9

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

9.1 INTRODUCTION

In your earlier classes, you have solved linear equations. You have also found the solution of a system of linear equations in two variables of the form

$$x + y = 3$$

$$2x + 3y = 5$$

On solving the above equations, we find that $x = 4$ and $y = -1$ are the solutions of this system.

But suppose you are asked to find the solutions of a system of linear equations in three variables of the form

$$2x + 3y + z = 9 \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$x - 2y - z = 4 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$3x + y + 2z = -1 \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

The usual method of finding the solution of such a system is like this —

Let us consider z to be a constant

Then equations (1) and (2) becomes

$$2x + 3y = 9 - z \quad \ldots\ldots\ldots\ldots \quad (4)$$

$x - 2y = 4 + z$ .... (5)

Solving the equations (4) and (5) for x and y, we get

$$x = \frac{30+z}{7} \text{ and } y = \frac{1-3z}{7}$$

Substituting these values of x and y in equation (3) ,

we get $x = \frac{23}{7}$, $y =$ and $z = -7$

But this method is very long and tedious. Now that we have studied determinants and matrices, we are equipped with a better tool to solve these equations in a simple and precise manner.

In this lesson, we are going to learn about these simple methods to solve a system of linear equations in more than one variable.

9.2 OBJECTIVES

After studying this lesson, you will be able to

- state Cramer's rule to solve a system of linear equations
- solve a given system of equations in two or three variables using Cramer's rule
- state the condition for a system of linear equations in two or three variables to have a unique solution
- solve a system of linear equations by the matrix method

9.3 PREVIOUS KNOWLEDGE

(a) Solution of a system of linear equations in two variables

(b) Concept of a determinant

(c) Finding the value of a determinant

(d) Concept of matrices

(e) Findg the inverse of a matrix

9.4 CRAMER'S RULE TO SOLVE A SYSTEM OF LINEAR EQUATIONS.

The Swiss Mathematician Gabriel Cramer (1704–1752) gave one of the methods to find the solutions of a system of linear equations with the help of determinants. Therefore, this method is called **Cramer's rule.**

In order to learn about this method, consider the following system of linear equations.

$$2x + 3y - 5 = 0 \quad \text{..........} \quad \text{(i)}$$

$$3x + 5y - 7 = 0 \quad \text{.........} \quad \text{(ii)}$$

Usually, to find solutions of such a system, we apply the method of elimination. Thus, first of all solving for y, we get

$$\begin{cases} 3(2x + 3y - 5) = 0 \\ 2(3x + 5y - 7) = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} 3(2x) + 3(3y) = 3(5) & \text{.........} \quad \text{(iii)} \\ 2(3x) + 2(5y) = 2(7) & \text{........} \quad \text{(iv)} \end{cases}$$

Subtracting (iv) from (iii), we get

$$\{\, 3(3) - 2(5) \,\}\ y = 3(5) - 2\ (7)$$

i.e.,
$$\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} \quad y \quad = \quad \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}$$

$$\Rightarrow \quad y \quad = \quad \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} \qquad \text{............} \quad A$$

Observe that in the determinant of the numerator, if the 1st column consists of the coefficient of x and the 2nd column consists of the constant terms.

Also, in the determinant of the denominator of y, the 1st column and the 2nd column consists of the coefficients of x and y respectively.

Similarly, solving (i) and (ii) for x, we get

$$5\ (2x + 3y - 5) = 0$$

$$3(3x + 5y - 7) = 0$$

$$\Rightarrow \quad \begin{cases} 5(2)\ x + 5(3y) - 5(5) = 0 & \text{.............} \quad \text{(v)} \\ 3(3x) + 3(5y) - 3(7) = 0 & \text{.............} \quad \text{(vi)} \end{cases}$$

Subtracting (vi) from (v), we get

$$\{5(2) - 3\ (3)\}\ x = 5(5) - 3(7)$$

i.e., $$\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} x = \begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}$$

$$\Rightarrow \quad x = \frac{\begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} \quad \text{..............} \quad B$$

Again observe that in the determinant of the numerator of x, the 1st column consists of the constant terms and the 2nd column consists of the coefficient of y.

Also, in the determinant of the denominator of x, the 1st column and the 2nd column consists f the coefficients of x and y respectively.

$$\text{Thus,} \quad x = \frac{\begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} = \frac{D_1}{D} \text{ (say)}$$

$$\text{and} \quad y = \frac{\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} \qquad \frac{D_2}{D} \ \text{(say)}$$

are the solutions of the given system of equations in determinant form.

Consider another system of equations

$$x + y = 3 \qquad \text{..........} \qquad \text{(i)}$$

$$2x - 3y = 1 \qquad \text{..........} \qquad \text{(ii)}$$

Again, solving (i) and (ii) by elimination method, for x, we get

$$-3(x + y) = -3(3)$$

$$1(2x - 3y) = 1$$

$$\Rightarrow \quad \left\{ \begin{array}{ll} -3(x) - 3(y) = -3(3) & \text{...........} \quad \text{(iii)} \\ 1(2x) - 1(3y) = 1(1) & \text{...........} \quad \text{(iv)} \end{array} \right.$$

Subtracting (iv) from (iii), we get

$$-3(x) - 1(2x) = -3(3) - 1(1)$$

$$\Rightarrow \quad \{-3(1) - 1(2)\}\ x = -3(3) - 1(1)$$

$$\text{i.e.} \quad \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} x = \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}$$

$$\Rightarrow \quad x = \frac{\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}} \qquad \frac{D_1}{D} \ \text{(say)}$$

Again, we observe that in the determinant D_1, the 1st column consists of the constant terms and the 2nd column consists of the coefficient of y. Also, in D, the 1st and the 2nd columns consist of the coefficients of x and y respectively.

Similarly, solving (i) and (ii) for y, we get

$$2(x) + 2(y) = 2(3) \qquad \ldots\ldots\ldots\ldots \quad (v)$$

$$1(2x) + 1(-3y) = 1(1) \qquad \ldots\ldots\ldots\ldots \quad (vi)$$

Subtracting (v) from (vi), we get

$$\{1(-3) - 2(1)\}\, y = 1(1) - 2(3)$$

i.e.,
$$\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} y = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$\Rightarrow \qquad y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}} = \frac{D_2}{D} \text{ (say)}$$

Again, we observe that the determinant D is nothing but the determinant of the coefficient of x and y and the determinant D_2 is obtained by replacing the coefficient of y by the constant terms.

Thus,
$$x = \frac{\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}}$$

are the solutions of the given equation

Therefore, if and
$$\left\{\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}\right.$$

is a given system of linear equations in two variables, then its solution will be

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_1}{D}$$

$$\text{and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_2}{D}$$

$$\text{provided } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

Let us now consider the following system of equations in three variables

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2x + y + z &= 1 \qquad \ldots\ldots\ldots\ldots \text{Ⓒ} \\ 3x + 3y + 5y &= 3 \end{aligned}$$

$$\text{Let } D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix}$$

i.e., D is the determinant of the coefficient of x, y and z.

$$\text{Then, } xD = \begin{vmatrix} x & 2 & 3 \\ 2x & 1 & 1 \\ 3x & 3 & 5 \end{vmatrix}$$

Multiplying the 2nd column to y and the 3rd column to z and adding these to the 1st column, we get

$$xD = \begin{vmatrix} x + 2y + 3z & 2 & 3 \\ 2x + y + z & 1 & 1 \\ 3x + 3y + 5z & 3 & 5 \end{vmatrix}$$

$\therefore$ from Ⓒ, we get

$$xD = \begin{vmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix} = D_1 \text{ (say)}$$

i.e., $xD = D_1 \Rightarrow x =$

Similarly, we will get

$$D_2 \begin{vmatrix} 1 & 4 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix} \quad \text{and} \quad D_3 = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 1 \\ 3 & 3 & 3 \end{vmatrix}$$

Then, as before, we can see that

$$yD = D_2 \text{ and } z\mathrm{D} = D_3$$

Thus, $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$, $z = \dfrac{D_3}{D}$, where $D \neq 0$

are the solutions of the given system of linear equations in three variables.

Therefore, if

$$\left| \begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array} \right|$$

is a given system of linear equations in three variables, then

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_1}{D}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_2}{D}$$

and

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_3}{D}$$

are the solutions of the given system provided

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

The method used for solving the system of equations in three variables can be used in exactly the same way to solve a system of 'n' equations in 'n' unknowns.

⇒ Below, we state the theorem for the general case, known as Cramer's rule, after the Swiss Mathematician Gabriel Cramer (1704–1752).

Theorem: Consider the system of n linear equations in n unknowns given by

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ a_{2n}x_n = b_2 \\ &. \\ &. \\ &. \\ &. \\ &. \\ &a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n \end{aligned}$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix}$

i.e., D is the determinant of the coefficients of $x_1, x_2, \ldots x_n$

Let D_1 be the determinant obtained from D after replacing the 1st column, i.e.

$$\begin{vmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{n1} \end{vmatrix} \quad by \quad \begin{vmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{vmatrix}$$ i.e., the constant terms.

Let D_2 be the determinant obtained from D after replacing the 2nd column, i.e.,

$$\begin{vmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{vmatrix} \quad \text{i.e., the constant terms.}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column

$$\begin{vmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{nj} \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{vmatrix} \quad \text{i.e.,}$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad \ldots.. \quad x_n = \frac{D_n}{D} \quad \text{provided } D \neq 0$$

will be the solutions of the given system of equations.

Note: Cramer's rule does not apply if $D = 0$

Example A:

Solve the following system of equations by Cramer's rule:

$$2x + 3y = 5$$

$$3x + 5y = 7$$

Solution: Now

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1$$

Using Cramer's rule

$$D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix} = 25 - 21 = 4$$

$$D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix} = 14 - 15 = -1$$

Thus, $x = \dfrac{D_1}{D} = \dfrac{4}{1} = 4$

and $y = \dfrac{D_2}{D} = \dfrac{-1}{1} = -1$

are the solutions of the given system of equations.

Example B:

Solve the following system of equations by Cramer's rule:

$$2x + y - 3z = 3$$

$$x + 2y + z = 5$$

$$3x - 5y + 2z = 1$$

Solution : Now

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \mathrm{a}_3 & \mathrm{b}_3 & \mathrm{c}_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -3 \\ 1 & 2 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= 2(4+5) - 1(2-3) - 3(-5-6) = 18+1+33$$

$$= 52 \neq 0$$

Also, to find D_1, the 1st column will be replaced by constants

$$\therefore \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & 1 & -3 \\ 5 & 2 & 1 \\ 1 & -5 & 2 \end{vmatrix}$$

$$= 3(4 + 5) -1 (10 - 1) -3 (-25 - 2)$$

$$= 27 - 9 + 81 = 99$$

To find D_2, 2nd column will be replaced by constants.

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -3 \\ 1 & 5 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= 2(10 - 1) -3\ (2 - 3) -3\ (1 - 15)$$

$$= 18 + 3 + 42 = 63$$

To find D_3, 3rd column will be replaced by constants

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 5 \\ 3 & -5 & 1 \end{vmatrix}$$

$$= 2(2 + 25) -1\ (1 - 15) +3\ (-5 - 6)$$

$$= 54 + 14 - 33 = 35$$

Thus, $x = \dfrac{D_1}{D} = \dfrac{99}{52}$

$y = \dfrac{D_2}{D} = \dfrac{63}{52}$

and $z = \dfrac{D_3}{D} = \dfrac{35}{52}$

are the solutions of the given system of equations.

Check–point: Select the right answer.

The given system of equations

$$x - 2y = 3$$

$$3x - 6y = 9$$

(i) can be solved by Cramer's rule

(ii) cannot be solved by Cramer's rule

{ **Ans :** (ii) }

9.5 CONDITION FOR A SYSTEM OF LINEAR EQUATIONS IN TWO OR THREE VARIABLES TO HAVE A UNIQUE SOLUTION

Consider the system of equations

$$2x + 3y = 4$$

$$x - 2y = 3$$

Now $D = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4 - 3 = -7 \neq 0$

Also $D_1 = \begin{vmatrix} 4 & 3 \\ 3 & -2 \end{vmatrix} = -8 - 9 = -17$

$D_2 = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 6 - 4 = 2$

$\therefore$ by Cramer's rule

$$x = \frac{D_1}{D} = \frac{-17}{-7} = \frac{17}{7}$$

and $$y = \frac{D_2}{D} = \frac{2}{-7} = \frac{-2}{7}$$

Thus, we find that for $\boldsymbol{D \neq 0}$ **and** $\boldsymbol{D_1 \neq 0, D_2 \neq 0,}$ the given system of equations have **non–zero, unique solution** $x = \frac{17}{7}$ and $y = \frac{-2}{7}$

In this case, we say that the given system of equations are consistent.

Now consider the equations

$$x + 2y = 0$$

$$-2x + 3y = 0$$

Here, $D = \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} = 3 + 4 = 7 \neq 0$

Also $D_1 = \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0 - 0 = 0$

and $D_2 = \begin{vmatrix} 1 & 0 \\ -2 & 0 \end{vmatrix} = 0 - 0 = 0$

Hence , $x = \dfrac{D_1}{D} = \dfrac{0}{7} = 0$

and $x = \dfrac{D_2}{D} = \dfrac{0}{7} = 0$

Thus, we find that for **$D \neq 0$ and $D_1 = D_2 = 0$** the given system of equations will have only the **trivial solution $x = y = 0$**

We already know that Cramer's rule does not apply if $D = 0$. The two cases arise namely (i) no solution, (ii) infinitely many solutions.

Consider the system of equations

$$2x + 4y = 5$$

$$x + 2y = 3$$

Here $D = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 4 - 4 = 0$

Since $D = 0$, Cramer's rule does not apply here.

Now, $D_1 = \begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix} = 10 - 12 = -2$

and $D_2 = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 6 - 5 = 1$

Hence, since **$D \neq 0$ and $D_1 \neq 0$, $D_2 \neq 0$,** therefore the equations will have **no solution**.

Similar is the case for a system of three equations in three variables. For that, consider the system of equations

$$x + y + z = 2$$

$$x + 2y + 3z = 3$$

$$x + 3y + 5z = 5$$

Here, $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} \begin{array}{l} \\ = 1(10 - 9) - 1(5 - 3) + 1(3 - 2) \\ = 1 - 2 + 1 = 0 \end{array}$

Now $xD = \begin{vmatrix} x & 1 & 1 \\ x & 2 & 3 \\ x & 3 & 5 \end{vmatrix}$

Multiplying the 2nd column by y and the 3rd column by z and adding them to the 1st column, we get

$$xD = \begin{vmatrix} x + y + z & 1 & 1 \\ x + 2y + 3z & 2 & 3 \\ x + 3y + 5z & 3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 5 & 3 & 5 \end{vmatrix} = D_1$$

Thus, $D_1 = 2\ (10 - 9) - 1\ (15 - 15) + 1\ (9 - 10)$

$= 2 - 0 - 1$

$= 1$

so, $xD = D_1 \Rightarrow xD = 1$

$$\Rightarrow \quad x = \frac{1}{D} = \frac{1}{0} \quad \text{which is undefined.}$$

Similarly, we will get

$$yD = D_2 \text{ and } zD = D_3$$

where $D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 5 & 5 \end{vmatrix} = 1\ (15-15)\ -2(5-3)\ +1(5-3)$

$= 0 - 4 + 2$

$= -2$

and $D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 1(10-9) -1(5-3) +2(3-2)$

$= 1 - 2 + 2$

$= 1$

$$\therefore \quad yD = D_2 \Rightarrow y = \frac{D_2}{D} \Rightarrow y = \frac{-2}{0} \quad \text{which is undefined.}$$

$$\text{and} \quad zD = D_3 \Rightarrow z = \frac{D_3}{D} \Rightarrow z = \frac{-1}{0} \quad \text{which is undefined.}$$

Thus, **if $D = 0$ and $D_1 \neq 0$, $D_2 \neq 0$ and $D_3 \neq 0$,** then the system of equations will have **no solution.** In this case, we say that the system of equations are **inconsistent.**

Now, consider the system of equations

$$x - y + 3z = 6$$

$$x + 3y - 3z = -4$$

$$5x + 3y + 3z = 10$$

Here ,

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{vmatrix} \begin{aligned} &= 1(9 + 9) + 1(3 + 15) + 3(3 - 15) \\ &= 18 + 18 - 36 \\ &= 0 \end{aligned}$$

Also

$$D_1 = \begin{vmatrix} 6 & -1 & 3 \\ -4 & 3 & -3 \\ 10 & 3 & 3 \end{vmatrix} \begin{aligned} &= 6(9 + 9) + 1(-12 + 30) + 3(-12 - 30) \\ &= 108 + 18 - 126 \\ &= 0 \end{aligned}$$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 1 & -4 & -3 \\ 5 & 10 & 3 \end{vmatrix} \begin{aligned} &= 1(-12 + 30) - 6(3 + 15) + 3(10 + 20) \\ &= 18 - 108 + 90 \\ &= 0 \end{aligned}$$

$$D_3 = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 3 & -4 \\ 5 & 3 & 10 \end{vmatrix} \begin{aligned} &= 1(30 + 12) + 1(10 + 20) + 6(3 - 15) \\ &= 42 + 30 - 72 \\ &= 0 \end{aligned}$$

Thus, $D = 0$ and $D_1 = D_2 = D_3 = 0$

Therefore **the given system of equations will have infinitely many solutions.**

Consider the first two equation, i.e.,

$$x - y + 3z = 6$$

$$x + 3y - 3z = -4$$

These can be written as

$$x - y = 6 - 3z$$

$$x + 3y = -4 + 3z$$

Solving these equations by determinants, we get

$$x = \frac{\begin{vmatrix} 6-3z & -1 \\ -4+3z & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{3(6-3z)+1(-4+3z)}{3+1}$$

$$= \frac{18-9z-4+3z}{4}$$

$$= \frac{14-6z}{4}$$

$$\therefore \qquad x = \frac{7-3z}{2}$$

and

$$y = \frac{\begin{vmatrix} 1 & 6-3z \\ 1 & -4+3z \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{1(-4+3z)-1(6-3z)}{3+1}$$

$$= \frac{-4+3z-6+3z}{4}$$

$$= \frac{-10+6z}{4}$$

$$\therefore \qquad y = \frac{-5+3z}{2}$$

Let $z = k$, where k is any number, then we get

$$x = \frac{7-3k}{2}, \qquad y = \frac{-5+3k}{2} \text{ and } z = k$$

Thus, the equations has many solutions.

So we conclude that for a given system of equations

(i) if $D \neq 0$ and atleast one of D_1, D_2, ..., D_n is not equal to zero, then the system will have **non–zero, unique solution.**

(ii) if $D \neq 0$ and each $D_i = 0$, then the system has only **the trivial solution** $x_1 = x_2 = \ldots. = x_n = 0$

(iii) Cramer's rule does not apply if $D = 0$

(iv) If $D = 0$ and some $D_i \neq 0$, then the system has **no solution**

(v) If $D = 0$ and each $D_i = 0$, then the system has **infinitely many solutions.**

Example C:

Solve the following system of equations

$$x + y + z = 2$$

$$2x + 7y - 3z = 5$$

$$3x + 5y - z = 4$$

Solution:

Here,

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 7 & -3 \\ 3 & 5 & -1 \end{vmatrix} = 1(-7 + 15) -1\,(-2 + 9) +1\,(10 - 21)$$

$$= 8 - 7 - 11 = -10 \neq 0$$

Now

$$D_1 = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 7 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 2(-7 + 15) -1\,(-5 + 12) +1\,(25 - 28)$$

$$= 16 - 7 - 3$$

$$= 6$$

$$D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -3 \\ 3 & 4 & -1 \end{vmatrix} = 1(-5 + 12) -2\,(-2 + 9) + 1\,(8 - 15)$$

$$= 7 - 14 - 7$$

$$= -14$$

$$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 7 & 5 \\ 3 & 5 & 4 \end{vmatrix} = 1(28 - 25) - 1(8 - 15) + 2(10 - 21)$$

$$= 3 + 7 - 22$$

$$= -12$$

Since $D \neq 0$ and $D_1 \neq 0$, $D_2 \neq 0$, $D_3 \neq 0$, therefore the system of equations will have non–zero, unique solution.

Thus,

$$x = \frac{D_1}{D} = \frac{6}{-10} = \frac{-3}{5}$$

$$y = \quad = \frac{-14}{-10} =$$

$$z = \quad = \frac{-12}{-10} =$$

$\frac{6D_3}{5D}$

are the solutions of the given system of equations.

Check–point: Select the correct answer

If $x - 2y + 3z = 6$

$2x + y - 2z = -3$

$-3x + 4y + z = 5$

is a given system of equations,

$$D = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

then in order to get the value of y, we will

(i) replace the 1st column by constants

(ii) replace the 2nd column by constants

(iii) replace the 3rd column by constants

{ **Ans** : (ii) }

Example D:

Determine which of the following system of equations will have a unique solution and also find the solutions in each cases:

(i) $2x - 3y + 4z = -9$

$-3x + 4y + 2z = -12$

$4x - 2y - 3z = -3$

(ii) $x + 2y - z = 0$

$2x + y + 2z = 0$

$x - 3y + z = 0$

(iii) $x + 2y + z = 2$

$2x + y + 2z = 3$

$x - 3y + z = 4$

(iv) $-x + 2y + 7z = 1$

$3x - y + z = 5$

$2x + 5y - 5z = 11$

(v) $x + 2y + 3z = 1$

$3x - y + 2z = 1$

$4x + y + 5z = 2$

Solution:

(i) $2x - 3y + 4z = -9$

$-3x + 4y + 2z = -12$

$4x - 2y - 3z = -3$

Here

$$D = \begin{vmatrix} 2 & -3 & 4 \\ -3 & 4 & 2 \\ 4 & -2 & -3 \end{vmatrix} = 2(-12 + 4) + 3(9 - 8) + 4(6 - 16)$$

$$= -16 + 3 - 40$$

$$= -53 \neq 0$$

Also

$$D_1 = \begin{vmatrix} -9 & -3 & 4 \\ -12 & 4 & 2 \\ -3 & -2 & -3 \end{vmatrix} = -9(-12 + 4) + 3(36 + 6) + 4(24 + 12)$$

$$= 72 + 126 + 144$$

$$= 342$$

$$D_2 = \begin{vmatrix} 2 & -9 & 4 \\ -3 & -12 & 2 \\ 4 & -3 & -3 \end{vmatrix} \begin{aligned} &= 2(36 + 6) + 9\ (9 - 8) + 4\ (9 + 48) \\ &= 84 + 9 + 228 \\ &= 321 \end{aligned}$$

$$D_3 = \begin{vmatrix} 2 & -3 & -9 \\ -3 & 4 & -12 \\ 4 & -2 & -3 \end{vmatrix} \begin{aligned} &= 2(-12 - 24) + 3\ (9 + 48) - 9\ (6 - 16) \\ &= -72 + 171 + 90 \\ &= 189 \end{aligned}$$

Since $D \neq 0$ and $D_1 \neq 0$, $D_2 \neq 0$, $D_3 \neq 0$,

∴ The system of equations will have a non–zero unique solution which are

$$x = \qquad = \frac{342}{-53} = \frac{-342}{53}$$

$$\frac{D_B}{D}$$

$$y = \frac{D_2}{D} = \frac{321}{-53} = \frac{-321}{53}$$

and $$z = \qquad = \frac{189}{-53} = \frac{-189}{53}$$

(ii) $x + 2y - z = 0$

$2x + y + 2z = 0$

$x - 3y + z = 0$

Here $$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} \begin{aligned} &= 1(1 + 6) - 2\ (2 - 2) - 1\ (-6 - 1) \\ &= 7 + 7 \\ &= 14 \neq 0 \end{aligned}$$

Also $D_1 = \begin{vmatrix} 0 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & -3 & 1 \end{vmatrix} = 0$ (expanding by the 1st column)

$D_2 = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 0$

$D_3 = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & -3 & 0 \end{vmatrix} = 0$

Thus, we find that $D \neq 0$ and $D_1 = D_2 = D_3 = 0$

$\therefore$ The system of linear equations will not have a unique solution but in fact, it will have trivial solutions $x = y = z = 0$

(iii) $x + 2y + z = 2$

$2x + y + 2z = 3$

$x - 3y + z = 4$

Here $D = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 0$

= ($C_1 = C_3$ see lesson 5)

Also, $D_1 = \begin{vmatrix} 2 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & -3 & 1 \end{vmatrix}$

$= 2(1 + 6) - 2(3 - 8) + 1(-9 - 4)$

$= 14 + 10 - 13$

$= 11$

$$D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 4 & 1 \end{vmatrix} = 0 \quad (\quad C_1 = C_3)$$

$$D_3 = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 1 & -3 & 4 \end{vmatrix} \begin{aligned} &= 1(4 + 9) - 2(8 - 3) + 2(-6 - 1) \\ &= 13 - 10 - 14 \\ &= -11 \end{aligned}$$

Since $D = 0$ and $D_1 \neq 0$, $D_2 = 0$ and $D_3 \neq 0$,

$\therefore$ the system of equations has no solution

(iv) $-x + 2y + 7z = 1$

$3x - y + z = 5$

$2x + 5y - 5z = 11$

$\because$

Here $$D = \begin{vmatrix} -1 & 2 & 7 \\ 3 & -1 & 1 \\ 2 & 5 & -5 \end{vmatrix} \begin{aligned} &= -1(5 - 5) - 2(-15 - 2) + 7(15 + 2) \\ &= 0 + 34 + 119 \\ &= 153 \neq 0 \end{aligned}$$

Also $$D_1 = \begin{vmatrix} 1 & 2 & 7 \\ 5 & -1 & 1 \\ 11 & 5 & -5 \end{vmatrix} \begin{aligned} &= 1(5 - 5) - 2(-25 - 11) + 7(25 + 11) \\ &= 0 + 72 + 252 \\ &= 324 \end{aligned}$$

$$D_2 = \begin{vmatrix} -1 & 1 & 7 \\ 3 & 5 & 1 \\ 2 & 11 & -5 \end{vmatrix} \begin{aligned} &= -1(-25 - 11) - 1(-15 - 2) + 7(33 - 10) \\ &= 36 + 17 + 161 \\ &= 214 \end{aligned}$$

$$D_3 = \begin{vmatrix} -1 & 2 & 1 \\ 3 & -1 & 5 \\ 2 & 5 & 11 \end{vmatrix} = -1(-11 - 25) - 2\ (33 - 10)\ + 1\ (15+2)$$

$$= 36 - 46 + 17$$

$$= 7$$

Thus, since $D \neq 0$, and $D_1 \neq 0$, $D_2 \neq 0$ and $D_3 \neq 0$,

therefore, the system of linear equations will have non–zero, unique solution which are

$$x = \qquad = \frac{324}{153} = \frac{36}{17}$$

$$y = \frac{D_2}{D} = \frac{214}{153}$$

and $$z = \frac{D_3}{D} = \frac{7}{153}$$

(v) $x + 2y + 3z = 1$

$3x - y + 2z = 1$

$4x + y + 5z = 2$

Here $$D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & 5 \end{vmatrix} = 1(-5 - 2)\ - 2\ (15 - 8)\ + 3\ (3 + 4)$$

$$= -7 - 14 + 21$$

$$= 0$$

Also $$D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 5 \end{vmatrix} = 1(-5 - 2)\ - 2\ (5 - 4)\ + 3\ (1 + 2)$$

$$= -7 - 2 + 9$$

$$= 0$$

$$D_2 = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 1 & 2 \\ 4 & 2 & 5 \end{vmatrix} \begin{aligned} &= 1(5-4) - 1(15-8) + 3(6-4) \\ &= 1 - 7 + 6 \\ &= 0 \end{aligned}$$

$$D_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -1 & 1 \\ 4 & 1 & 2 \end{vmatrix} \begin{aligned} &= 1(-2-1) - 2(6-4) + 1(3+4) \\ &= -3 - 4 + 7 \\ &= 0 \end{aligned}$$

Since $D = 0$ and $D_1 = D_2 = D_3 = 0$, therefore, the given system of equations will have infinitely many solutions.

Check–point: Select the correct answer

For a given system of equations, if $D = 0$ and each $D_i = 0$, then the system of equations have

(i) non–zero, unique solution

(ii) trivial solution

(iii) no solution

(iv) infinitely many solutions

{ **Ans : (ii)** }

INTEXT QUESTIONS 9.1

1. Solve the following system of equations by Cramer's rule

(a) $2x - 4y = 3$
$3x + y = 5$

(b) $x + 2y = 1$
$2x + 5y = 3$

2. Obtain the solutions of the system of equations using Cramer's rule:

(a) $2x + y + 3z = 1$
$x + 4y + 6z = 9$
$4x + 3y + 9z = 5$

(b) $2x - 3y + 2z = 1$
$x + 3y - z = -2$
$x - y + 3z = 3$

(c) $3x - 4y + 5z = -6$

$x + y - 2z = -1$

$2x + 3y + z = 5$

3. Solve the following system of equations:

(a) (i) $3x + 2y = 4$

$2x + y = 3$

(ii) $6x - 3y = -1$

$2x + 2y = -3$

(b) (i) $2x + 3y + 4z = 8$

$3x + y - z = -2$

$4x - y - 5z = -9$

(ii) $x + 3y - z = 4$

$3x - 2y + 5z = -4$

$5x - y - 4z = -9$

(iii) $x + 3y - 2z = 5$

$2x + y + 4z = 8$

$6x + y - 3z = 5$

(iv) $5x - 7y + z = 11$

$6x - 8y - z = 15$

$3x + 2y - 6z = 7$

4. Determine which of the following system of equations will have a unique solution. Also, find the solution in such a case.

(a) $x - 2y = 4$

$-3x + 5y = -7$

(b) $2x - y + z = 0$

$x + y - 2z = 0$

$3x + 2y - z = 0$

(c) $6x + y - 3z = 5$

$x + 3y - 2z = 5$

$2x + y + 4z = 8$

(d) $x - y - z = -5$

$3x - y + 2z = 1$

$2x + z = 2$

(e) $x + 2y + 3z = 1$

$3x - y + 2z = 5$

$4x + y + 5z = 3$

9.6 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY MATRIX METHOD

Consider the system of equations

$$\left.\begin{array}{l} 4x - 3y = 11 \\ 3x + 7y = -1 \end{array}\right\} \qquad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

This system can be expressed in the matrix form as

$$\begin{bmatrix} 4x - 3y \\ 3x + 7y \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad \ldots\ldots\ldots \quad \text{(ii)}$$

If $A = \begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $D = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$

then (ii) reduces to

$$AX = D \qquad \ldots\ldots\ldots\ldots \quad \text{(iii)}$$

Now, $$|A| = \begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix} = 28 + 9 = 37 \neq 0$$

Since $|A| \neq 0$, therefore, A^{-1} exists.

Now multiplying both sides of (iii) on the left by A^{-1}, we get.

$$A^{-1}\ (AX) = A^{-1}\mathrm{D}$$

$\Rightarrow$ $$(A^{-1}\ A)X = A^{-1}D$$

i.e., $$IX = A^{-1}D$$

$\therefore$ $$X = A^{-1}D$$

Hence, $$X = \frac{1}{|A|}\ \mathrm{Adj}A\ D$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{37} \begin{bmatrix} 7 & 3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 77 & -3 \\ -33 & -4 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 74 \\ -37 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So, $x = 2$, $y = -1$ are the non–zero, unique solutions of this system of equations.

Thus, we find that if $\neq$ **0** i.e., if A is non–singular and **Adj *A* *D* ≠ 0**, then the system of equations have **a non–zero unique solution.**

Now, consider the system of equations

$$2x + 5y - 3z = 0$$

$$x - 2y + z = 0$$

$$3x - y - 6z = 0$$

Such a system of equations is called a **homogenous system of linear equations** because in each of these equations, the constant term is zero.

In matrix form, the above system can be written as

$$\begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $AX = 0$

where

$$A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -6 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} -2 & 5 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -6 \end{vmatrix} = 2(12 + 1) -5\ (-\ 6 - 3)\ -3\ (-1 + 6)$$

$$= 26 + 45 - 15$$

$$= 56 \neq 0$$

But $D = 0 \Rightarrow \text{Adj}A\ D = 0$

Thus,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \text{Adj}AD$$

$|A|$

$$= \frac{0}{56} = 0$$

$\therefore \qquad x = 0,\ y = 0,\ z = 0$

i.e., the system of equations will have a trivial solution. Hence, for a **homogenous system of linear equations where $\neq 0$ and Adj.A D = 0,** there will be **only trivial solution $x_1 = x_2 = \cdot\ \cdot = x_n = 0$**

Consider again the homogenous system of equations

$$2x + y - 3z = 0$$

$$x - 2y + z = 0$$

$$3x - y - 2z = 0$$

In matrix form, the above system can be written as

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $AX = 0$

where, $A = 2 \begin{bmatrix} 1 & -3 & \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, \quad X = x \begin{bmatrix} \\ y \\ z \end{bmatrix},$

$$D = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $A = \begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2(4 + 1) -1(-2 - 3) -3(-1 + 6)$

$= 10 + 5 - 15$

$= 0$

Also, $D = 0 \Rightarrow \text{Adj.}A \; D = 0$

Thus $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \text{Adj.}A \; D$$

$$= \frac{0}{0}$$

$\Rightarrow$ The system of equations will have infinitely many solutions which will be non–trivial. Considering the first two equations, we get

$$2x + y = 3z$$

$$x - 2y = -z$$

Solving, we get $x = z$, $y = z$

Now, for any number k, let $z = k$

Then $x = k$, $y = k$ and $z = k$ are the solutions of this system.

Hence, for a system of **homogenous equations, where $|A|$ = 0 and Adj$A$ D = 0,** there will be **infinitely many solutions.**

Till now, we have seen that

(i) if $|A| \neq 0$ and Adj$A$ $D \neq 0$, then the system of equations have a non–zero unique solution.

(ii) if $|A| \neq 0$ and Adj$A$ $D = 0$, then the system of equations have only trivial solution $x = y = z = 0$

(iii) if $|A| = 0$ and Adj$A$ $D = 0$, then the system of equations have infinitely many solutions.

Let us now take up an example of a system of linear equations where $|A| = 0$ and Adj$A$ $D = 0$. Consider the following system of equations

$$x + 2y + z = 5$$

$$2x + y + 2z = -1$$

$$x - 3y + z = 6$$

In matrix form, the above system of equations can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$$

i.e., $AX = D$

where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$

Now, $$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix} \begin{matrix} \\ = 0 \\ = (\because C_1 = C_3) \end{matrix}$$

Also $$\text{Adj}A \; D = \begin{bmatrix} 7 & -5 & 3 \\ 0 & 0 & 0 \\ -7 & 5 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$$ (find Adj$A$ yourself)

$$= \begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix} \neq 0$$

Since $= 0$ and Adj$A$ $D \neq 0$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \text{Adj}A \; D$$

$$= \frac{\begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix}}{0}$$ which is undefined

$\therefore$ the given system of linear equation will have no solution.

Thus, we find that if $|A|$ **= 0 and Adj*A* *D* ≠ 0** then the system of equations will have **no solution.**

We can summarise the above finding as:

For a given system of equations

(i) if $|A| \neq 0$ and **AdjA** $D \neq 0$ then the system of equation will have a **non–zero**, **unique solution.**

(ii) if $|A| \neq 0$ and **Adj*A*** $D = 0$, then the system of equations will have **trivial solutions**

(iii) if $|A| = 0$ and **Adj*A*** $D = 0$, then the system of equations will have **infinitely many solutions.**

(iv) if $|A| = 0$ and **Adj*A*** $D \neq 0$, the the system of equations will have **no solution**

Example E:

Solve the given system of equations using matrix method:

$$2x - 3y = 7$$

$$x + 2y = 3$$

Solution:

The given system of equation is

$$2x - 3y = 7$$

$$x + 2y = 3$$

It can be expressed in the matrix form as

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

i.e., $A\ X = D$ .... (i)

where $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $D = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$$\text{Now,} \quad A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = 4 + 3 = 7 \neq 0$$

$$\therefore \quad A^{-1} \text{ exists}$$

$$\text{Also, Adj } A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$\text{and } A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \text{........ (ii)}$$

From (i), we get

$$X = \mathrm{A}^{-1}D$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix} \qquad (\text{ from (ii) })$$

$$= \begin{bmatrix} 23 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$\therefore$ The solution of the system of equations is $x =$ and $y =$

Example F:

Solve the given system of equations using the matrix method.

$$x + 2y + z = 2$$

$$2x - y + 3z = 3$$

$$x + 3y - z = 0$$

Solution:

The given system of equations can be represented in the matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

i.e., $AX = D$ .... (i)

where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$|A|$

Now $= \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} = 1(1 - 9) -2 (- 2 - 3) +1 (6 + 1)$

$= -8 + 10 + 7$

$= 9 \neq$

Hence, A^{-1} exists

Also, $\text{Adj}A = \begin{bmatrix} -8 & 5 & 7 \\ 5 & -2 & -1 \\ 7 & -1 & -5 \end{bmatrix}$

$$A^{-1} = 1 \frac{1}{|A|} \text{Adj } A$$

$$= \frac{1}{9}\begin{bmatrix} -8 & 5 & 7 \\ 5 & -2 & -1 \\ 7 & -1 & -5 \end{bmatrix}$$

From (i), we have

$$X = A^{-1}\ \mathrm{D}$$

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -8 & 5 & 7 \\ 5 & -2 & -1 \\ 7 & -1 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -1 \\ 4 \\ 11 \end{bmatrix}$$

Hence $x = \frac{-1}{9}$, $y =$ and $z =$ are solutions of the given system of equations.

Check–point

Select the correct answer

The system of equations

$$2x - y + z = 3$$

$$x + 3y - 2z = 1$$

$$x + y + z = 6$$

has a unique solution because

(i) $\neq 0$ and Adj$A$ $D \neq 0$

(ii) $|A| \neq 0$ and Adj$A$ $D = 0$