b) $15 + 9i$

   (c) $1 + \quad i$

   (d) $(1 - \sqrt{3}) + (2 + \sqrt{2})i$

   (e) $5i^2 - 4i + 3$

4. Illustrate with examples

   $|z_1| = |z_2|$ does not always imply $z_1 = z_2$.

5. $|z|$ is always greater than or equal to zero. True or false. Give reason for your answer.

6. Write geometrical interpretation of

   $|z_1 + z_2| \leq |z_1| + |z_2|$

7. Find the modulus of the following complex numbers.

   (a) $(8 + 9i)i$ (b) $(8i + 8i^2)7i$ (c) $\frac{7-i}{6+i}$

   (d) (e) $i + 1$ (f) $\frac{(1+i)(2+i^2)}{3+1}$

   (g) $\frac{i}{i^3-1}$ (h) $\frac{i+\sqrt{3}}{3+\sqrt{5}i}$ (i) $\frac{(4-3i)i}{(2+i)(1+i)}$

   (j) i^{13}

8. For the following pairs of complex numbers

   verify that $|z_1 + z_2| \leq |z_1| + |z_2|$

   (i) $z_1 = i - 5$ $z_2 = 3i + 2$

   (ii) $z_1 = 4 + 3i$ $z_2 = 9 + 8i$

9. For the following pair of complex numbers

   verify that $|z_1 - z_2| \geq |z_1| - |z_2|$

   (a) $z_1 = 1 + i$ $z_2 = i + 3$

   (b) $z_1 = 4i + i^2$ $z_2 = 3 - 2i$

10. For the following pair of complex numbers

    verify that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

    (a) $z_1 = 2 + 6i$, $z_2 = 1 - 4i$

(b) $z_1 = 7 - i$, $z_2 = 3 + 4i$

11. For the following pair of complex numbers verify that $|z_1 z_2| = |z_1| \; |z_2|$

(a) $z_1 = 3 + 2i$, $z_2 = 1 - 5i$

(b) $z_1 = 7 + 3i$, $z_2 = 4 - 8i$

12. Express the following in the polar form

(i) $2+2\sqrt{3}\,i$ (b) $-5 + 5i$ (c) $-\sqrt{6}-\sqrt{2}\,i$

(d) $-3i$ (e) $2 - 2i$ (f) $-1+\sqrt{3}\,i$

g) $2\sqrt{2}+2\sqrt{2}\,i$ (h) -4

13. Polar representation of a complex number is not unique. Support the above statement with example.

14. Write the following in the form $a + bi$

(a) cos 60° – i sin 60°

(b) $\sqrt{5}$ (cos 210° + i sin 210°)

(c) $\sqrt{2}$ (cos 60° + i sin 60°)

ANSWERS TO CHECK POINTS

Check-point 1: 1. (iv) 2. (ii)

Check-point 2: 1. (i) 2. (i) 3. (ii)

Check-point 3: 1. (ii) 2. (i) 3. (iii)

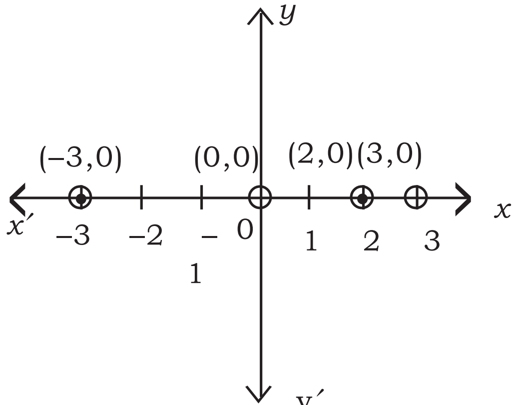
Check-point 4: 1. (iii) 2. Modulus, argument

Check-point 5: 1. (ii) 2. (i)

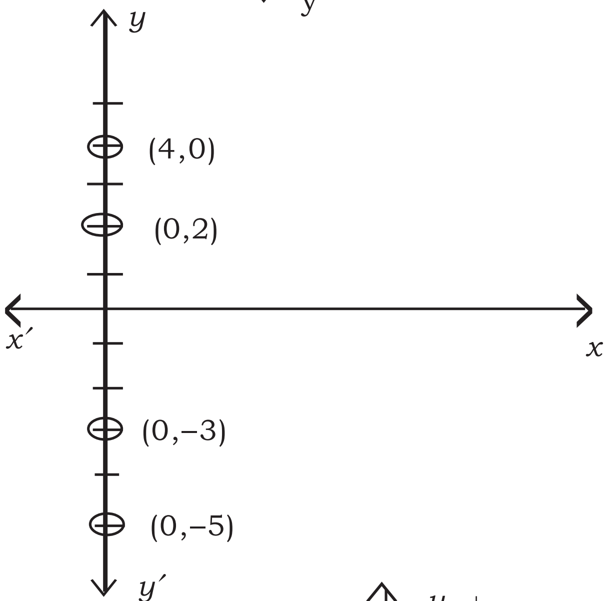
ANSWERS TO INTEXT QUESTION

3.1

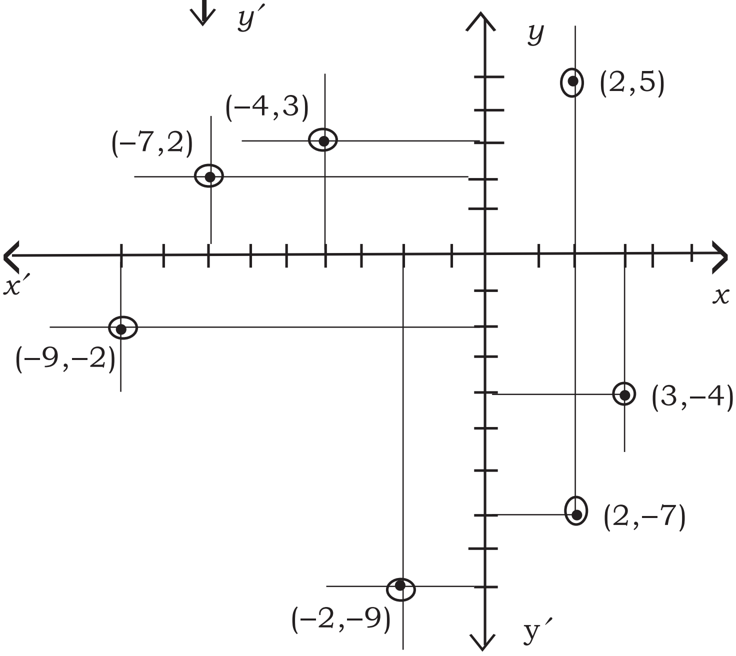
1 (a)



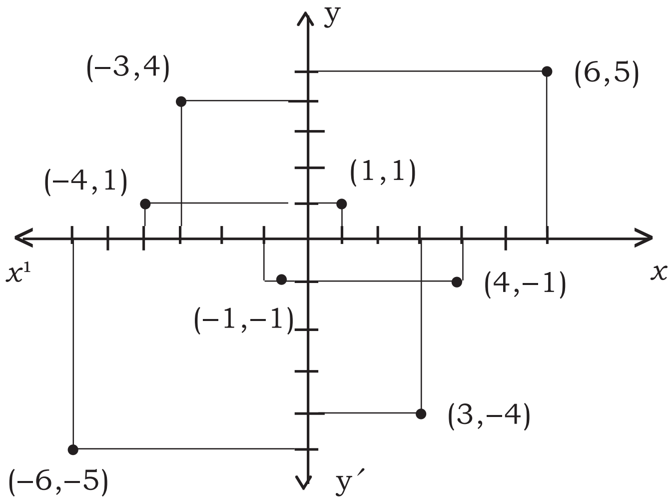
(b)



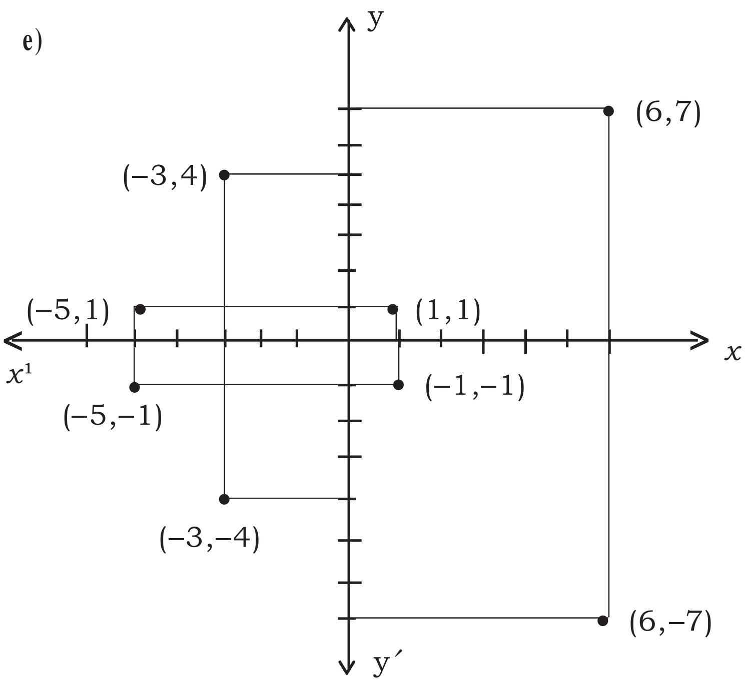
(c)



d)



e)



2. (a) (i) $\sqrt{2}$ (ii) 34 (iii) $\sqrt{13}$

(iv) $\sqrt{89}$ (v) $6\sqrt{2}$

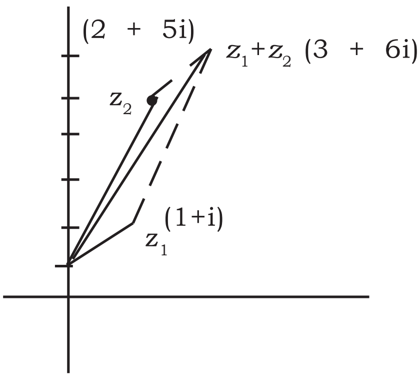
3. P(5,3) Q(0,4) R(−2,0) S(1,−3) T(−5,3)

4. (a) 2 (b) 2 c) 3 d) $\sqrt{29}$

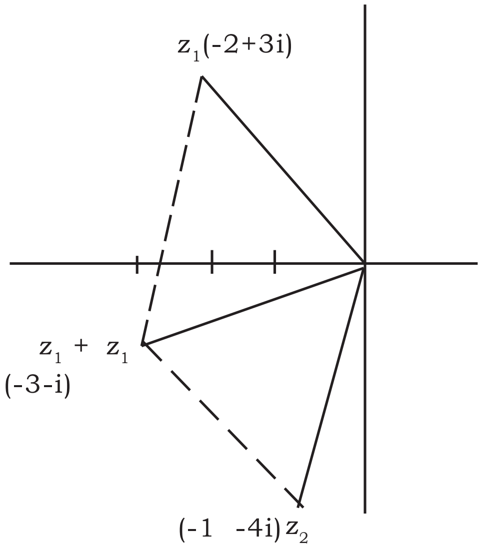
e) $\sqrt{2}$ f) 1 g) $\sqrt{10}$

3.2

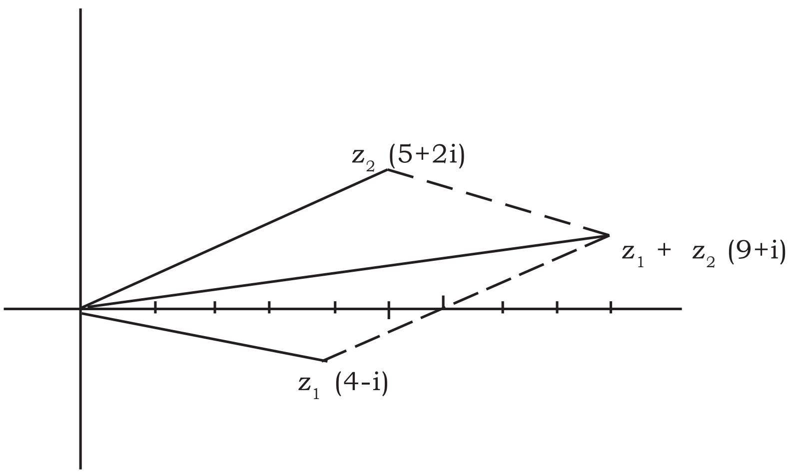
(a)



(b)

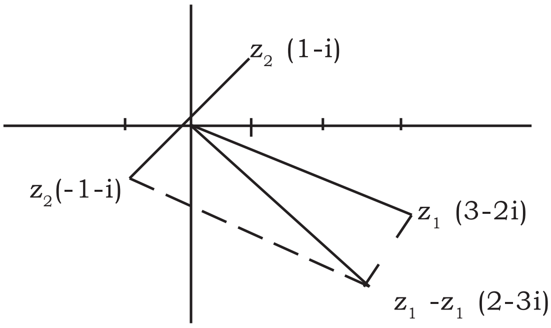


(c)

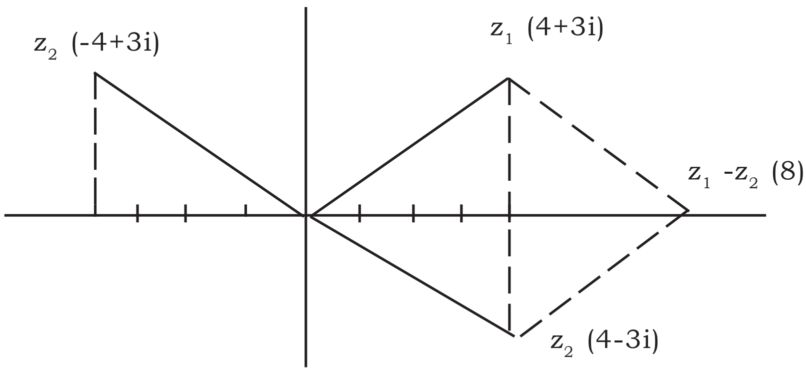


3.

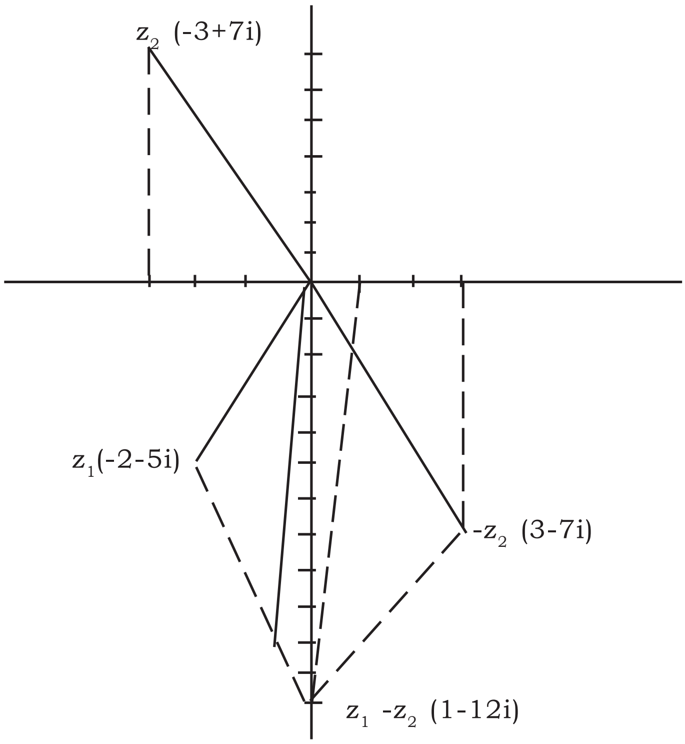
(a)


z2 (1-i)
z2(-1-i)
z1 (3-2i)
z1 -z1 (2-3i)

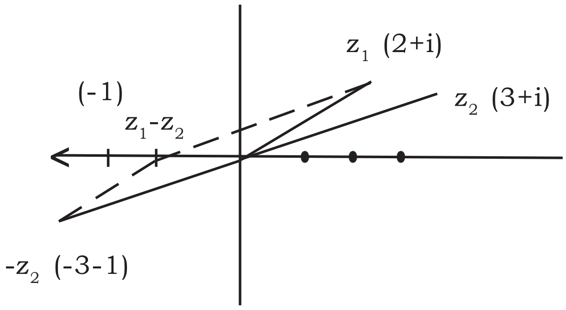
(b)


z2 (-4+3i)
z1 (4+3i)
z1 -z2 (8)
z2 (4-3i)

(c)


z2 (-3+7i)
z1(-2-5i)
-z2 (3-7i)
z1 -z2 (1-12i)

(d)



3.3

1. (a) (i) $4\sqrt{2}$ (cos 45° + i sin 45°)

(ii) 2 (cos 30° + i sin 30°)

(b) (i) $\sqrt{2}$ { cos (–45°) + i sin (–45°) }

(ii) 2 { cos (–60°) + i sin (60°) }

2. (a) (i) 2 { cos (–60°) + i sin (–60°) }

2 { cos (120°) + i sin (120°) }

(ii) $2\sqrt{2}$ { (cos (–45°) + i sin (–45°) }

$2\sqrt{2}$ { (cos (135°) + i sin (135°) }

(iii) $\sqrt{2}$ { cos (45°) + i sin (45°) }

$\sqrt{2}$ (cos 225° + i sin 225°)

(iv) $6\sqrt{3}$ (cos 45° + i sin 45°)

$6\sqrt{3}$ (cos 225° + i sin 225°)

3 (a) $\frac{5\sqrt{3}}{2}+\frac{5}{2}i$ (b) $\frac{-11}{2}+\frac{11\sqrt{3}}{2}i$

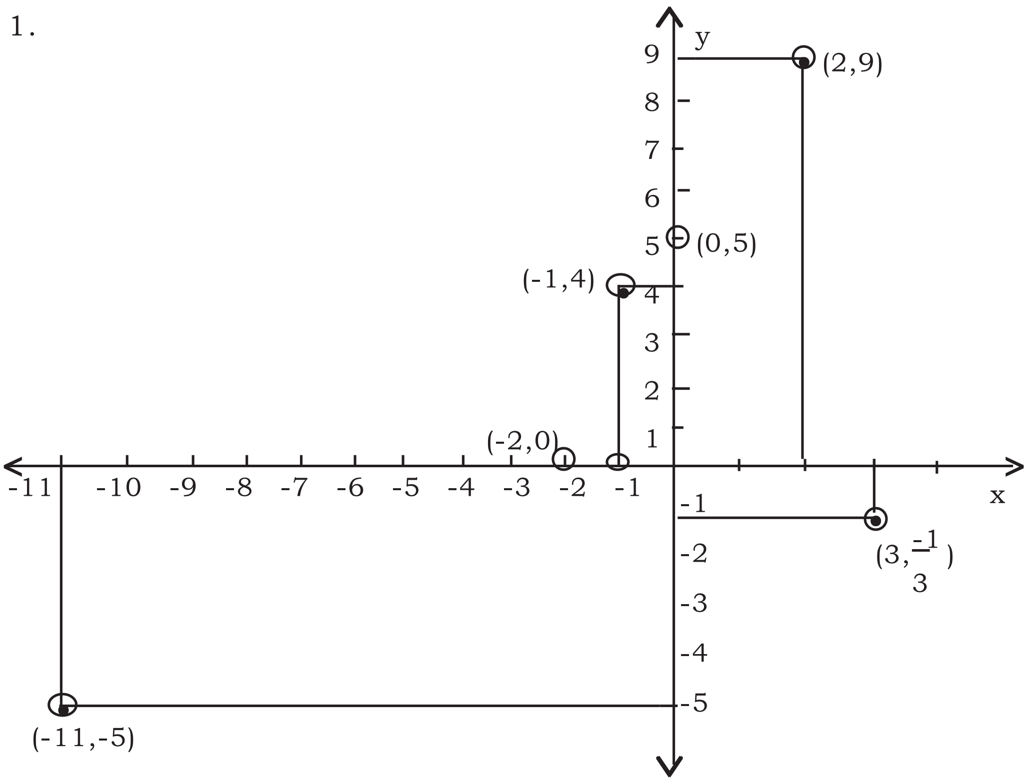
(c) $\frac{\sqrt{3}-1}{2\sqrt{2}}+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)i$ (d) $\frac{-3}{\sqrt{2}}+\frac{3}{\sqrt{2}}i$

3.4

1	(a)	(i) $\frac{1}{\sqrt{5}}$	(ii) $\sqrt{\frac{29}{5}}$	(iii) $\sqrt{\frac{82}{53}}$	
	(b)	(i) $\sqrt{130}$	(ii) $\sqrt{10}$	(iii) $3\sqrt{410}$	
	(c)	(i) $\frac{1}{\sqrt{2}}$	(ii) $\sqrt{\frac{3}{14}}$	(iii) $\sqrt{26}$	(iv) $5\sqrt{13}$

3.11 ANSWERS TO TERMINAL QUESTIONS

1.



2. A→ 2i, B→ −3+3.5i, C→ −3−1.5i, D→3, E→ 4−2i

3. (a) $\sqrt{5}$, (b) $3\sqrt{34}$, (c) 2 (d) $\sqrt{10+4\sqrt{2}-2\sqrt{3}}$,

   (e) $2\sqrt{5}$

4.

5. True

7. (a) $\sqrt{145}$, (b) $55\sqrt{2}$ (c) $\frac{5\sqrt{2}}{\sqrt{27}}$ (d) 1

   (e) $\frac{3}{\sqrt{43}}$ (f) $\frac{1}{\sqrt{5}}$ (g) $\frac{1}{\sqrt{2}}$ (h) $\frac{2}{\sqrt{14}}$

   (i) $\frac{5}{\sqrt{10}}$ (j) 1

12. (a) 4(cos 60° + i sin 60°),

    (b) $5\sqrt{2}$(cos 135 + i sin 135°)

    (c) $2\sqrt{2}$(cos 210° + i sin 210°)

    (d) 3(cos 270° + i sin 270°)

    (e) $2\sqrt{2}$(cos 315° + i sin 315°)

    (f) 2 cos (120° + i sin 120°)

    (g) 4 (cos 45° + i sin 45°)

    (h) 4 (cos 180° + i sin 180°)

14. (a) $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ (b) $\frac{\sqrt{15}}{2}-\frac{\sqrt{5}}{2}i$ (c) $\frac{1}{\sqrt{2}}+\sqrt{\frac{3}{2}}i$