Senior Secondary Course

MATHEMATICS (311)

1

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विद्याधनम् सर्वधनं प्रधानम्

NATIONAL INSTITUTE OF OPEN SCHOOLING

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Chairman's Message

Dear learner,

As the needs of the society in general, and some groups in particular, keep on changing with time, the methods and techniques required for fulfilling those aspirations also have to be modified accordingly. Education is an instrument of change. The right type of education at right time can bring about positivity in the outlook of society, attitudinal changes to face the new/fresh challenges and the courage to face difficult situations.

This can be very effectively achieved by regular periodic curriculum renewal. A static curriculum does not serve any purpose, as it does not cater to the current needs and aspirations of the individual and society.

For this purpose only, educationists from all over the country come together at regular intervals to deliberate on the issues of changes needed and required. As an outcome of such deliberations, the National Curriculum Framework (NCF 2005) came out, which spells out in detail the type of education desirable/needed at various levels of education – primary, elementary, secondary or senior secondary?

Keeping this framework and other national and societal concerns in mind, we have currently revised the curriculum of Mathematics course at senior secondary level, as per the Common Core Curriculum provided by National Council of Educational Research and Traning (NCERT) and the Council of Boards of School Education in India (COBSE) making it current and need based. Textual material production is an integral and essential part of all NIOS programmes offered through open and distance learning system. Therefore, we have taken special care to make the learning material user friendly, interesting and attractive for you.

I would like to thank all the eminent persons involved in making this material Interesting and relevant to your needs. I hope you will find it appealing and absorbing.

On behalf of National Institute of Open Schooling, I wish you all a bright and Successful future.

(Dr. S. S. Jena)
Chairman, NIOS

A Note from the Director

Dear Learner,

The Academic Department at the National Institute of Open Schooling tries to bring you new programmes every now and then in accordance with your needs and requirements.

The Mathematics course at Senior Secondary level has now been revised as per the Common Core Curriculum developed by COBSE (Council of Boards of School Education) and NCERT (National Council for Educational Research and Training) making it current and need based.

The National Curriculum Framework developed by the National Council for Educational Research and Training was kept as a reference point. Leading experts in the subject of the country were involved and with their active involvement, study materials based on the new curriculum have been updated.

Old, outdated information has been removed and new, relevant things have been added.

I am happy to place this new revised study material in Senior Secondary Mathematics in your hand. I hope you will find the new material that is now in your hands interesting and exciting. Any suggestions for further improvement are welcome.

Let me wish you all a happy and successful future.

(Dr. Kuldeep Agarwal)
Director (Academic), NIOS

Letter to Learner

Dear Learner,

I welcome all of you to the Senior Secondary course in Mathematics. It gives me a great pleasure that you have opted for Mathematics as one of your subjects of study. Study of Mathematics contributes to the development of precision, rational and analytical thinking, reasoning and scientific temper. These qualities, no doubt, are essential for success in life, whatever career you choose. Mathematics is important fields in many professions like-Engineering, Architecture, Statisticians, Commerce and Accountancy related profession, Econometrics etc.

The present curriculum in Mathematics has been divided into two Parts.

***Part-1** contains 19 Lessons under five modules. These modules are Sets, Relations and Functions; Sequences and Series; Algebra-I; Co-ordinate Geometry; and Statistics & Probability. Similarly **Part-2** contains 19 Lessons from five modules. These modules are Algebra-II; Relations and Functions; Calculus; Vectors and Three Dimensional Geometry; and Linear Programming & Mathematical Reasoning.*

*All efforts have been made to give related illustrations and examples for your better understanding. You should go through all solved examples and try to solve all problems under **"Check Your Progress"** and **"Terminal Exercise"** independently given at the end of each lesson.*

If you face any difficulty, please do not hesitate to write to me. Your suggestions and doubts are most welcome.

Wish you a bright future.

Yours,

Dr. Rajendra Kumar Nayak

Academic Officer (Mathematics), NIOS

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MATHEMATICS IN INDIA

Mathematics is considered to be a system of logic. It is the subject of systematic study of quantitative phenomena around us. It is based on certain logical connotation of numbers and integral part of human civilisation. Mathematics is a creative activity and is one of the most useful, fascinating and stimulating divisions of human knowledge. It is a process of managing and communicating information and has the power to predict and provide solutions to practical problems as well as enabling the individual to create new imaginative worlds to explore. We use Mathematics in everyday life, in science, in industry, in business and in our free time. Mathematics education is concerned with the acquisition, understanding and application of skills. Mathematical literacy is of central importance in providing the learner with the necessary skills to live a full life as a child and later as an adult. Society needs people who can think and communicate quantitatively and who can recognize situations where Mathematics can be applied to solve problems. It is necessary to make sense of data encountered in the media, to be competent in terms of vocational mathematical literacy and to use appropriate technology to support such applications.

The earliest traces of Mathematical knowledge in the Indian subcontinent appear right from the ancient period. The three main contributions in the field of Mathematics were the notation system, the decimal system and the use of zero. **Aryabhata** worked on the place value system using letters to signify numbers. The most significant contribution of Indian Mathematicians were the introduction of zero (0) to the Mathematics, which is understood as "nothing". The concept itself was one of the most significant inventions in the ascent of humans for the growth of culture and civilization. **Brahmagupta** introduced negative numbers and operations on zero into Mathematics. He wrote Brahm, Sputa Siddantika through which Arabs came to know them the Mathematical system. **Bhaskaracharya** otherwise known as Bhaskara-II was one of the most powerful and creative mathematicians of ancient India. He contributed the idea of infinity, negative numbers and Zero rules in the field of Mathematics.

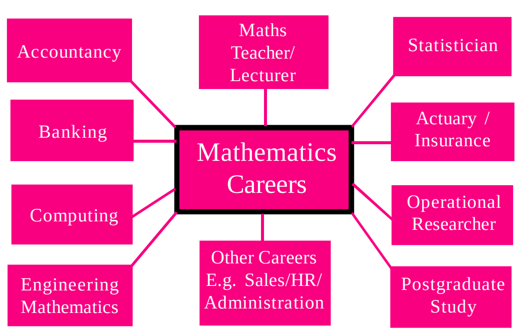
Baudhayan was the first one ever to arrive at several concepts in Mathematics, which were later rediscovered by the western world. The value of **Pi** (®) was first calculated by him. Pythagoras theorem is already found in Baudhayan's Sulvasutra, which was written several years before the age of Pythagoras. **Mahaviraacharya** another prominent Mathematician in India contributed on trigonometric functions and cubic equations. He described Fraction, algebraic equations, logarithms and exponents in a very interesting manner. **Sridhar** another highly esteemed Indian Mathematician who has contributed to the solution quadratic equation.

Astronomy is the applied Mathematics that used Mathematical equations to describe the universe or to predict various aspects of the universe. Mathematics is and always has been of central importance to astronomy. In ancient India, **Nagarjuna** a famous astronomer as well as mathematician used different equations of Mathematics to describe motion of stars and planets. **Varahamihira** was an Indian astrologer whose main work was on mathematical astronomy. He discovered a version of Pascal's triangle and worked on magic square much before Pascal's period. He was also aware of gravity over a millennium before Newton.

During modern period, **Srinivasa Ramanujan** was one of India's greatest Mathematical geniuses. Ramanujan's contribution was mostly on Number theory and he had obtained over mastery the number 1729. Ramanujan discovered that 1729 is the smallest number which could be represented in two different ways as a sum of two cubes. After that,1729 has been called "Ramanujan-Hardy number". **Shakuntala Devi** is a world renowned Indian Mathematician. She has been nicknamed 'Human Computer" because of her innate ability to solve complex mathematical problems without using any kind of aid.

The main goal of Mathematics education in schools is the Mathematisation of the child's thinking. Clarity of thought and pursuing assumptions to logical conclusions is central to the Mathematical enterprise. There are many ways of thinking, and the kind of thinking one learns in Mathematics is an ability to handle abstractions, and an approach to problem solving. The procedural knowledge i.e. to solve problems for getting answers only is not enough. At the Senior Secondary stage one needs to develop both procedural as well as conceptual knowledge for better understanding of Mathematics. Success in Mathematics requires more than just computational skills. It also requires the ability to apply Mathematics in solving problems, to more information from a variety of sources. Senior Secondary stage is the launching pad from which the learner is guided towards career choices, whether learners aspire in university education or otherwise. By this time, the learners' interests and aptitude have been largely determined, and Mathematics education in these two years can help in sharpening their abilities.

Many university degrees require Mathematics as a pre requirement. Learners who choose not to take Mathematics seriously or to ignore it in secondary and senior secondary level forfeit many future career opportunities that they could have. They essentially turn their backs on more than half the job market. The importance of mathematics for potential careers cannot be over emphasized. To get degrees in the following areas i.e. the Physical Sciences (Chemistry, Physics, Engineering), the life and health sciences (Biology, Psychology, Pharmacy, Nursing, Optometry), Social Sciences, including Anthropology, Communications, Economics, Linguists, Education, Geography the Tech Sciences, like Computer Science, Networking, Software development, Business and Commerce Medicine, Actuarial science, used by insurance companies, one need to have good knowledge of Mathematics and Statistics. Learning Mathematics at Senior Secondary stage helps to choose a career in several areas. The diagram below depicts some of these fields.



How to Use the Study Materials

You should note that taking admission in the National Institute of Open Schooling (NIOS), you have entered in a system where studying is different than in a formal school.

You Are a Self Learner Now

In a school, a teacher is always available to take classes, clarify doubts, guide and encourage. There you also must be discussing with your peer-group, going to the library, doing practicals, participate in co-curricular activities, watching educational TV, Radio programmes etc. All this was also contributing to your learning.

However, in the NIOS, there is no teacher available and you have to learn on your own, It means that you have become a self-learner. Responsibilities of a self-learner are much more than a normal learner who is dependent on a teacher; but at the same time it is also challenging. Here it is solely you, who is responsible for your learning. It means that you have to organize your study, learn regularly, keep up your motivation and achieve your goal.

Understanding Your Learning Material

The NIOS will help you by providing learning material, part of which is in your hands at this moment. We call them learning materials because these are different from the textbooks you have read in your schools. Here the textbooks and the teachers have been put together. You will find that the contents, concepts and topics have been explained here in a way the teacher does in a classroom. You will also find examples and illustrations to help you understand the things properly.

This is the reason you will find them bulky, but let this shall not frighten you. You will find a few sections in your lessons. Let us know the purpose of these:

Introduction: It introduces the topic to you.

Objectives: Here you will find the list of objectives, which you shall achieve after learning the lesson. You can actually check whether or not you have achieved these because these are presented in measurable terms.

Check Your Progress: These will appear after every section of the lesson. They will contain objective, very short answer, short answer and long answer type questions based on each lesson. This will help you to find whether you have learnt the section or not. Your will find the key to these questions at the end of the lesson. If you are able to answer the questions, then you can proceed further otherwise you should learn the section again.

Let us Sum up: Here you will find the summary of the main concepts of the lesson for recapitulation and revision.

Terminal Exercise: Here short answer and long answer type questions are given to help you enhance your learning and give you an opportunity to practice for examination.

Answers: Answers of all “Check Your Progress” and “Terminal Exercise” are given at the end of each lesson. Some hints are also given in case of difficult questions.

Apart from the learning material, you will also get, sample papers, previous year question papers, etc., for help in your studies and for practice for examination.

Personal Contact Programmes

A few sessions/ classes would be provided to you at the center. You should, however, note that those are not meant for teaching you as it happens in a normal school. Here you will have an opportunity to clarify your doubts, solve your problems and get guidance and advice on your study. So go to the classes well prepared to get maximum benefit.

Use of Audio and Video Programmes

NIOS has also developed a few audio and video programmes, which will be very interesting for you and will help you in your studies. You can take copies of this from your Center.

Planning and Organizing Your Study

Let me also give you some tips for planning and organizing your study.

First of all, you need to understand that there is no substitute for hard work. Harder you work better you achieve. Also, there are no short cuts to success. If someone has given you assurance of helping you in passing, then that will not work as there will be strict checking and vigilance in the exam. Even if you are able to get through somehow, you will not gain any learning. So in order to succeed in an honest way and get benefit of your learning in your life, you need to learn.

As you must have understood by now, the NIOS provides a lot of freedom and flexibilities in your study. For example, there is no need to take examination in all the subjects simultaneously. So, first of all, think about the time available with you and decide whether you wish to study all the subjects together or you wish to learn them one after the other. Just gambling in all the subjects will bring you to a situation where you will not be able concentrate on any subject.

Now fix up a time for study, evening, morning or daytime whichever is convenient Draw a timetable giving justified time to the subjects you have taken up for study and follow this as far as possible. While you study, underline the concepts you feel are important. You should study from NIOS learning material. In addition you may read other-books, if you have time. However, for your purpose, this material will be sufficient. Keep a copy each for the subjects you are preparing. Note down the points you did not understand. Discuss these with your parents, friends or teachers at the centres.

Solve all the exercises, and practice exercises of each module appearing in the material. This will not only help you in learning, but also provide practice for your examination. You may also like to solve sample papers and previous years' question papers. Show your answers to your parents and friends and discuss with them.

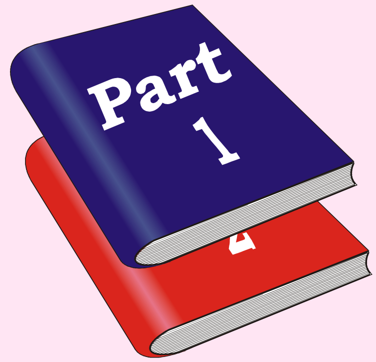
These were a few hints to help you. You may find some other techniques which may work better with you. Follow those, if you like. I am sure you will be successful in your endeavor.

Course Overview

Part-1: [For Tutor Marked Assignment]

Module-I: Sets, Relations and Functions

1. Sets
2. Relations and Functions-I
3. Trigonometric Functions-I
4. Trigonometric Functions-II
5. Relation between Sides and Angles of A triangle



Module- II: Sequences and Series

6. Sequences and Series
7. Some Special Sequences

Module -III: Algebra-I

8. Complex Numbers
9. Quadratic Equations and Linear inequalities
10. Principle of Mathematical Induction
11. Permutations and Combinations
12. Binomial Theorem

Module-IV: Co-ordinate Geometry

13. Cartesian System of Rectangular Co-ordinates
14. Straight Lines
15. Circles
16. Conic Sections

Module-V: Statistics and Probability

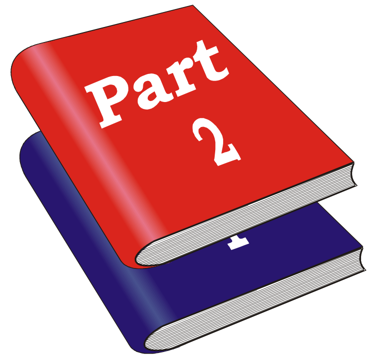
17. Measures of Dispersion
18. Random Experiments and Events
19. Probability

IMPORTANT NOTE: All contents of Part-1 will be assessed/examined through Tutor Marked Assignment (TMA). TMA is compulsory and contains 20% marks as weightage. The Marks/ Grades of TMA will be reflected in the mark sheet.

Part-2: [For Public Examination]

Module-VI: Algebra-II

20. Matrices
21. Determinants
22. Inverse of a Matrix and its Applications



Module-VII: Relations and Functions

23. Relations and Functions-II
24. Inverse Trigonometric Functions

Module-VIII: Calculus

25. Limits and Continuity
26. Differentiation
27. Differentiation of Trigonometric functions
28. Differentiation of Exponential and Logarithmic functions
29. Application of Derivatives
30. Integration
31. Definite Integrals
32. Differential Equations

Module-IX: Vectors and Three Dimensional Geometry

33. Introduction to Three Dimentional Geometry
34. Vectors
35. Plane
36. Straight Line

Module-X: Linear Programming and Mathematical Reasoning

37. Linear Programming
38. Mathematical Reasoning

IMPORTANT NOTE: All Contents of Part-2 will be assessed/examined through Public/Final Examination. Public Examination is compulsory and contains 80% marks as weightage.

CONTENTS

CURRICULUM OF SENIOR SECONDARY COURSE IN MATHEMATICS (311)

1. RATIONALE

Mathematics is an important discipline of learning at the Senior Secondary stage. It helps the learner in acquiring decision making ability through its applications to real life both in familiar and unfamiliar situations. It predominantly contributes to the development of precision, rational reasoning and analytical thinking. In the Senior Secondary stage is most crucial where learners for the first time move towards diversification. At this stage, the learners start thinking to take important decisions concerning their future career by choosing suitable courses. It is the stage, from where learners would either go for higher academic education in Mathematics or for Professional courses or it may be the end of their academic career. One of the basic aims of learning Mathematics at Senior Secondary level is to be developed problem solving skills and quantificational experiences around the learners. It includes a way of doing things, and the ability and attitude to formulate and solve problems (NCF, 2005). The idea is to allow the learner to realize how and why Mathematics is all around us. In view of these facts, it is important to make Mathematics Education at this level broad based and meaningful. The revised curriculum in Mathematics has been designed to meet the needs of diversity of learners. The contents and design of the revised curriculum broadly based on the common curriculum suggested by Council of Boards of School Education in India (COBSE). In order to relate Mathematics to real life and work situations of NIOS learners, greater emphasis has been put on applications of various concepts.

2. OBJECTIVES

The main objectives of learning Mathematics at Senior Secondary level are to enables the learners to:

- acquire knowledge and understanding of basic concepts, facts, principles, terms, symbols and processes of Mathematics.
- acquire the skills of quantification of experiences around them and make linkage with their life.
- articulate logically and use the same to prove results.
- convert the word problems in the mathematical forms and solve them.
- introduce learners to different ways of processing the given data and help them in arriving at conclusions
- provide learners with an appreciation of the wide variety of application of Mathematics and equip them with the basic device that enable such application
- develop appreciation for the influence and exquisiteness of Mathematics for its applications in Science, Commerce, Economics and daily life
- apply mathematical knowledge and skills to solve variety problems.
- develop positive attitude towards Mathematics and its application.

3. COURSE STRUCTURE:

The present curriculum in Mathematics has been distributed into two parts and ten modules. Part-1 consists of five modules namely as: Sets; Relation and Function; Sequences and Series; Algebra-I; Co-ordinate Geometry; Statistics and Probability. Similarly five modules are inside the Part-2 namely as: Algebra-II; Relations and Functions; Calculus; Vectors and Three Dimensional Geometry; Linear Programming and Mathematical Reasoning. Each module has been divided further into different lessons.

The number of Lesson and suggested study time allotted for each module is as follows:

Part-1

Module/ Lesson	No. of Lessons	Study Time (in hours)
Module-I: Sets, Relations and Functions	05	30
1. Sets		
2. Relations and Functions-I		
3. Trigonometric Functions-I		
I4. Trigonometric Functions-II		
5. Relation between Sides and Angles of a Triangle		
Module- II: Sequences and Series	02	15
6. Sequences and Series		
7. Some Special Sequences		
Module -III: Algebra-I	05	30
8. Complex Numbers		
9. Quadratic Equations and Linear Inequalities		
10. Principle of Mathematical Induction		
11. Permutations and Combinations		
12. Binomial Theorem		
Module-IV: Co-ordinate Geometry	04	30
13 Cartesian System of Rectangular Co-ordinates		
14 Straight Lines		
15 Circles		
16 Conic Sections		
Module-V: Statistics and Probability	03	15
17 Measures of Dispersion		
18.Random Experiments and Events		
19.Probability		
Total	**19**	**120**

Part-2

Module/ Lesson	No. of Lessons	Study Time (in hours)
Module-VI: Algebra-II	03	30
20. Matrices		
21. Determinants		
22. Inverse of a Matrix and its Applications		
Module-VII: Relations and Functions	02	30
23. Relation and Functions-II		
24. Inverse Trigonometric Functions		
Module-VIII: Calculus	08	60
25. Limits and Continuity		
26. Differentiation		
27. Differentiation of Trigonometric Functions		
28. Differentiation of Exponential and Logarithmic Functions		
29. Application of Derivatives		
30. Integration		
31.Definite Integrals		
32. Differential Equations		
Module-IX: Vectors and Three Dimensional Geometry	04	30
33. Introduction to Three Dimensional Geometry		
34. Vectors		
35. Plane		
36. Straight Line		
Module-X: Linear Programming and Mathematical Reasoning	02	30
37. Linear Programming		
38. Mathematical Reasoning		
Total	**19**	**180**

4. COURSE DESCRIPTION

Part-1

Module I : Sets, Relation and Function.

Lesson 1 : Sets

Sets and their representations, Classification of Sets, Sub- sets, Intervals as subsets of real numbers, Power Set , Universal set, Venn Diagram, Difference of sets, Complement of a set and its properties, Union and Intersection of sets,

Lesson 2 : Relations and Functions -I

Cartesian product of two sets, Cartesion product of the reals with itself, Definition of relation, Domain, Co-domin and range of a relation, Definition of a Function, Domain and Co-domain and Range of a Function, Graphical representation of a function, some special functions, sum, difference, product and quo tient of functions.

Lesson 3 : Trigonometric Functions -I

Circular measure of angle, Trigonometric functions, Trigonometric functions of some specific real numbers, Graphs of Trigonometric functions, Periodicity of the Trigonometric functions,

Lesson 4 : Trigonomertic Functions-II

Addition and Multiplication of Trigonometric Functions, Trans formation of products into sums and vice versa, Trigonometric functions of multiples and submultiples of angles, Trigonometric Equations,

Lesson 5 : Relation between sides and Angles of a Triangle

Sine formula, Cosine formula, Projection formula, Simple applications of sine and cosine formula

Module-II : Sequences and Series

Lesson 6 : Sequences and Series

Sequences, Arithmetic Progression, Arithmetic Mean, Gemetric Progression, General terms of A.P and G.P., sum of n terms of a A.P. and G.P. Infinite G.P. and its sum, Geometric Mean, relation between A.M. and G.M.

Lesson 7 : Some Special Sequences

Series, sum of n terms of the special series: $\sum n, \sum n^2$ and $\sum n^3$

Module-III : Algebra-I

Lesson 8 : Complex Numbers

Understanding complex numbers, Powers of i, Congugate of a comolex number, Geometrical representation of a Complex number, Modulus of a Complex number, Equality of a Complex number, Addition and substraction of complex numbers, Polar representation and Argument of a Complex Number; Multiplication and Division of two Complex numbers, Square Root of a Complex number.

Lesson 9 : Quadratic Equations and Linear Inequalities

Roots of a Quadratic Equation, Solution of Quadratic Equation by factorization and by quadratic formula; Relation between roots and cofficients of a Quadratic equation, Fundamental theorem of algebra; Linear inequalities, Algebraic solutions of linear inequalities in one variable and their representation on number line. Graphical solution of linear inequalities in two Variables. Solution of system of linear inequalities in two variables-graphically.

Lesson 10 : Principle of Mathematical Induction

Understanding Statement, Principle of Mathematical induction and simple applications.

Lesson 11 : Permutations and Combinations

Fundamental Principle of Counting , Factorial n$\left(n!or\,\underline{|n}\right)$; Permutations and Combinations, derivations of formulae and their connections, simple applications;

Lesson 12 : Bionomial Theorem

Binomial theorem for a natural exponent; General and middle term in a binomial expansion, simple Applications.

Module IV : Co-ordinate Geometry

Lesson 13 : Cartesian System of Rectangular Co- ordinates

Rectangular Co-rdinate axes, Distance between two points, Section formula, Area of a triangle, Condition for collinearity of Three points, Inclination and Slope of a line, slope of a line joining two distinct points, conditions for Parallelism and Perpendicularity of lines, Intercepts made by a line on axes, Angle between two lines, shifting of origin.

Lesson 14 : Straight Lines

Straight line paralled to an axis, Straight line in various standerd forms, General equation of first degree in two variables, distance of a given point from a given line, Equation of Parallel or Perpendicular lines, Equation of family of lines passing through the point of intersection of two lines.

Lesson 15 : Circles

Defining Circle, Equation of a Circle in Standard form, General equation of a circle.

Lesson 16 : Conic Sections

Sections of a cone, Ellipse, parabola, hyperbola, Rectangular Hyperbola,

Standar equations and simple properties of parabola, ellipse and hyperbola

Module V : Statistics and Probability

Lesson 17 : Measures of Dispersion

Understanding dispersion, Measure of dispersion, Mean Deviation, Variance and Standard Deviation of ungrouped / grouped data, Analysis of frequency distributions with equal means but different variances.

Lesson 18 : Random Experiments and Events

Random experiments: out comes, sample space, Events: occurrence of events, 'not, 'and' & 'or'events, exclusive events mutrally exclusive events, Independent and dependent events, equally likely events

Lesson 19 : Probability

Events and their probability, Calculation of probability - using Permutation and Combination, complement of an event, addition and multiplication law of Porbability, Conditional probability, independent events, law of probability. Baye's theorem, Random variable. and its probability distribution, Mean and variance of random varianle, Bernoulli trials and Binomial distribution.

Part-2

Module-VI : Algebra-II

Lesson : 20 Matrices

Matrices ann their representations; order, equality and types of matrices, Zero matrix, Transpose of a matrix, Symmetric and Skew-symmetric matrices, Addition, substraction, Multiplication and Scalar multiplication of matrices, Simple properties of addition, subtration, multiplication and scalar Multiplication, Invertible matrices, Elementry operations, Inverse of a matrix by Elementary operations.

Lesson 21 : Determinants

Determinants of a square matrix (upto 3 x 3 matrices), Properties of Determinants, Minors, Cofactors, Evaluation of a determinants using properties, Application of determinants.

Lesson 22 : Inverse of a Matrix and Its Applications

Singular and Non - singular matrix, Adjoint and Inverse of a matrix, solution of a system of linear equations, criterian for consistency of a system of equations, Solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix

Module VII : Relations and Functions

Lesson 23 : Relations and Functions-II

Understanding relation, Typs of relation, Equivalance relation, Ont to One and Onto functions, Composition of functions, Inverse of a function, Binary operations,

Lesson 24 : Inverse Trigonometric Functions

Definition, Range, Domain, Principle value branches, Graph of inverse trigonometric functions, Elementery properties of Inverse trigonometric functions.

Module VIII : Calculus

lesson 25 : Limit and Continuity

Limit of a function, Left hand and right hand Limits, Basic theorems of Limits, limit of some important functions, Continuity of a function at a point, Properties of continious functions,

Lesson 26 : Differentiation

Derivative of a function, velocity as limit, Geometrical interpretation of $\frac{dy}{dx}$, derivative of constant function, derivative of a function from first principle, Algebra of derivatives, derivatives of sum and difference of functions, and Product of functions, Quotient rule, chain rule

Lesson 27 : Differentiation of Trigonometric Functions

Derivative of Trigonometric Functions. Derivatives of Inverse Trigonometric Functions, Second Order derivatives.

Lesson 28 : Differentiation of Exponential and Logarithmic Functions

Derivatives of Exponential and Logarithmic functions, Second order derivatives. Derivative of Parametric functions, Second order derivative of Parametric functions, Rolle's theorem

Lesson 29 : Application of Derivatives

Rate of change of quantities, Approximations, Slope of Tangent and Normal, Equation of Tangent and Normal to a curve, Mathematical formulation of Rolle's theorem, Langrange's mean value theorem, Increasing and Decreasing functions, relation between the sign of the derivative and Monotonicity of function, Maximum and minimum values of a function, conditions for maximum and minimum; Use of second derivative for determination of maximumand minimum values of a function, Application of Maxima and Minima,

Lesson 30 : Integration

Integration as inverse of differentiation, Properties of integrals, Techniques of integration, Integration by parts, integration of the form $\int e^x \left[f(x) + f^1(x) \right] dx$, integration by using partial fractions,

Lesson 31 : Definite Integrals

Definite integral as a limit of sum, evaluation of definite integral by subsitution, Basic properties of definite integrals, Application of integration,

Lesson 32 : Differential Equations

Definition, Order and Degree, Linear and Non-linear differential equations, Formations of differential equations, General and Particular solution of a differential equation. Techniques of solving a differential equation.

Module IX : Vectors and Three Dimensional Geometry

Lesson 33 : Introduction to Three Dimensional Geometry

Coordinate system and coordinates of a point on space, Distance between two points, coordinates of a point of division of a line segment.

Lesson 34 : Vectors

Scalars and Vectors, Vector as a directed line segment, Classification of vectors, addition of vectors, position vector of a point, Negative of a vector, Components of a vector, multiplication of a vector by a scalar, Co-planarity of vectors, resolution of a vector, section formula. Direction cosine and ratios of a vector. Scalar and vector product of vectors, Scalar Tripple Product.

Lesson 35 : Plane

Vector equation of a Plane, Equation of a plane in normal form, conversion of vector form into cartesian form, equation of a plane passing through a given point and perpendicular to a given vector, equation of plane passing through three non-collinear points, equation of plane in the intercept form. Angle between two planes, distane of a point from plane,

Lesson 36 : Straight Line

Vector equation of a line, Reduction of the equations of a line in to symmetric form, Perpendicular distance of a point from a line, angle between a line and a plane, condition of coplanarity of two lines.

Module X : Linear Programming & Mathematical Reasoning

Lesson 37 : Linear Programming

Introduction , definition of various terms involved in Linear Programming, Formulation of Linear Programming Problem, Geometric approach of linear programming Problem, Solution of Linear Programming Problems,

Lesson 38 : Mathematical Reasoning

Mathematically acceptable statements, connecting words/ phrases Consolidating the understanding of "if and only if" (necessary and sufficient) condition, "Implies" ,"and/or," "implied by" "and","or", "there exists" and their use through variety of example related to real life and Mathematics, Validating the statements involving the connecting words - difference between contradiction, Converse and Contrapositive.

5. SCHEME OF STUDY

The course in Mathematics provides you with package of learning opportunities which comprise of:

- Printed Self Learning Material (SLM) in two parts i.e. Part-1 and Part-2.
- Supplementary Materials in the form of Audio and Video Programmes.
- Video tutorials in Mathematics available on the NIOS website (www.nios.ac.in) as well as YouTube. The links of these tutorials have been mentioned within the SLM in the concerned lesson.
- 30 Personal Contact Programme (PCP) sessions at your study centre. Please contact your study centre for the PCP schedule
- Apart from Face-to-Face Personal Contact Programme (PCP) at your study centre, live Personal Contact Programmes (PCPs) through audio streaming are webcast on Mukta Vidya Vani, which can be accessed through NIOS website (www.nios.ac.in).

6. SCHEME OF EVALUATION

The learner will be assessed through Continuous and Comprehensive Evaluation (CCE) in the form of Tutor Marked Assignment (TMA) as well as Public Examination. The following table shows the details:

Mode of Evaluation	Syllabus/Contents	Duration	Weightage
Tutor Marked Assignment (TMA)	All Contents under SLM Part-1	Self Paced	20%
Public/Final Examination	All Contents under SLM Part-2	3 Hours	80%

1

SETS

Let us consider the following situation : One day Mrs. and Mr. Mehta went to the market. Mr. Mehta purchased the following objects/items. "a toy, one kg sweets and a magazine". Where as Mrs. Mehta purchased the following objects/items. "Lady fingers, Potatoes and Tomatoes".

In both the examples, objects in each collection are well defined. What can you say about the collection of students who speak the truth ? Is it well defined? Perhaps not. A set is a collection of well defined objects. For a collection to be a set it is necessary that it should be well defined.

The word well defined was used by the German Mathematician George Cantor (1845- 1918 A.D) to define a set. He is known as father of set theory. Now-a-days set theory has become basic to most of the concepts in Mathematics. In this lesson we shall discuss some basic definitions and operations involving sets.

OBJECTIVES

After studying this lesson, you will be able to :

- define a set and represent the same in different forms;
- define different types of sets such as, finite and infinite sets, empty set, singleton set, equivalent sets, equal sets, sub sets and cite examples thereof;
- define and cite examples of universal set, complement of a set and difference between two sets;
- define union and intersection of two sets;
- represent union and intersection of two sets, universal set, complement of a set, difference between two sets by Venn Diagram;

EXPECTED BACKGROUND KNOWLEDGE

- Number systems,

1.1 SOME STANDARD NOTATIONS

Before defining different terms of this lesson let us consider the following examples:

(i) collection of tall students in your school.	(i) collection of those students of your school whose height is more than 180 cm.
(ii) collection of honest persons in your colony.	(ii) collection of those people in your colony who have never been found involved in any theft case.
(iii) collection of interesting books in your school library.	(iii) collection of Mathematics books in your school library.
(iv) collection of intelligent students in your school.	(iv) collection of those students in your school who have secured more than 80% marks in annual examination.

In all collections written on left hand side of the vertical line the term tallness, interesting, honesty, intelligence are not well defined. In fact these notions vary from individual to individual. Hence these collections can not be considered as sets.

While in all collections written on right hand side of the vertical line, 'height' 'more than 180 cm.' 'mathematics books' 'never been found involved in theft case,' ' marks more than 80%' are well defined properties. Therefore, these collections can be considered as sets.

If a collection is a set then each object of this collection is said to be an element of this set. A set is usually denoted by capital letters of English alphabet and its elements are denoted by small letters.

For example, A = {toy elephant, packet of sweets, magazines.}

Some standard notations to represent sets :

N : the set of natural numbers

W : the set of whole numbers

Z : the set of integers

Z^+: the set of positve integers

Z^-: the set of negative integers

Q : the set of rational numbers

I : the set of irrational numbers

R : the set of real numbers

C : the set of complex numbers

Other frequently used symbols are :

$\in$: 'belongs to'

$\notin$: 'does not belong to'

$\exists$: There exists, $\nexists$: There does not exist.

For example N is the set of natural numbers and we know that 2 is a natural number but -2 is not a natural number. It can be written in the symbolic form as $2 \in N$ and $-2 \notin N$.

1. 2 REPRESENTATION OF A SET

There are two methods to represent a set.

1.2.1 (i) Roster method (Tabular form)

In this method a set is represented by listing all its elements, separating them by commas and enclosing them in curly bracket.

If V be the set of vowels of English alphabet, it can be written in Roster form as :

V = { a, e, i, o, u}

(ii) If A be the set of natural numbers less than 7. then

A={1, 2, 3, 4, 5, 6}, is in the Roster form.

Note : To write a set in Roster form elements are not to be repeated i.e. all elements are taken as distinct. For example if A be the set of letters used in the word mathematics, then

A = {m, a, t, h, e, i, c, s}

1.2.2 Set-builder form

In this form elements of the set are not listed but these are represented by some common property.

Let V be the set of vowels of English alphabet then V can be written in the set builder form as:

V = {x : x is a vowel of English alphabet}

(ii) Let A be the set of natural numbers less than 7. then $A = \{x : x \in N \text{ and } x < 7\}$

Note : Symbol ':' read as 'such that'

Example: 1.1 Write the following in set -builder form :

(a) $A=\{-3,-2,-1,0,1,2,3\}$ (b) $B = \{3,6,9,12\}$

Solution : (a) $A = \{x : x \in Z \text{ and } -3 \le x \le 3\}$

(b) $B = \{x : x = 3n \text{ and } n \in N, n \le 4\}$

Example: 1.2 Write the following in Roster form.

(a) $C = \{x : x \in N \text{ and } 50 \le x \le 60\}$

(b) $D = \{x : x \in R \text{ and } x^2 - 5x + 6 = 0\}$

Solution : (a) C ={50, 51, 52,53,54,55,56,57,58,59,60}

(b) $x^2 - 5x + 6 = 0$

$\Rightarrow \quad (x-3)(x-2)=0 \quad \Rightarrow \quad x=3,2.$

$\therefore \quad D = \{2,3\}$

1.3 CLASSIFICATION OF SETS

1.3.1 Finite and infinite sets

Let A and B be two sets where

A = {x : x is a natural number}

B = {x : x is a student of your school}

As it is clear that the number of elements in set A is not finite while number of elements in set B is finite. A is said to be an infinite set and B is said to be a finite set.

A set is said to be finite if its elements can be counted and it is said to be infinite if it is not possible to count upto its last element.

1.3.2 Empty (Null) Set : Consider the following sets.

$A = \{x : x \in R \text{ and } x^2 + 1 = 0\}$

B = {x : x is number which is greater than 7 and less than 5}

Set A consists of real numbers but there is no real number whose square is -1. Therefore this set consists of no element. Similiarly there is no such number which is less than 5 and greater than 7. Such a set is said to be a null (empty) set. It is denoted by the symbol ϕ or { }

A set which has no element is said to be a null/empty/void set, and is denoted by ϕ. or {}

1.3.3 Singleton Set : Consider the following set :

A = {x : x is an even prime number}

As there is only one even prime number namely 2, so set A will have only one element. Such a set is said to be singleton. Here A = $\{2\}$.

A set which has only one element is known as singleton.

1.3.4 Equal and equivalent sets : Consider the following examples.

(i) $A = \{1,2,3\}$, $B = \{2,1,3\}$ (ii) $D = \{1,2,3\}$, $E = \{a,b,c\}$.

In example (i) Sets A and B have the same elements. Such sets are said to be equal sets and it is written as A = B. In example (ii) sets D and E have the same number of elements but elements are different. Such sets are said to be equivalent sets and are written as $A \approx B$.

Two sets A and B are said to be equivalent sets if they have same number of elements but they are said to be equal if they have not only the same number of elements but elements are also the same.

1.3.5 Disjoint Sets : Two sets are said to be disjoint if they do not have any common element. For example, sets A= { 1,3,5} and B = { 2,4,6 } are disjoint sets.

Example 1.3 Given that $A = \{2, 4\}$ and $B = \{x : x \text{ is a solution of } x^2 + 6x + 8 = 0\}$

Are A and B disjoint sets ?

Solution : If we solve $x^2 + 6x + 8 = 0$, we get

$x = -4, -2$. $\therefore$ $B = \{-4, -2\}$ and $A = \{2, 4\}$

Clearly, A and B are disjoint sets as they do not have any common element.

Example 1.4 If $A = \{x : x \text{ is a vowel of English alphabet}\}$

and $B = \{y : y \in N \text{ and } y \leq 5\}$ Is (i) $A = B$ (ii) $A \approx B$?

Solution : $A = \{a, e, i, o, u\}$, $b = \{1, 2, 3, 4, 5\}$.

Each set is having five elements but elements are different

$\therefore$ $A \neq B$ but $A \approx B$.

Example 1.5 Which of the following sets

$A = \{x : x \text{ is a point on a line}\}$, $B = \{y : y \in N \text{ and } y \leq 50\}$ are finite or infinite ?

Solution : As the number of points on a line is uncountable (cannot be counted) so A is an infinite set while the number of natural numbers upto fifty can be counted so B is a finite set.

Example 1.6 Which of the following sets

$A = \{x : x \text{ is irrational and } x^2 - 1 = 0\}$.

$B = \{x : x \in z \text{ and } -2 \leq x \leq 2\}$ are empty?

Solution : Set A consists of those irrational numbers which satisfy $x^2 - 1 = 0$. If we solve $x^2 - 1 = 0$ we get $x = \pm 1$. Clearly ± 1 are not irrational numbers. Therefore A is an empty set.

But $B = \{-2, -1, 0, 1, 2\}$. B is not an empty set as it has five elements.

Example 1.7 Which of the following sets are singleton ?

$A = \{x : x \in Z \text{ and } x - 2 = 0\}$ $\qquad$ $B = \{y : y \in R \text{ and } y^2 - 2 = 0\}$.

Solution : Set A contains those integers which are the solution of $x - 2 = 0$ or $x = 2$. $\therefore$ $A = \{2\}$.

$\Rightarrow$ A is a singleton set.

B is a set of those real numbers which are solutions of $y^2 - 2 = 0$ or $y = \pm\sqrt{2}$

$\therefore$ $B = \{-\sqrt{2}, \sqrt{2}\}$ Thus, B is not a singleton set.

CHECK YOUR PROGRESS 1.1

1. Which of the following collections are sets ?
   - (i) The collection of days in a week starting with S.
   - (ii) The collection of natural numbers upto fifty.
   - (iii) The collection of poems written by Tulsidas.
   - (iv) The collection of fat students of your school.
2. Insert the appropriate symbol in blank spaces. If A= $\{1, 2, 3\}$.
   - (i) 1..........A (ii) 4........A.
3. Write each of the following sets in the Roster form :
   - (i) $A = \{x : x \in z \text{ and } -5 \le x \le 0\}$.
   - (ii) $B = \{x : x \in R \text{ and } x^2 - 1 = 0\}$.
   - (iii) C = {x : x is a letter of the word banana}.
   - (iv) D = {x : x is a prime number and exact divisor of 60}.
4. Write each of the following sets in the set builder form ?
   - (i) A = {2, 4, 6, 8, 10} (ii) B = {3, 6, 9,...... ∞ }
   - (iii) C = {2, 3, 5, 7} (iv) $D = \{-\sqrt{2}, \sqrt{2}\}$

   Are A and B disjoints sets ?
5. Which of the following sets are finite and which are infinite ?
   - (i) Set of lines which are parallel to a given line.
   - (ii) Set of animals on the earth.
   - (iii) Set of Natural numbers less than or equal to fifty.
   - (iv) Set of points on a circle.
6. Which of the following are null set or singleton ?
   - (i) $A = \{x : x \in R \text{ and } x \text{ is a solution of } x^2 + 2 = 0\}$.
   - (ii) $B = \{x : x \in Z \text{ and } x \text{ is a solution of } x - 3 = 0\}$.
   - (iii) $C = \{x : x \in Z \text{ and } x \text{ is a solution of } x^2 - 2 = 0\}$.
   - (iv) D = {x : x is a student of your school studying in both the classes XI and XII }
7. In the following check whether A = B or $A \approx B$.
   - (i) A = {a}, B = {x : x is an even prime number}.
   - (ii) A = {1, 2, 3, 4}, B = {x : x is a letter of the word guava}.
   - (iii) A = {x : x is a solution of $x^2 - 5x + 6 = 0$ },B = {2 , 3}.

1.4 SUB- SET

Let set A be a set containing all students of your school and B be a set containing all students of class XII of the school. In this example each element of set B is also an element of set A. Such a set B is said to be subset of the set A. It is written as $B \subseteq A$

Consider $D = \{1, 2, 3, 4, ...\}, E = \{..... -3 -2, -1, 0, 1, 2, 3, ...\}$

Clearly each element of set D is an element of set E also $\therefore\ D \subseteq E$

If A and B are any two sets such that each element of the set A is an element of the set B also, then A is said to be a subset of B.

Remarks

(i) Each set is a subset of itself i.e. $A \subseteq A$.

(ii) Null set has no element so the condition of becoming a subset is automatically satisfied. Therefore null set is a subset of every set.

(iii) If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(iv) If $A \subseteq B$ and $A \neq B$ then A is said to be a proper subset of B and B is said to be a super set of A. i.e. $A \subset B$ or $B \supset A$.

Example 1.8 If A = {x : x is a prime number less than 5} and

B = {y : y is an even prime number}, then is B a proper subset of A ?

Solution : It is given that

$A = \{2, 3\}, \quad B = \{2\}$.

Clearly $B \subseteq A$ and $B \neq A$

We write $B \subset A$ and say that B is a proper subset of A.

Example 1.9 If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5\}$. is $A \subseteq B$ or $B \subseteq A$?

Solution : Here $1 \in A$ but $1 \notin B \Rightarrow A \not\subseteq B$. Also $5 \in B$ but $5 \notin A \Rightarrow B \not\subseteq A$.

Hence neither A is a subset of B nor B is a subset of A.

Example 1.10 If $A = \{a, e, i, o, u\}$, $B = \{e, i, o, u, a\}$

Is $A \subseteq B$ or $B \subseteq A$ or both ?

Solution : Here in the given sets each element of set A is an element of set B also

$\therefore \quad A \subseteq B$ .... (i)

and each element of set B is an element of set A also. $\therefore\ B \subseteq A$ ......(ii)

From (i) and (ii) $A = B$

1.4.1 Number of Subsets of a Set :

Let A = {x}, then the subsets of A are ϕ, A.

Note that n(A) = 1, number of subsets of A = 2 = 2^1

Let A = {2, 4}, then the subsets of A are ϕ, {4}, {2}, {2, 4}.

Note that n(A) = 2, number of subsets of A = 4 = 2^2

Let A = {1, 3, 5}, then subsets of A are ϕ, {1}, {3}, {5}, {1, 3}, {1, 5}, {3, 5}, {1, 3, 5}.

Note that n(A) = 3, number of subsets of A = 8 = 2^3

> If A is a set with n(A) = p, then the number of subsets of A = 2^p and number of proper subsets of A = $2^p - 1$.

Subsets of real Numbers :

We know some standard sets of numbers as-

The set of natural numbers N = {1, 2, 3, 4, ...}

The set of whole numbers W = {0, 1, 2, 3, 4, ...}

The set of Integers Z = {........., –4, –3, –2, –1, 0, 1, 2, 3, 4, ...}

The set of Rational numbers Q = $\left\{x : x = \dfrac{p}{q}, p, q \in Z \text{ and } q \neq 0\right\}$

The set of irrational numbers denoted by I.

I = $\{x : x \in \text{R} \text{ and } x \notin \text{Q}\}$ i.e. all real numbers that are not rational

These sets are subsets of the set of real numbers. Some of the obvious relations among these subsets are

$\text{N} \subset \text{W} \subset \text{Z} \subset \text{Q}, \text{Q} \subset \text{R}, \text{I} \subset \text{R}, \text{N} \not\subset \text{I}$

1.4.2 INTERVALS AS SUBSETS OF REAL NUMBERS

An interval I is a subset of R such that if $x, y \in$ I and z is any real numbers between x and y then $z \in$ I.

> Any real number lying between two different elements of an interval must be contained in the interval.

If $a, b \in$R and $a < b$, then we have the following types of intervals :

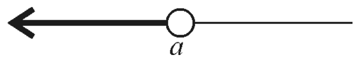
(i) The set $\{x \in \text{R} : a < x < b\}$ is called an open interval and is denoted by (a, b). On the number line it is shown as :

(ii) The set $\{x \in \text{R} : a \le x \le b\}$ is called a closed interval and is denoted by $[a, b]$. On the number line it is shown as :

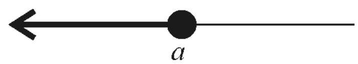
(iii) The set $\{x \in R : a < x \leq b\}$ is an interval, open on left and closed on right. It is denoted by $(a, b]$. On the number line it is shown as :

(iv) The set $\{x \in R : a \leq x < b\}$ is an interval, closed on left and open on right. It is denoted by $[a, b)$. On the number line it is shown as :

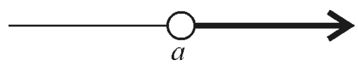
(v) The set $\{x \in R : x < a\}$ is an interval, which is dentoed by $(-\infty, a)$. It is open on both sides. On the number line it is shown as :



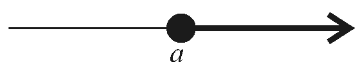
(vi) The set $\{x \in R : x \leq a\}$ is an interval which is denoted by $(-\infty, a]$. It is closed on the right. On the number line it is shown as :



(vii) The set $\{x \in R : x > a\}$ is an interval which is denoted by (a, ∞). It is open on the both sides. On the number line it is shown as :



(viii) The set $\{x \in R : x \geq a\}$ is an interval which is denoted by $[a, \infty)$. It is closed on left. On the number line it is shown as :



First four intervals are called finite intervals and the number $b - a$ (which is always positive) is called the length of each of these four intervals (a, b), $[a, b]$, $(a, b]$ and $[a, b)$.

The last four intervals are called infinite intervals and length of these intervals is not defined.

1.5 POWER SET

Let $A = \{a, b\}$ then, Subset of A are ϕ, $\{a\}$, $\{b\}$ and $\{a, b\}$.

If we consider these subsets as elements of a new set B (say) then, $B = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

B is said to be the power set of A.

Notation : Power set of a set A is denoted by P(A).
and it is the set of all subsets of the given set.

Example 1.11 Write the power set of each of the following sets :

(i) $A = \{x : x \in R \text{ and } x^2 + 7 = 0\}$.

(ii) $B = \{y : y \in N \text{ and } 1 \leq y \leq 3\}$.

Solution :

(i) Clearly A = ϕ (Null set), $\therefore$ ϕ is the only subset of given set, $\therefore$ P (A)={ ϕ }

(ii) The set B can be written as {1, 2, 3}

Subsets of B are ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}.

$\therefore$ P (B) = { ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} }.

Example 1.12 Write each of the following sets as intervals :

(i) $\{x \in R : -1 < x \le 2\}$ (ii) $\{x \in R : 1 \ge 2x - 3 \ge 0\}$

Solution : (i) The given set = $\{x \in R : -1 < x \le 2\}$

Hence, Interval of the given set = (–1, 2]

(ii) The given set = $\{x \in R : 1 \ge 2x - 3 \ge 0\}$

$\Rightarrow \{x \in R : 4 \ge 2x \ge 3\}$, $\Rightarrow \left\{x \in R : 2 \ge x \ge \frac{3}{2}\right\}$

$\Rightarrow \left\{x \in R : \frac{3}{2} \le x \le 2\right\}$, Hence, Interval of the given set = $\left[\frac{3}{2}, 2\right]$

1.6 UNIVERSAL SET

Consider the following sets.

A = {x : x is a student of your school}

B = {y : y is a male student of your school}

C = {z : z is a female student of your school}

D = {a : a is a student of class XII in your school}

Clearly the set B, C, D are all subsets of A. A can be considered as the universal set for this particular example. Universal set is generally denoted by U. In a particular problem a set U is said to be a universal set if all the sets in that problem are subsets of U.

Remarks

(i) Universal set does not mean a set containing all objects of the universe.

(ii) A set which is a universal set for one problem may not be a universal set for another problem.

Example 1.13 Which of the following sets can be considered as a universal set ?

X = {x : x is a real number}

Y = {y : y is a negative integer}

Z = {z : z is a natural number}

Solution : As it is clear that both sets **Y** and **Z** are subset of **X**.

$\therefore$ **X** is the universal set for this problem.

1.7 VENN DIAGRAM

British mathematician John Venn (1834 – 1883 AD) introduced the concept of diagrams to represent sets. According to him universal set is represented by the interior of a rectangle and other sets are represented by interior of circles.

For example if U= {1, 2, 3, 4, 5}, A = {2, 4} and B = {1,3}, then these sets can be represented as

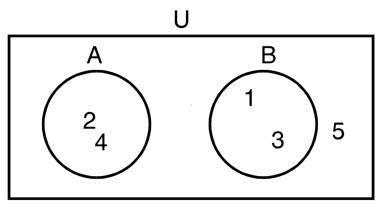


Fig. 1.1

Diagramatical representation of sets is known as a Venn diagram.

1.8 DIFFERENCE OF SETS

Consider the sets

A = {1, 2, 3, 4, 5} and B= {2, 4, 6}.

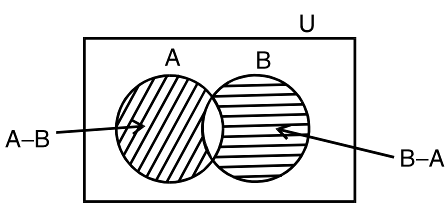
A new set having those elements which are in A but not in B is said to be the difference of sets A and B and it is denoted by A – B. $\therefore$ A – B= {1, 3, 5}

Similiarly a set of those elements which are in B but not in A is said to be the difference of B and A and it is devoted by B – A. $\therefore$ B – A = {6}

In general, if A and B are two sets then

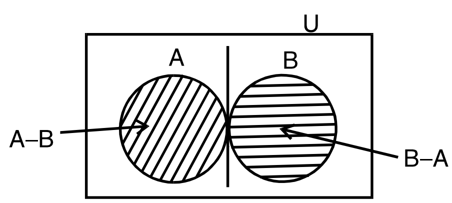
$$A - B = \{ x : x \in A \text{ and } x \notin B \} \text{ and } B - A = \{ x : x \in B \text{ and } x \notin A \}$$

Difference of two sets can be represented using Venn diagram as :



When A and B are not **disjoint** sets

or



When A and B are **disjoint** sets

Fig. 1.2 **Fig. 1.3**

1.9. COMPLEMENT OF A SET

Let **X** denote the universal set and **Y, Z** its sub sets where

X = {x : x is any member of a family}

Y = {x : x is a male member of the family}

Z = {x : x is a female member of the family}

X – Y is a set having female members of the family.

X – Z is a set having male members of the family.

X – Y is said to be the complement of Y and is usally denoted by Y' or Y^c.

X – Z is said to be complement of Z and denoted by Z' or Z^c.

If U is the universal set and A is its subset then the complement of A is a set of those elements which are in U but not in A. It is denoted by A' or A^c.

$A' = U - A = \{x : x \in U \text{ and } x \notin A\}$

The complement of a set can be represented using Venn diagram as :

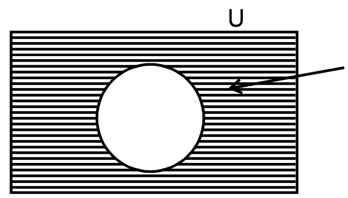


Fig. 1.4

Remarks

(i) Difference of two sets can be found even if none is a subset of the other but complement of a set can be found only when the set is a subset of some universal set.

(ii) $\phi^c = U$. (iii) $U^c = \phi$.

Example 1.14 Given that

A = {x : x is a even natural number less than or equal to 10}

and B = {x : x is an odd natural number less than or equal to 10}

Find (i) A – B (ii) B – A (iii) is A – B = B – A ?

Solution : It is given that

A = {2, 4, 6, 8, 10}, B = {1, 3, 5, 7, 9}

Therefore, (i) A – B = {2, 4, 6, 8, 10}, (ii) B – A = {1, 3, 5, 7, 9}

(iii) Clearly from (i) and (ii) $A - B \neq B - A$.

Example 1.15 Let U be the universal set and A its subset where

$U = \{x : x \in N \text{ and } x \leq 10\}$

A = {y : y is a prime number less than 10}

Find (i) A^c (ii) Represent A^c in Venn diagram.

Solution : It is given

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. and A = { 2, 3, 5, 7}

(i) $A^c = U - A = \{1, 4, 6, 8, 9, 10\}$

(ii)

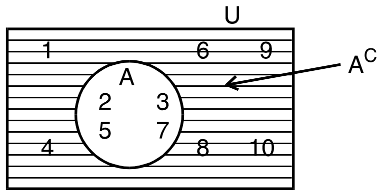


Fig. 1.5

1.9.1 Properties of complement of sets

1. Complement Law's

   (i) $A \cup A' = U$ (ii) $A \cap A' = \phi$

2. De Morgan's Law

   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

3. Law of double complementation $(A')' = A$

4. Law of empty set and universal set i.e $\phi' = U$ and $U' = \phi$

1. Verification of Complement Law

Let $U = \{1, 2, 3, ... 10\}$ and $A = \{2, 4, 6, 8, 10\}$

Then $A' = \{1, 3, 5, 7, 9\}$

Now, $A \cup A' = \{1, 2, 3, 4, ..., 10\} = U$ and $A \cap A' = \phi$

Hence, $A \cup A' = U$ and $A \cap A' = \phi$

2. Verification of De Morgan's Law

Let $U = \{1, 2, 3, ..., 9\}$ and $A = \{2, 4, 6, 8\}$, $B = \{2, 3, 5, 7\}$

Hence, $A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$

and $(A \cup B)' = U - (A \cup B) = \{1, 9\}$ ...(1)

Now $A' = U - A = \{1, 3, 5, 7, 9\}$ and $B' = U - B = \{1, 4, 6, 8, 9\}$

$\therefore$ $A' \cap B' = \{1, 9\}$ ...(2)

From (1) and (2), $(A \cup B)' = A' \cap B'$

Also $A \cap B = \{2\}$

$\therefore$ $(A \cap B)' = U - (A \cap B) = \{1, 3, 4, 5, 6, 7, 8, 9\}$ ...(3)

and $A' \cup B' = \{1, 3, 4, 5, 6, 7, 8, 9\}$ ...(4)

From (3) and (4), we get $(A \cap B)' = A' \cup B'$

Verification of $(A')' = A$

Let $U = \{1, 2, 3, ..., 10\}$ and $A = \{1, 2, 3, 5, 7, 9\}$

Then $A' = U - A = \{4, 6, 8, 10\}$ $(A')' = U - A' = \{1, 2, 3, 5, 7, 9\} = A$

Thus from the definition of the complement of a subset A of the universal set U it follows that $(A')' = A$

CHECK YOUR PROGRESS 1.2

1. Insert the appropriate symbol in the blank spaces, given that A={1, 3, 5, 7, 9}

   (i) ϕ ....A (ii) {2, 3, 9}..........A

   (iii) 3...........A (iv) 10...................A

2. Given that A = {a, b}, how many elements P(A) has ?

3. Let A = { ϕ, {1} ,{2}, {1,2}}. Which of the following is true or false ?

   (i) $\{1,2\} \subset A$ (ii) $\phi \in A$.

4. Which of the following statements are true or false ?

   (i) Set of all boys, is contained in the set of all students of your school.

   (ii) Set of all boy students of your school, is contained in the set of all students of your school.

   (iii) Set of all rectangles, is contained in the set of all quadrilaterals.

   (iv) Set of all circles having centre at origin is contained in the set of all ellipses having centre at origin.

5. If A = { 1, 2, 3, 4, 5 } , B= {5, 6, 7} find (i) A – B (ii) B – A.

6. Let N be the universal set and A, B, C, D be its subsets given by

   A = {x : x is a even natural number}, B = {x : $x \in N$ and x is a multiple of 3}

   C = {x : $x \in N$ and $x \geq 5$}, D = {x : $x \in N$ and $x \leq 10$}

   Find complements of A, B, C and D respectively.

7. Write the following sets in the interval form.

   (a) $\{x \in R : -8 < x < 3\}$ (b) $\{x \in R : 3 \leq 2x \angle 7\}$

8. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {1, 2, 3, 4} and B = {2, 4, 6, 8}, then verify the following

   (i) $(A')' = A$ (ii) $(B')' = B$

   (iii) $A \cap A' = \phi$ (iv) $(A \cup B)' = A' \cap B'$

1.10. INTERSECTION OF SETS

Consider the sets

A = {1, 2, 3, 4} and B = { 2, 4, 6}

It is clear, that there are some elements which are common to both the sets A and B. Set of these common elements is said to be interesection of A and B and is denoted by $A \cap B$.

Here $A \cap B = \{2, 4\}$

If A and B are two sets then the set of those elements which belong to both the sets is said to be the intersection of A and B. It is denoted by $A \cap B$. $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A \cap B$ can be represented using Venn diagram as :

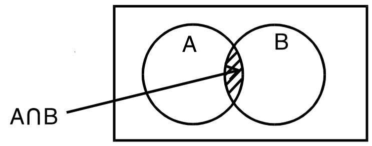


Fig. 1.6

Remarks

If $A \cap B = \phi$ then A and B are said to be **disjoint sets**. In Venn diagram disjoint sets can be represented as

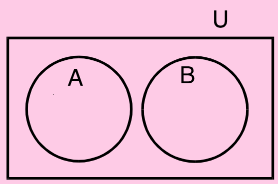


Fig.1.7

Example 1.16 Given that

A = {x : x is a king out of 52 playing cards}

and B = { y : y is a spade out of 52 playing cards}

Find (i) $A \cap B$ (ii) Represent $A \cap B$ using Venn diagram .

Solution : (i) As there are only four kings out of 52 playing cards, therefore the set A has only four elements. The set B has 13 elements as there are 13 spade cards but out of these 13 spade cards there is one king also. Therefore there is one common element in A and B.

$\therefore A \cap B$ = { King of spade}.

(ii)

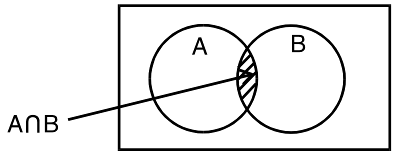


Fig. 1.8

1.11 UNION OF SETS

Consider the following examples :

(i) A is a set having all players of Indian men cricket team and B is a set having all players of Indian women cricket team. Clearly A and B are disjoint sets. Union of these two sets is a set having all players of both teams and it is denoted by $A \cup B$.

(ii) D is a set having all players of cricket team and E is the set having all players of Hockety team, of your school. Suppose three players are common to both the teams then union of D and E is a set of all players of both the teams but three common players to be written once only.

If A and B are any two sets then union of A and B is the set of those elements which belong to A or B.

In set builder form : $A \cup B = \{x : x \in A \text{ or } x \in B\}$

OR

$$A \cup B = \{x : x \in A - B \text{ or } x \in B - A \text{ or } x \in A \cap B\}$$

$A \cup B$ can be represented using Venn diagram as :

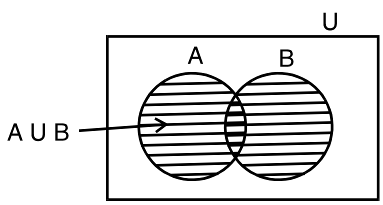


Fig. 1.9

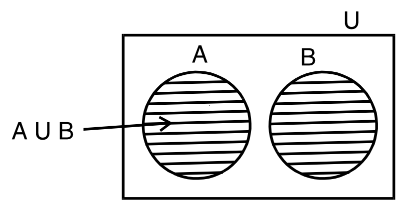


Fig. 1.10

$$n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B).$$

or $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

where n $A \cup B$ stands for number of elements in $A \cup B$.

Example 1.17 $A = \{x : x \in Z^+ \text{ and } x \le 5\}$, B = {y : y is a prime number less than 10}

Find (1) $A \cup B$ (ii) represent $A \cup B$ using Venn diagram.

Solution : We have,

A={1, 2, 3, 4, 5} B = {2, 3, 5, 7}. $\therefore$ $A \cup B$ = {1, 2, 3, 4, 5, 7}.

(ii)

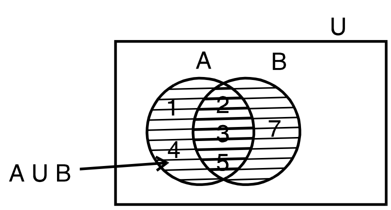


Fig.1.11

CHECK YOUR PROGRESS 1.3

1. Which of the following pairs of sets are disjoint and which are not ?

   (i) {x : x is an even natural number}, {y : y is an odd natural number}

   (ii) {x : x is a prime number and divisor of 12}, { y : $y \in N$ and $3 \le y \le 5$}

   (iii) {x :x is a king of 52 playing cards}, { y : y is a diamond of 52 playing cards}

   (iv) {1, 2, 3, 4, 5}, {a, e, i, o, u}

2. Find the intersection of A and B in each of the following :

   (i) $A = \{x : x \in Z\}$, $B = \{x : x \in N\}$ (ii) A = {Ram, Rahim, Govind, Gautam}
   B = {Sita, Meera, Fatima, Manprit}

3. Given that A = {1, 2, 3, 4, 5}, B={5, 6, 7, 8, 9, 10}

   find (i) $A \cup B$ (ii) $A \cap B$.

4. If $A = \{x : x \in N\}$, $B = \{y : y \in z$ and $-10 \le y \le 0\}$, find $A \cup B$ and write your answer in the Roster form as well as in set-builder form.

5. If A ={2, 4, 6, 8, 10}, B {8, 10, 12, 14}, C ={14, 16, 18, 20}.

   Find (i) $A \cup (B \cup C)$ (ii) $A \cap (B \cap C)$.

6. Let U = {1, 2, 3,.......10}, A = {2, 4, 6, 8, 10}, B = { 1, 3, 5, 7, 9, 10}

   Find (i) $(A \cup B)'$ (ii) $(A \cap B)'$ (iii) (B')' (iv) $(B - A)'$.

7. Draw Venn diagram for each of the following :

   (i) $A \cap B$ when $B \subset A$ (ii) $A \cap B$ when A and B are disjoint sets.

   (iii) $A \cap B$ when A and B are neither subsets of each other nor disjoint sets.

8. Draw Venn diagram for each of the following :

   (i) $A \cup B$ when $A \subset B$. (ii) $A \cup B$ when A and B are disjoint sets.

   (iii) $A \cup B$ when A and B are neither subsets of each other nor disjoint sets.

9. Draw Venn diagram for each the following :

   (i) $A - B$ and $B - A$ when $A \subset B$.

   (ii) $A - B$ and $B - A$ when A and B are disjoint sets.

   (iii) $A - B$ and $B - A$ when A and B are neither subsets of each other nor disjoint sets.

LET US SUM UP

- Set is a well defined collection of objects.
- To represent a set in Roster form all elements are to be written but in set builder form a set is represented by the common property of its elements.

- If the elements of a set can be counted then it is called a finite set and if the elements cannot be counted, it is infinite.
- If each element of set *A* is an element of set *B*, then *A* is called sub set of *B*.
- For two sets *A* and *B*, $A - B$ is a set of those elements which are in *A* but not in *B*.
- Complement of a set *A* is a set of those elements which are in the universal set but not in *A*. i.e. $A^c = U - A$
- Intersection of two sets is a set of those elements which belong to both the sets.
- Union of two sets is a set of those elements which belong to either of the two sets.
- Any set 'A' is said to be a subset of a set 'B' if every element of A is contained in B.
- Empty set is a subset of every set.
- Every set is a subset of itself.
- The set 'A' is a proper subset of set 'B' iff $A \subseteq B$ and $A \neq B$
- The set of all subsets of a given set 'A' is called power set of A.
- Two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$
- If $n(A) = p$ then number of subsets of $A = (2)^p$
- (a, b), $[a, b]$, $(a, b]$ and $[a, b)$ are finite intervals as their length b – a is real and finite.
- Complement of a set A with respect to U is denoted by A' and defined as $A' = \{x : x \in U \text{ and } x \notin A\}$
- $A' = U - A$
- If $A \subset U$, then $A' \subset U$
- Properties of complement of set A with respect to U
  - $A \cup A' = U$ and $A \cap A' = \phi$
  - $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$
  - $(A')' = A$
  - $\phi' = U$ and $U' = \phi$

SUPPORTIVE WEB SITES

http://www.mathresource.iitb.ac.in/project/indexproject.html

http://mathworld.wolfram.com/SetTheory.html

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Beginnings_of_set_theory.html

TERMINAL EXERCISE

1. Which of the following statements are true or false :

   (i) $\{1, 2, 3\} = \{1, \{2\}, 3\}$.   (ii) $\{1, 2, 3\} = \{3, 1, 2\}$.

   (iii) $\{a, e, o\} = \{a, b, c\}$.   (iv) $\{\phi\} = \{\ \}$

2. Write the set in Roster form represented by the shaded portion in the following.

   (i) $A = \{1, 2, 3, 4, 5\}$

   $B = \{5, 6, 7, 8, 9\}$

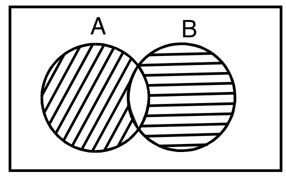


Fig. 1.12

   (ii) $A = \{1, 2, 3, 4, 5, 6\}$

   $B = \{2, 6, 8, 10, 12\}$

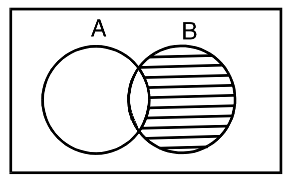


Fig. 1.13

3. Represent the follwoing using Venn diagram.

   (i) $(A \cup B)'$ provided A and B are not disjoint sets.

   (ii) $(A \cap B)'$ provided A and B are disjoint sets.

   (iii) $(A - B)'$ provided A and B are not disjoint sets.

4. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A\{2, 4, 6, 8\}$, $B = \{1, 3, 5, 7\}$

   Verify that

   (i) $(A \cup B)' = A' \cap B'$

   (ii) $(A \cap B)' = A' \cup B'$

   (iii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

5. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

   $A = \{1, 3, 5, 7, 9\}$.

   $B = \{2, 4, 6, 8, 10\}$, $C = \{1, 2, 3\}$.

   Find (i) $A' \cap (B - C)$.   (ii) $A \cup (B \cup C)$

   (iii) $A' \cap (B \cup C)'$ (iv) $(A \cap B)' \cup C'$

6. What does the shaded portion represent in each of the following Venn diagrams :

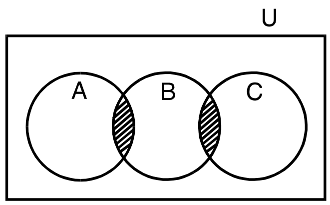
(i)


Fig. 1.14

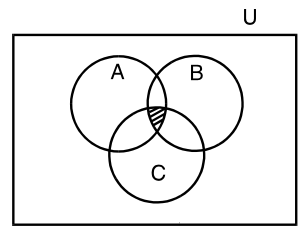
(ii)


Fig. 1.15

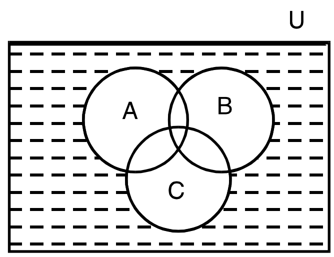
(iii)


Fig. 1.16

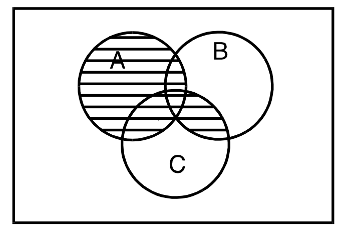
(iv)


Fig. 1.17

7. Draw Venn diagram for the following :

(i) $A' \cap (B \cup C)$ (ii) $A' \cap (C - B)$

Where A,B,C are not disjoint sets and are subsets of the universal set U.

8. Verify De Morgan's Law if $U = \{x : x \in N \text{ and } x \leq 10\}$

$A = \{x: x \in U \text{ and } x \text{ is a prime number}\}$ and

$B = \{x : x \in U \text{ and } x \text{ is a factor of } 24\}$

9. Examine whether the following statements are true or false :

(*a*) $\{a, e\} \subset \{x : x$ is a vowel in the English alphabet]

(*b*) $\{1, 2, 3\} \subset \{1, 3, 5\}$ (*c*) $\{a, b, c\} \subset \{a, b, c\}$ (*d*) $\phi \subset \{1, 3, 5\}$

10. Write down all the subsets of the following sets :

(*a*) $\{a\}$ (*b*) $\{1, 2, 3\}$ (*c*) ϕ

11. Write down the following as intervals :

(*a*) $\{x : x \in R, -4 < x \leq 6\}$ (*b*) $\{x : x \in R, -12 < x < -10\}$

(*c*) $\{x : x \in R, 0 \leq x < 7\}$ (*d*) $\{x : x \in R, 3 \leq x \leq 4\}$

ANSWERS

CHECK YOUR PROGRESS 1.1

1. (i), (ii), (iii) are sets.
2. (i) $\in$ (ii) $\notin$
3. (i) $A = \{-5, -4, -3, -2, -1, 0\}$ (ii) $B = \{-1, 1\}$,
   (iii) $C = \{a, b, n\}$ (iv) $D = \{2, 3, 5\}$.
4. (i) $A = \{x : x \text{ is a even natural number less than or equal to ten}\}$.
   (ii) $B = \{x : x \in N \text{ and } x \text{ is a multible of } 3\}$.
   (iii) $C = \{x : x \text{ is a prime number less than } 10\}$.
   (iv) $D = \{x : x \in R \text{ and } x \text{ is a solution of } x^2 - 2 = 0\}$.
5. (i) Infinite (ii) Finite (iii) Finite (iv) Infinite
6. (i) Null (ii) Singleton (iii) Null (iv) Null
7. (i) $A \approx B$ (ii) $A \approx B$ (iii) $A = B$.

CHECK YOUR PROGRESS 1.2

1. (i) $\subset$ (ii) $\not\subset$ (iii) $\in$ (iv) $\notin$
2. 4
3. (i) False (ii) True
4. (i) False (ii) True (iii) True (iv) False
5. (i) $\{1, 2, 3, 4\}$ (ii) $\{6, 7\}$.
6. $A^c = \{x : x \text{ is an odd natural number}\}$

   $B^c = \{x : x \in N \text{ and x is not a multiple of } 3\}$

   $C^c = \{1, 2, 3, 4\}$. $D^c = \{11, 12, 13, ...\}$
7. (a) (-8,3) (b) $\left[\frac{3}{2}, \frac{7}{2}\right)$

CHECK YOUR PROGRESS 1.3

1. (i) Disjoint (ii) Not disjoint (iii) Not disjoint (iv) Disjoint
2. (i) $\{x : x \in N\}$ (ii) ϕ
3. (i) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (ii) $\{5\}$

4. Roster from $\{-10, -9, -8, ...0, 1, 2, 3, ...\}$

Set builder from $\{x : x \in Z \text{ and } -10 \leq x \leq \infty\}$

5. (i) $\{x : x \text{ is a even natural number less than equal to } 20\}$. (ii) ϕ

6. (i) ϕ (ii) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(iii) $\{1, 3, 5, 7, 9, 10\}$ (iv) $\{2, 4, 6, 8, 10\}$

7. (i)

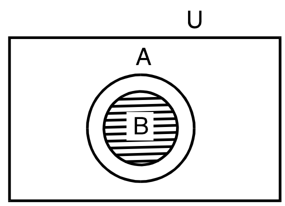


Fig. 1.18

(ii)

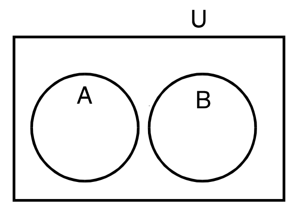


Fig. 1.19

(iii)

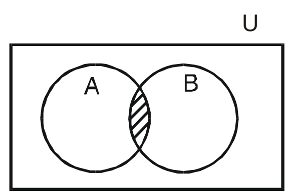


Fig. 1.20

8. (i)

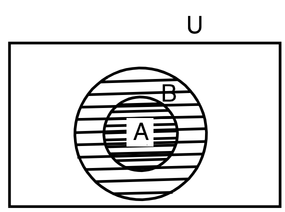


Fig. 1.21

(ii)

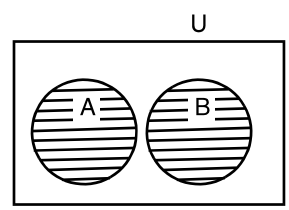


Fig. 1.22

(iii)

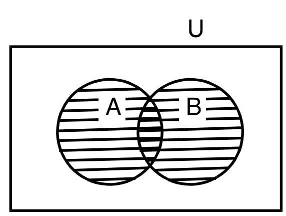
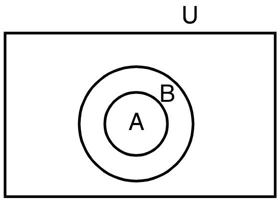
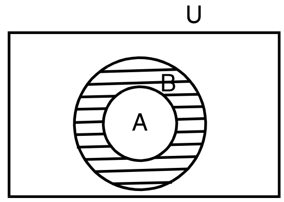


Fig. 1.23

9. (i)



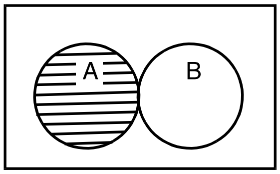
$A - B = \phi$



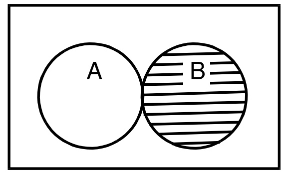
$B - A =$ Shaded portion.

Fig. 1.24

(ii)



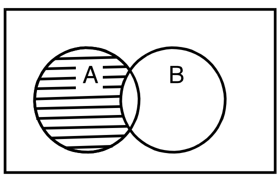
$A - B = A$



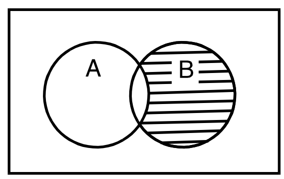
$B - A = B$

Fig. 1.25

(iii)



$A - B =$ Shaded Portion



$B - A =$ Shaded Portion

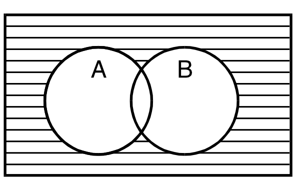
Fig. 1.26

TERMINAL EXERCISE

1. (i) False (ii) True (iii) False (iv) False

2. (i) $\{1, 2, 3, 4, 6, 7, 8, 9\}$ (ii) $\{8, 10, 12\}$

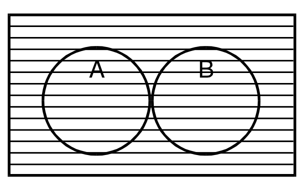
3. (i)



$(A \cup B)' =$ Shaded Portion

Fig. 1.27

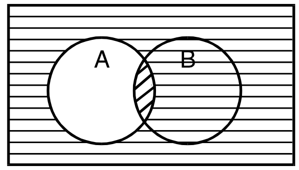
(ii)



$(A \cap B)' =$ Shaded Portion

Fig. 1.28

(iii)



$(A - B)' =$ Shaded Portion

Fig. 1.29

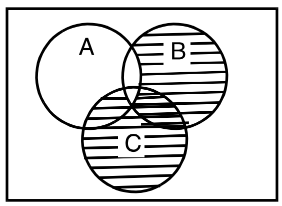
5. (i) $\{4, 6, 8, 10\}$ (ii) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

   (iii) ϕ (iv) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

6. (i) $(A \cap B) \cap (B \cap C)$ (ii) $(A \cap B) \cap C$

   (iii) $[(A \cup B) \cup C]'$ (iv) $A \cup (B \cap C)$.

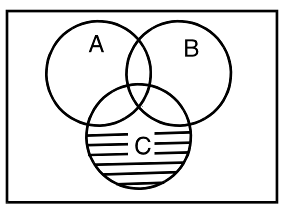
7. (i)



$A' \cap (B \cup C) =$ Shaded Portion

Fig.1.30

(ii)



$A' \cap (C - B) =$ Shaded Portion

Fig. 1.31

9. (*a*) True (*b*) False

   (*c*) True (*d*) True

10. (*a*) ϕ, $\{a\}$ (*b*) ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$

    (c) ϕ

11. (*a*) $(-4, 6]$ (*b*) $(-12, -10)$

    (*c*) $[0, 7)$ (*d*) $[3, 4]$

2

RELATIONS AND FUNCTIONS-I

In our daily life, we come across many patterns that characterise relations such as brother and sister, father and son, teacher and student etc. In mathematics also, we come across many relations such as number m is greater than number n, line ℓ is perpendicular to line m etc. the concept of relation has been established in mathematical form. The word "function" was introduced by leibnitz in 1694. Function is defiend as a special type of relation. In the present chapter we shall discuss cartesion product of sets, relation between two sets , conditions for a relation to be a function, different types of functions and their properties.

OBJECTIVES

After studying this lesson, you will be able to :

- define cartesion product of two sets.
- define. relation, function and cite examples there of
- find domain and range of a function
- draw graph of functions.
- define and cite examples of even and odd turnotions.
- determine whether a function is odd or even or neither
- define and cite examples of functions like $|x|, [x]$ the greatest integer functions, polynomial functions, logarithmic and exponential functions
- to Find sum. difference, product and quotient of real functions.

EXPECTED BACKGROUND KNOWLEDGE

- concept of ordered pairs.

2.1 CARTESIAN PRODUCT OF TWO SETS

Consider two sets A and B where

A={1, 2}, B= {3, 4, 5}.

Set of all ordered pairs of elements of A and B

is $\{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

This set is denoted by $A \times B$ and is called the cartesian product of sets A and B.

i.e. $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$

Cartesian product of sets B and A is denoted by $B \times A$.

In the present example, it is given by

$B \times A = \{(3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)\}$, Clearly $A \times B \neq B \times A$.

In the set builder form :

$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$ and $B \times A = \{(b,a) : b \in B \text{ and } a \in A\}$

Note : If $A = \phi$ or $B = \phi$ or $A, B = \phi$

then $A \times B = B \times A = \phi$.

Example 2.1

(1) Let $A=\{a,b,c\}$, $B=\{d,e\}$, $C=\{a,d\}$.

Find (i) $A \times B$ (ii) $B \times A$ (iii) $A \times (B \cup C)$ (iv) $(A \cap C) \times B$

(v) $(A \cap B) \times C$ (vi) $A \times (B - C)$.

Solution : (i) $A \times B = \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$.

(ii) $B \times A = \{(d, a), (d, b), (d, c), (e, a) (e, b), (e, c)\}$.

(iii) $A = \{a, b, c\}$, $B \cup C = \{a,d,e\}$.

$\therefore$ $A \times (B \cup C) = \{(a, a), (a, d), (a, e), (b, a), (b, d), (b, e), (c, a), (c, d), (c, e)$.

(iv) $A \cap C = \{a\}$, $B=\{d, e\}$. $\therefore$ $(A \cap C) \times B = \{(a, d), (a, e)\}$

(v) $A \cap B = \phi$, $C=\{a,d\}$, $\therefore$ $A \cap B \times C = \phi$

(vi) $A = \{a,b,c\}$, $B - C = \{e\}$. $\therefore$ $A \times (B - C) = \{(a,e), (b,e), (c,e)\}$.

2.1.1 Number of elements in the Cartesian product of two finite sets

Let A and B be two non-empty sets. We know that $A \times B = \{(a, b); a \in A \text{ and } b \in B\}$

Then number of elements in Cartesian product of two finite sets A and B

i.e. $n(A \times B) = n(A).\ n(B)$

Example 2.2 Suppose $A = \{1,2,3\}$ and $B=\{x, y\}$, show that $n(A \times B) = n(A) \times n(B)$

Solution : Here $n(A) = 3$, $n(B) = 2$

$\therefore$ $A \times B = \{(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)\}$

$n(A \times B) = n(A) \times n(B)$, $= 3 \times 2 = 6$

Example 2.3 If $n(A) = 5$, $n(B) = 4$, find $n(A \times B)$

Solution : We know that $n(A \times B) = n(A) \times n(B)$

$$n(A \times B) = 5 \times 4 = 20$$

2.1.2 Cartesian product of the set of real numbers R with itself upto R × R × R

Ordered triplet $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$

Here (a, b, c) is called an ordered triplet.

The Cartesian product R × R represents the set $R \times R = \{(x, y) : x, y \in R\}$ which represents the coordinates of all the points in two dimensional plane and the Cartesian product R × R × R represent the set $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$ which represents the coordinates of all the points in three dimensional space.

Example 2.4 If A = {1, 2}, form the set A × A × A.

Solution : A × A × A ={(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2)
(2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)}

2.2 RELATIONS

Consider the following example :

A={Mohan, Sohan, David, Karim} and B={Rita, Marry, Fatima}

Suppose Rita has two brothers Mohan and Sohan, Marry has one brother David, and Fatima has one brother Karim. If we define a relation R " is a brother of" between the elements of A and B then clearly.

Mohan R Rita, Sohan R Rita, David R Marry, Karim R Fatima.

After omiting R between two names these can be written in the form of ordered pairs as :

(Mohan, Rita), (Sohan, Rita), (David, Marry), (Karima, Fatima).

The above information can also be written in the form of a set R of ordered pairs as

R= {(Mohan, Rita), (Sohan, Rita), (David, Marry), Karim, Fatima}

Clearly $R \subseteq A \times B$, i.e.$R = \{(a, b) : a \in A, b \in B$ and $aRb\}$

If A and B are two sets then a relation R from A toB is a sub set of A×B.

If (i) $R = \phi$, R is called a void relation.

(ii) R=A×B, R is called a universal relation.

(iii) If R is a relation defined from A to A, it is called a relation defined on A.

(iv) $R = \{(a, a) \ \forall \ a \in A\}$, is called the identity relation.

2.2.1 Domain and Range of a Relation

If R is a relation between two sets then the set of first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation. In the previous example. Domain = {Mohan, Sohan, David, Karim}, Range = {Rita, Marry, Fatima}

Example 2.5 Given that $A = \{2, 4, 5, 6, 7\}$, $B = \{2, 3\}$.

R is a relation from A to B defined by

$$R = \{(a, b) : a \in A,\ b \in B \text{ and a is divisible by b}\}$$

find (i) R in the roster form (ii) Domain of R (iii) Range of R (iv) Repersent R diagramatically.

Solution : (i) R = {(2, 2), (4, 2), (6, 2), (6, 3)}

(ii) Domain of R = {2, 4, 6} (iii) Range of R = {2, 3}

(iv)

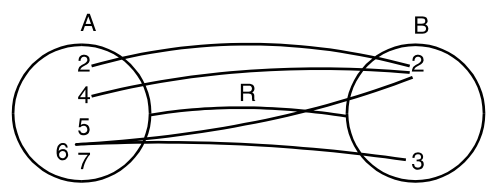


Fig. 2.1

Example 2.6 If R is a relation 'is greater than' from A to B, where

A= {1, 2, 3, 4, 5} and B = {1,2,6}.

Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.

Solution :

(i) R = {(2,1), (3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)}

(ii) Domain of R = {2, 3, 4, 5} (iii) Range of R = {1, 2}

2.2.2 Co-domain of a Relation

If R is a relation from A to B, then B is called codomain of R.

For example, let A = {1, 3, 4, 5, 7} and B = {2, 4, 6, 8} and R be the relation 'is one less than' from A to B, then R = {(1, 2), (3, 4), (5, 6), (7, 8)}

so codomain of R = {2, 4, 6, 8}

Example 2.7 Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$ and write down the domain, range and codomain of R.

Solution : R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

Domain of A = {1, 2, 3, 4, 5}

Range of R = {2, 3, 4, 5, 6} and Codomain of R = {1, 2, 3, 4, 5, 6}

CHECK YOUR PROGRESS 2.1

1. Given that A = {4, 5, 6, 7}, B = {8, 9}, C = {10}

   Verify that $A \times (B - C) = (A \times B) - (A \times C)$.

2. If U is a universal set and A, B are its subsets. Where U= {1, 2, 3, 4, 5}.
A = {1,3,5}, B = {x : x is a prime number}, find A' × B'

3. If A = {4, 6, 8, 10}, B = {2, 3, 4, 5}, R is a relation defined from A to B where

R= {(a, b) : $a \in A,\ b \in B$ and a is a multiple of b}

find (i)R in the Roster form (ii) Domain of R (iii) Range of R .

4. If R be a relation from N to N defined by R= {(x,y) : $4x + y = 12,\ x, y \in N$ }

find (i) R in the Roster form (ii) Domain of R (iii) Range of R.

5. If R be a relation on N defined by R={ (x, x^2) : x is a prime number less than 15}

Find (i) R in the Roster form (ii) Domain of R (iii) Range of R

6. If R be a relation on set of real numbers defined by R={(x,y) : $x^2 + y^2 = 0$ }, Find

(i) R in the Roster form (ii) Domain of R (iii) Range of R.

7. If $(x + 1, y - 2) = (3, 1)$, find the values of x and y.

8. If A = {–1, 1} find A × A × A.

9. If A × B = {(a, x), (a, y), (b, x), (b, y)}. Find A and B.

10. If n(A) = 6 and n(B) = 3, then find n(A × B).

2.3 DEFINITION OF A FUNCTION

Consider the relation f : {(a,1), (b,2), (c,3), (d,5)}from set A = {a,b,c,d}to set B = {1,2,3,4}.

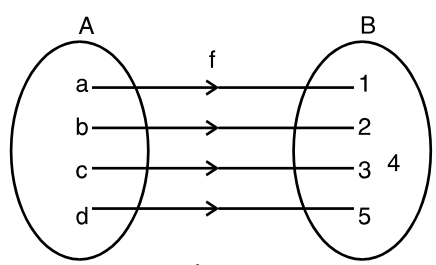


Fig. 2.2

In this relation we see that each element of A has a unique image in *B*. This relation *f* from set *A* to *B* where every element of *A* has a unique image in *B* is defined as a function from *A* to *B*. So we observe that ***in a function no two ordered pairs have the same first element.***

We also see that $\exists$ an element $\in$ B, i.e., 4 which does not have its preimage in A. Thus here:

(i) the set B will be termed as co-domain and (ii) the set {1, 2, 3, 5} is called the range. From the above we can conclude that ***range is a subset of co-domain.***

Symbolically, this function can be written as

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

2.3.1 Real valued function of a real variable

A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R, then it is called a real function.

Let R be the set of all real numbers and X, Y be two non-empty subsets of R, then a rule 'f' which associates to each $x \in$ X, a unique element y of Y is called a real valued function of the real variable or simply a real function and we write it as $f : X \to Y$

A real function 'f' is a rule which associates to each possible real number x, a unique real number $f(x)$.

Example 2.8 Which of the following relations are functions from A to B. Write their domain and range. If it is not a function give reason ?

(a) $\{(1,-2),(3,7),(4,-6),(8,1)\}$, $A = \{1,3,4,8\}$, $B = \{-2,7,-6,1,2\}$

(b) $\{(1,0),(1-1),(2,3),(4,10)\}$, $A = \{1,2,4\}$, $B = \{0,-1,3,10\}$

(c) $\{(a,b),(b,c),(c,b),(d,c)\}$, $A = \{a,b,c,d,e\}$ $B = \{b,c\}$

(d) $\{(2,4),(3,9),(4,16),(5,25),(6,36\}$, $A = \{2,3,4,5,6\}$, $B = \{4,9,16,25,36\}$

(e) $\{(1,-1),(2,-2),(3,-3),(4,-4),(5,-5)\}$, $A = \{0,1,2,3,4,5\}$,
$B = \{-1,-2,-3,-4,-5\}$

(f) $$\left\{\left(\sin\frac{\pi}{6},\frac{1}{2}\right),\left(\cos\frac{\pi}{6},\frac{\sqrt{3}}{2}\right),\left(\tan\frac{\pi}{6},\frac{1}{\sqrt{3}}\right),\left(\cot\frac{\pi}{6},\sqrt{3}\right)\right\},$$

$$A = \left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\} \quad B = \left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3},1\right\}$$

(g) $\{(a,b),(a,2),(b,3),(b,4)\}$, $A = \{a,b\}$, $B = \{b,2,3,4\}$.

Solution :

(a) It is a function. Domain $= \{1,3,4,8\}$, Range $= \{-2,7,-6,1\}$

(b) It is not a function. Because Ist two ordered pairs have same first elements.

(c) It is not a function. Because Domain $= \{a,b,c,d\} \neq A$, Range $= \{b,c\}$

(d) It is a function. Domain $= \{2,3,4,5,6\}$, Range $= \{4,9,16,25,36\}$

(e) It is not a function.

Because Domain $= \{1,2,3,4,5\} \neq A$, Range $= \{-1,-2,-3,-4,-5\}$

(f) It is a function.

$$\text{Domain} = \left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\}, \text{Range} = \left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3}\right\}$$

(g) It is not a function. Because first two ordered pairs have same first component and last two ordered pairs have also same first component.

Example 2.9 State whether each of the following relations represent a function or not.

(a)

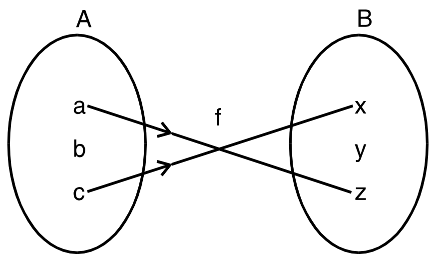


Fig. 2.3

(b)

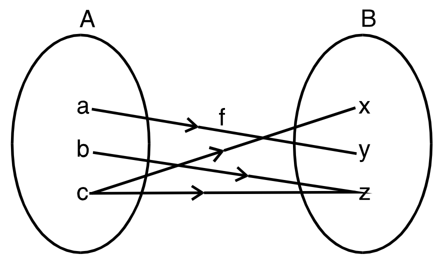


Fig. 2.4

(c)

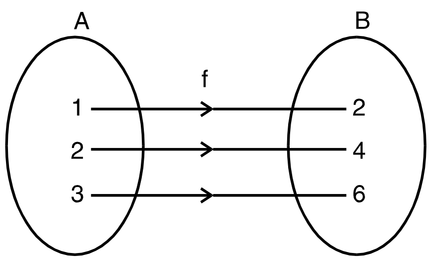


Fig. 2.5

(d)

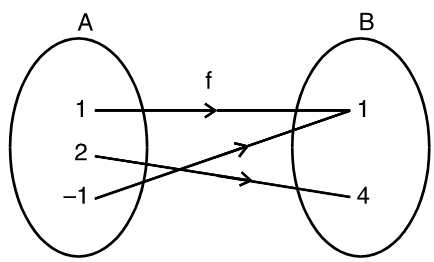


Fig. 2.6

Solution :

(a) *f* is not a function because the element *b* of *A* does not have an image in *B*.

(b) *f* is not a function because the element *c* of *A* does not have a unique image in *B*.

(c) *f* is a function because every element of *A* has a unique image in *B*.

(d) *f* is a function because every element in *A* has a unique image in *B*.

Example 2.10 Which of the following relations from $R \to .\ R$ are functions?

(a) $y = 3x + 2$ (b) $y < x + 3$ (c) $y = 2x^2 + 1$

Solution : (a) $y = 3x + 2$ Here corresponding to every element $x \in R, \exists$ a unique element $y \in R$. $\therefore$ It is a function.

(b) $y < x + 3$.

For any real value of x we get more than one real value of y. $\therefore$ It is not a function.

(c) $y = 2x^2 + 1$

For any real value of x, we will get a unique real value of y. $\therefore$ It is a function.

Example 2.11 Let N be the set of natural numbers. Define a real valued function $f : N \to N$ by $f(x) = 2x + 1$. Using the definition find $f(1)$, $f(2)$, $f(3)$, $f(4)$

Solution : $f(x) = 2x + 1$

$$f(1) = 2 \times 1 + 1 = 2 + 1 = 3, \quad f(2) = 2 \times 2 + 1 = 4 + 1 = 5$$

$$f(3) = 2 \times 3 + 1 = 6 + 1 = 7, \quad f(4) = 2 \times 4 + 1 = 6 + 1 = 9$$

CHECK YOUR PROGRESS 2.2

1. Which of the following relations are functions from A to B ?

(a) $\{(1,-2), (3,7), (4,-6), (8,11)\}$, $A = \{1,3,4,8\}$, $B = \{-2, 7, -6, 11\}$

(b) $\{(1,0), (1,-1), (2,3), (4,10)\}$, $A = \{1, 2, 4\}$, $B = \{1, 0, -1, 3, 10\}$

(c) $\{(a,2), (b,3), (c,2), (d,3)\}$, $A = \{a, b, c, d\}$, $B = \{2, 3\}$

(d) $\{(1,1), (1,2), (2,3), (-3,4)\}$, $A = \{1, 2, -3\}$, $B = \{1, 2, 3, 4\}$

(e) $\left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), ..., \left(10, \frac{1}{10}\right)\right\}$, $A = \{1, 2, 3, 4\}$, $B = \left\{\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{11}\right\}$

(f) $\{(1,1), (-1,1), (2,4), (-2,4)\}$, $A = \{0, 1, -1, 2, -2\}$, $B = \{1, 4\}$

2. Which of the following relations represent a function ?

(a)

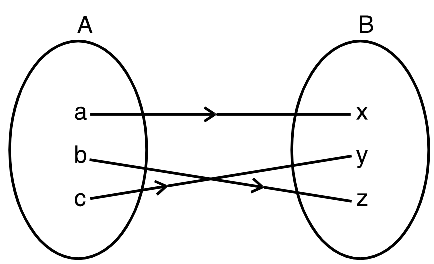


Fig. 2.7

(b)

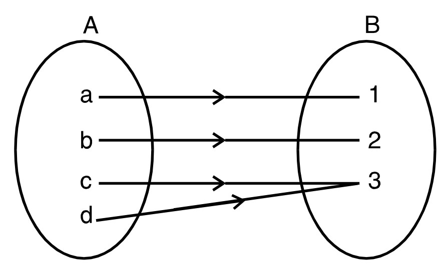


Fig. 2.8

(c)

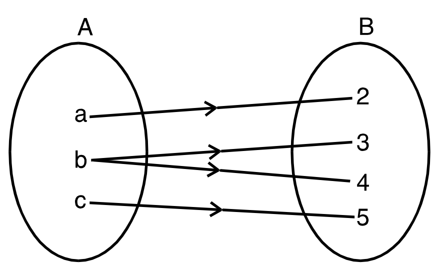


Fig. 2.9

(d)

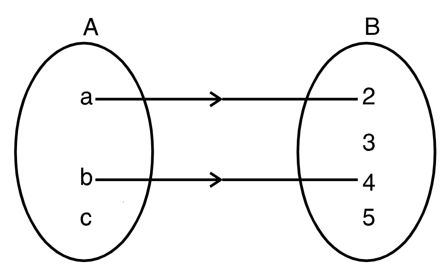


Fig. 2.10

3. Which of the following relations defined from $R \to R$ are functions ?

(a) $y = 2x + 1$ (b) $y > x + 3$ (c) $y < 3x + 1$ (d) $y = x^2 + 1$

4. Write domain and range for each of the following functions :

(a) $\{(\sqrt{2}, 2), (\sqrt{5}, -1), (\sqrt{3}, 5)\}$, (b) $\left\{\left(-3, \frac{1}{2}\right), \left(-2, \frac{1}{2}\right), \left(-1, \frac{1}{2}\right)\right\}$

(c) $\{(1,1), (0,0), (2,2), (-1,-1)\}$

(d) $\{(\text{Deepak}, 16), (\text{Sandeep}, 28), (\text{Rajan}, 24)\}$

5. Write domain and range for each of the following mappings :

(a)

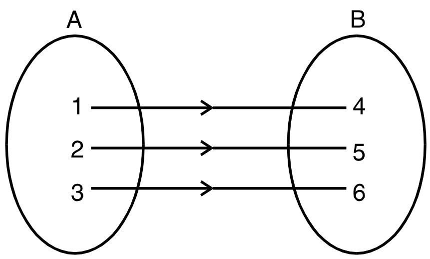


Fig. 2.11

(b)

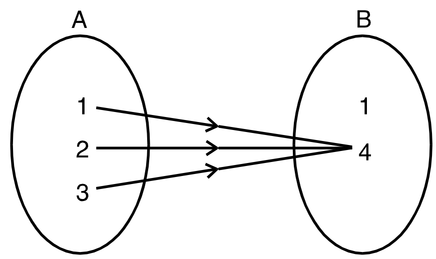


Fig. 2.12

(c)

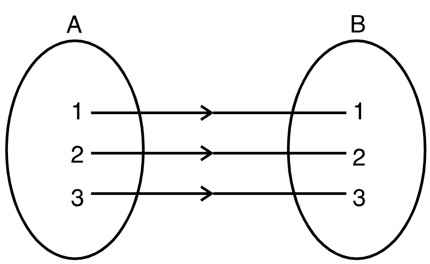


Fig. 2.13

(d)

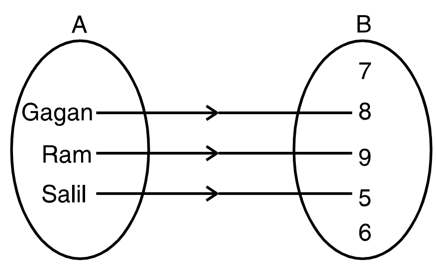


Fig. 2.14

(e)

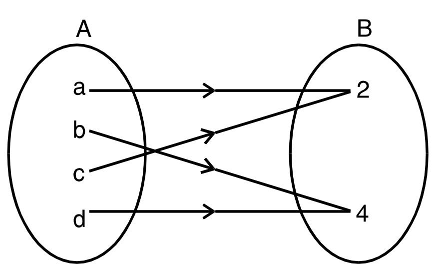


Fig. 2.15

2.3.2 Some More Examples on Domain and Range

Let us consider some functions which are only defined for a certain subset of the set of real numbers.

Example 2.12 Find the domain of each of the following functions :

(a) $y = \frac{1}{x}$ (b) $y = \frac{1}{x-2}$ (c) $y = \frac{1}{(x+2)(x-3)}$

Solution : The function $y = \frac{1}{x}$ can be described by the following set of ordered pairs.

$$\left\{ \ldots\ldots, \left(-2, -\frac{1}{2}\right), (-1,-1), (1,1) \left(2, \frac{1}{2}\right), \ldots. \right\}$$

Here we can see that x can take all real values except 0 because the corresponding image, i.e.,

$\frac{1}{0}$ is not defined. $\therefore$ Domain $= R - \{0\}$ [Set of all real numbers except 0]

Note : Here range $= R - \{0\}$

(b) *x* can take all real values except 2 because the corresponding image, i.e., $\frac{1}{(2-2)}$ does not exist. $\therefore$ Domain $= R - \{2\}$

(c) Value of y does not exist for $x = -2$ and $x = 3$ $\therefore$ Domain $= R - \{-2, 3\}$

Example 2.13 Find domain of each of the following functions :

(a) $y = +\sqrt{x-2}$ (b) $y = +\sqrt{(2-x)(4+x)}$

Solution :(a) Consider the function $y = +\sqrt{x-2}$

In order to have real values of y, we must have $(x-2) \geq 0$ i.e. $x \geq 2$

$\therefore$ Domain of the function will be all real numbers ≥ 2.

(b) $y = +\sqrt{(2-x)(4+x)}$

In order to have real values of y, we must have $(2-x)(4+x) \geq 0$

We can achieve this in the following two cases :

Case I : $(2-x) \geq 0$ and $(4+x) \geq 0 \Rightarrow x \leq 2$ and $x \geq -4$

$\therefore$ Domain consists of all real values of x such that $-4 \leq x \leq 2$

Case II : $2 - x \leq 0$ and $4 + x \leq 0 \Rightarrow 2 \leq x$ and $x \leq -4$.

But, x cannot take any real value which is greater than or equal to 2 and less than or equal to -4.

$\therefore$ From both the cases, we have Domain = $-4 \le x \le 2 \; \forall \; x \in R$

Example 2.14 For the function

$f(x) = y = 2x + 1$, find the range when domain $= \{-3, -2, -1, 0, 1, 2, 3\}$.

Solution : For the given values of *x*, we have

$$f(-3) = 2(-3) + 1 = -5, \; f(-2) = 2(-2) + 1 = -3$$

$$f(-1) = 2(-1) + 1 = -1, \; f(0) = 2(0) + 1 = 1$$

$$f(1) = 2(1) + 1 = 3, \; f(2) = 2(2) + 1 = 5, \; f(3) = 2(3) + 1 = 7$$

The given function can also be written as a set of ordered pairs.

i.e., $\{(-3, -5), (-2, -3), (-1, -1), (0, 1)(1, 3), (2, 5)(3, 7)\}$

$\therefore$ Range $= \{-5, -3, -1, 1, 3, 5, 7\}$

Example 2.15 If $f(x) = x + 3, \; 0 \le x \le 4$, find its range.

Solution : Here $0 \le x \le 4$ or $0 + 3 \le x + 3 \le 4 + 3$ or $3 \le f(x) \le 7$

$\therefore$ Range = $\{f(x) : 3 \le f(x) \le 7\}$

Example 2. 16 If $f(x) = x^2, \; -3 \le x \le 3$, find its range.

Solution : Given $-3 \le x \le 3$ or $0 \le x^2 \le 9$ or $0 \le f(x) \le 9$

$\therefore$ Range = $\{f(x) : 0 \le f(x) \le 9\}$

CHECK YOUR PROGRESS 2.3

1. Find the domain of each of the following functions $x \in R$:

(a) (i) $y = 2x$, (ii) $y = 9x + 3$, (iii) $y = x^2 + 5$

(b) (i) $y = \dfrac{1}{3x - 1}$, (ii) $y = \dfrac{1}{(4x + 1)(x - 5)}$

(iii) $y = \dfrac{1}{(x - 3)(x - 5)}$, (iv) $y = \dfrac{1}{(3 - x)(x - 5)}$

(c) (i) $y = \sqrt{6 - x}$, (ii) $y = \sqrt{7 + x}$, (iii) $y = \sqrt{3x + 5}$

(d) (i) $y = \sqrt{(3 - x)(x - 5)}$ (ii) $y = \sqrt{(x - 3)(x + 5)}$

(iii) $y = \dfrac{1}{\sqrt{(3+x)(7+x)}}$ (iv) $y = \dfrac{1}{\sqrt{(x-3)(7+x)}}$

2. Find the range of the function, given its domain in each of the following cases.

(a) (i) $f(x) = 3x + 10,\ x \in \{1, 5, 7, -1, -2\}$,

(ii) $f(x) = 2x^2 + 1,\ x \in \{-3, 2, 4, 0\}$

(iii) $f(x) = x^2 - x + 2,\ x \in \{1, 2, 3, 4, 5\}$

(b) (i) $f(x) = x - 2,\ 0 \le x \le 4$ (ii) $f(x) = 3x + 4,\ -1 \le x \le 2$

(c) (i) $f(x) = x^2,\ -5 \le x \le 5$ (ii) $f(x) = 2x,\ -3 \le x \le 3$

(iii) $f(x) = x^2 + 1,\ -2 \le x \le 2$ (iv) $f(x) = \sqrt{x},\ 0 \le x \le 25$

(d) (i) $f(x) = x + 5,\ x \in R$ (ii) $f(x) = 2x - 3,\ x \in R$

(iii) $f(x) = x^3,\ x \in R$ (iv) $f(x) = \dfrac{1}{x},\ \{x : x < 0\}$

(v) $f(x) = \dfrac{1}{x-2},\ \{x : x \le 1\}$ (vi) $f(x) = \dfrac{1}{3x-2},\ \{x : x \le 0\}$

(vii) $f(x) = \dfrac{2}{x},\ \{x : x > 0\}$ (viii) $f(x) = \dfrac{x}{x+5},\ \{x : x \ne -5\}$

2.4 GRAPHICAL REPRESENTATION OF FUNCTIONS

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

x	0	1	–1	2	–2	3	–3	4	–4
y	0	1	1	4	4	9	9	16	16

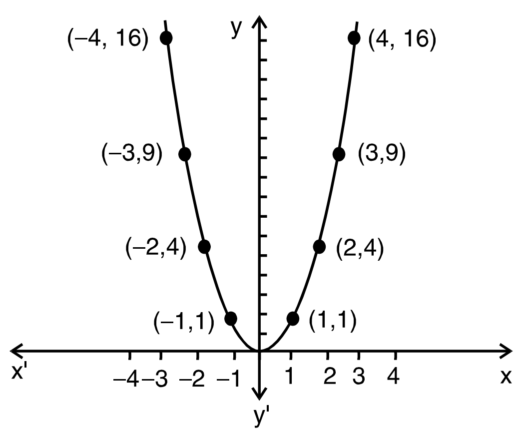


Fig. 2.16

Does this represent a function?

Yes, this represent a function because corresponding to each value of x $\exists$ a unique value of y.

Now consider the equation $x^2 + y^2 = 25$

x	0	0	3	3	4	4	5	–5	–3	–3	–4	–4
y	5	–5	4	–4	3	–3	0	0	4	–4	3	–3

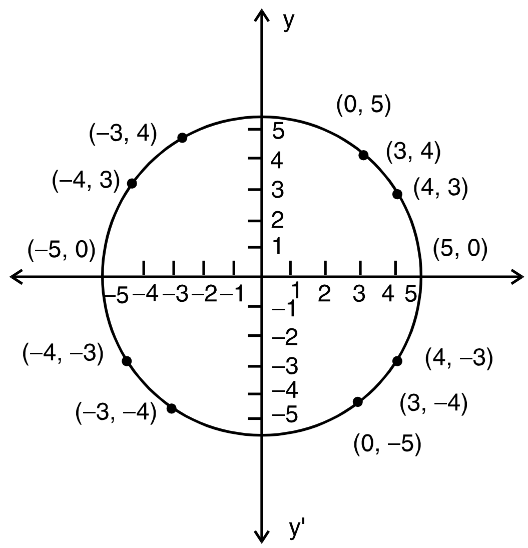


Fig. 2.17

This graph represents a circle. Does it represent a function ?

No, this does not represent a function because corresponding to the same value of x, there does not exist a unique value of y.

CHECK YOUR PROGRESS 2.4

1. (i) Does the graph represent a function?

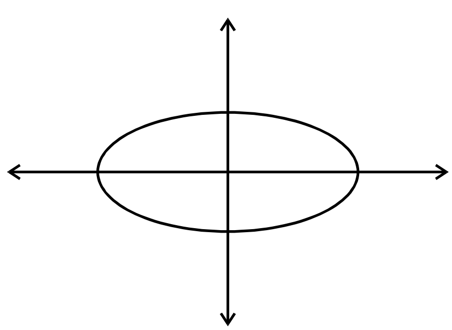


Fig. 2.18

(ii) Does the graph represent a function ?

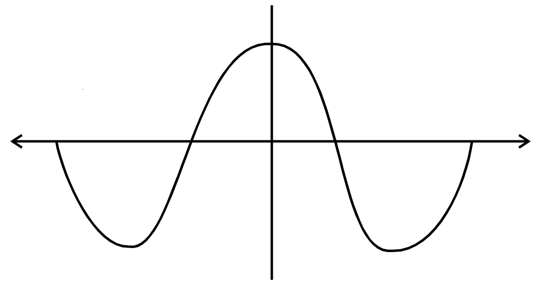


Fig. 2.19

2. Draw the graph of each of the following functions :

(a) $y = 3x^2$ (b) $y = -x^2$ (c) $y = x^2 - 2$

(d) $y = 5 - x^2$ (e) $y = 2x^2 + 1$ (f) $y = 1 - 2x^2$

3. Which of the following graphs represents a function ?

(a)

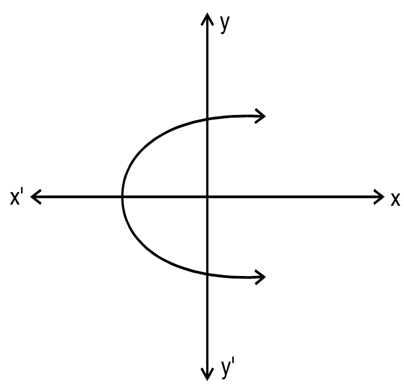


Fig. 2.20

(b)

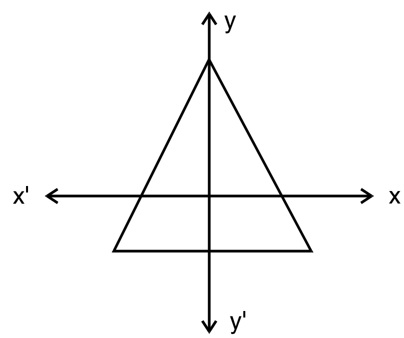


Fig. 2.21

(c)

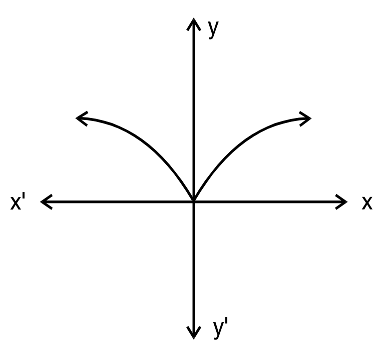


Fig. 2.22

(d)

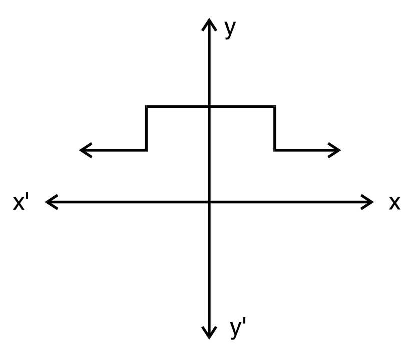


Fig. 2.23

(e)

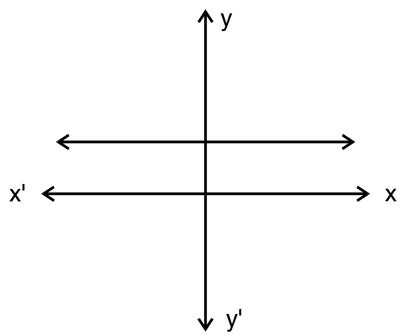


Fig . 2.24

Hint : If any line || to y-axis cuts the graph in more than one point, graph does not represent a function.

2.5 SOME SPECIAL FUNCTIONS

2.5.1 Monotonic Function

Let $F : A \to B$ be a function then F is said to be monotonic on an interval (a,b) if it is either increasing or decreasing on that interval.

For function to be increasing on an interval (a,b)

$$x_1 < x_2 \Rightarrow F(x_1) < F(x_2) \quad \forall \quad x_1 \; x_2 \in (a, b)$$

and for function to be decreasing on (a,b)

$$x_1 < x_2 \Rightarrow F(x_1) > F(x_2) \quad \forall \quad x_1 \; x_2 \in (a, b)$$

A function may not be monotonic on the whole domain but it can be on different intervals of the domain.

Consider the function $F : R \to R$ defined by $f(x) = x^2$.

Now $\quad \forall \quad x_1, x_2 \in [0, \infty], \; x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$

$\Rightarrow$ F is a ***Monotonic Function*** on $[0, \infty]$.

($\because$ It is only increasing function on this interval)

But $\quad \forall \; x_1, x_2 \in (-\infty, 0), \; x_1 < x_2 \Rightarrow F(x_1) > F(x_2)$

$\Rightarrow$ F is a ***Monotonic Function*** on $(-\infty, 0)$

($\because$ It is only a decreasing function on this interval)

Therefore if we talk of the whole domain given function is not monotonic on R but it is monotonic on $(-\infty, 0)$ and $(0, \infty)$. Again consider the function $F : R \to R$ defined by $f(x) = x^3$.

Clearly $\forall \; x_1 \; x_2 \in$ domain $x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$

$\therefore$ Given function is ***monotonic*** on R i.e. on the whole domain.

2.5.2 Even Function

A function is said to be an even function if for each x of domain $F(-x) = F(x)$

For example, each of the following is an ***even function***.

(i) If $F(x) = x^2$ then $F(-x) = (-x)^2 = x^2 = F(x)$

(ii) If $F(x) = \cos x$ then $F(-x) = \cos(-x) = \cos x = F(x)$

(iii) If $F(x) = |x|$ then $F(-x) = |-x| = |x| = F(x)$

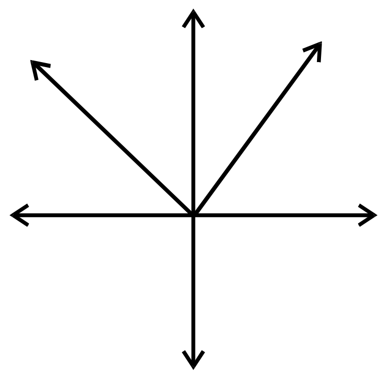


Fig. 2.25

The graph of this even function (modulus function) is shown in the figure above.

Observation Graph is symmetrical about y-axis.

2.5.3 Odd Function

A function is said to be an odd function if for each x

$$f(-x) = -f(x)$$

For example,

(i) If $f(x) = x^3$

then $f(-x) = (-x)^3 = -x^3 = -f(x)$

(ii) If $f(x) = \sin x$

then $f(-x) = \sin(-x) = -\sin x = -f(x)$

Graph of the odd function y = x is given in Fig.2.26

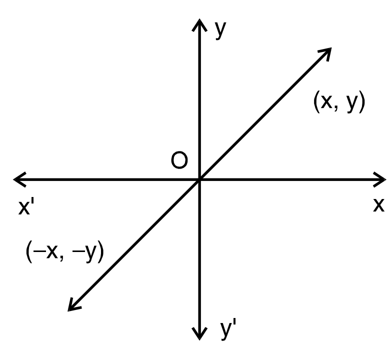


Fig. 2.26

Observation Graph is symmetrical about origin.

2.5.4 Greatest Integer Function (Step Function)

$f(x) = [x]$ which is the greatest integer less than or equal to x.

$f(x)$ is called Greatest Integer Function or Step Function. Its graph is in the form of steps, as shown in Fig. 2.27.

Let us draw the graph of $y = [x], x \in R$

$$[x] = 1, \quad 1 \le x < 2$$

$$[x] = 2, \quad 2 \le x < 3$$

$$[x] = 3, \quad 3 \le x < 4$$

$$[x] = 0, \quad 0 \le x < 1$$

$$[x] = -1, \quad -1 \le x < -0$$

$$[x] = -2, \quad -2 \le x < -1$$

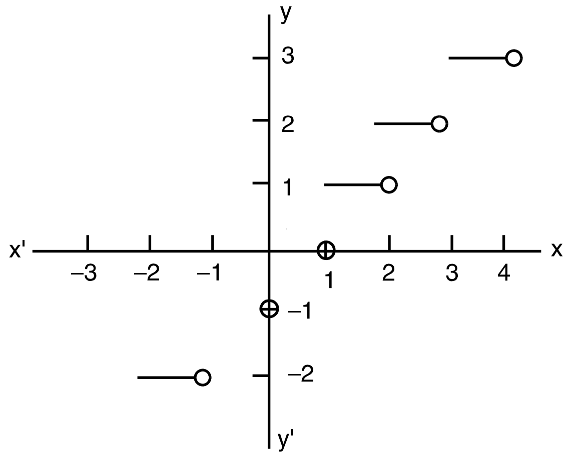


Fig. 2.27

- Domain of the step function is the set of real numbers.
- Range of the step function is the set of integers.

2.5.5 Polynomial Function

Any function defined in the form of a polynomial is called a polynomial function.

For example, (i) $f(x) = 3x^2 - 4x - 2$, (ii) $f(x) = x^3 - 5x^2 - x + 5$, (iii) $f(x) = 3$ are all polynomial functions.

Note : Functions of the type $f(x) = k$, where k is a constant is also called a constant function.

2.5.6 Rational Function

Function of the type $f(x) = \frac{g(x)}{h(x)}$, where $h(x) \ne 0$ and $g(x)$ and $h(x)$ are polynomial functions are called rational functions.

For example, $f(x) = \frac{x^2 - 4}{x + 1}, x \ne -1$ is a rational function.

2.5.7 Reciprocal Function: Functions of the type $y = \frac{1}{x}, x \ne 0$ is called a reciprocal function.

2.5.8 Exponential Function A swiss mathematician Leonhard Euler introduced a number e in the form of an infinite series. In fact

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots + \frac{1}{\lfloor n} + \ldots\ldots \qquad \ldots\ldots(1)$$

It is well known that the sum of this infinite series tends to a finite limit (i.e., this series is convergent) and hence it is a positive real number denoted by e. This number e is a transcendental irrational number and its value lies between 2 an 3.

Now consider the infinite series $1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + ... + \frac{x^n}{\lfloor n} + ...$

It can be shown that the sum of this infinite series also tends to a finite limit, which we denote by

e^x. Thus, $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + ... + \frac{x^n}{\lfloor n} + ...$ ......(2)

This is called the **Exponential Theorem** and the infinite series is called the **exponential series**. We easily see that we would get (1) by putting x = 1 in (2).

The function $f(x) = e^x$, where x is any real number is called an **Exponential Function**.

The graph of the exponential function $y = e^x$ is obtained by considering the following important facts :

(i) As x increases, the y values are increasing very rapidly, whereas as x decreases, the y values are getting closer and closer to zero.

(ii) There is no x-intercept, since $e^x \neq 0$ for any value of x.

(iii) The y intercept is 1, since $e^0 = 1$ and $e \neq 0$.

(iv) The specific points given in the table will serve as guidelines to sketch the graph of e^x (Fig. 2.28).

x	–3	–2	–1	0	1	2	3
$y = e^x$	0.04	0.13	0.36	1.00	2.71	7.38	20.08

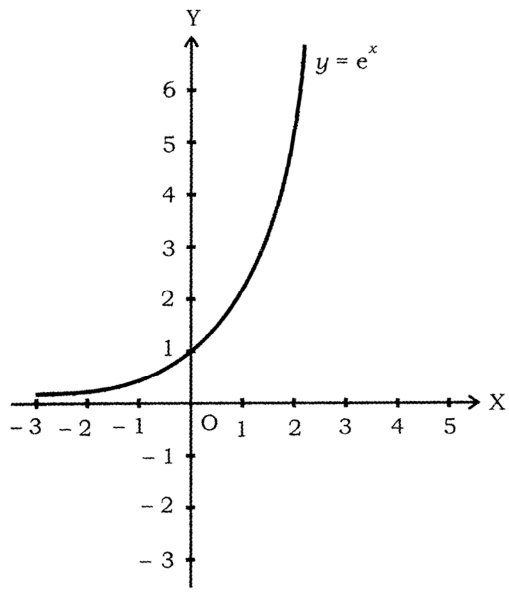


Fig. 2.28

If we take the base different from e, say a, we would get exponential function

$$f(x) = a^x, \text{provbided } a > 0, \quad a \neq 1.$$

For example, we may take a = 2 or a = 3 and get the graphs of the functions

$$y = 2^x \text{ (See Fig. 2.29)}$$

and $\quad y = 3^x$ (See Fig. 2.30)

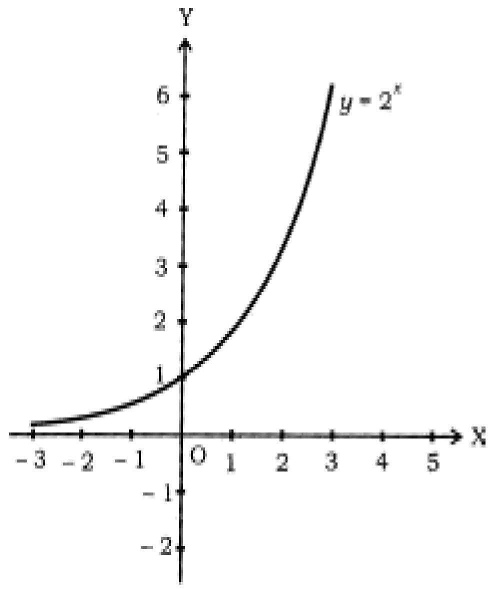


Fig. 2.29

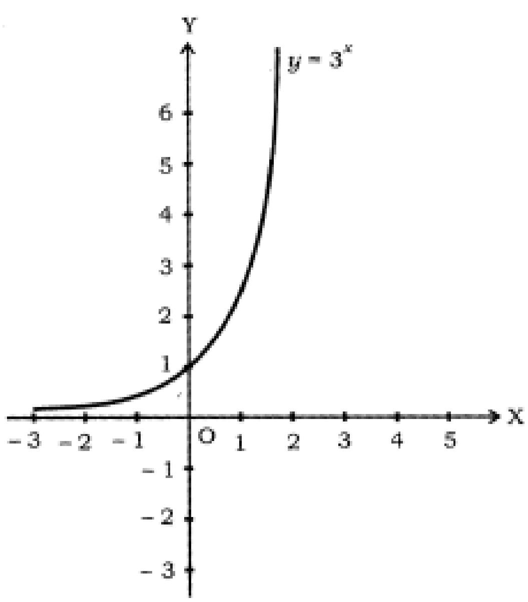


Fig. 2.30

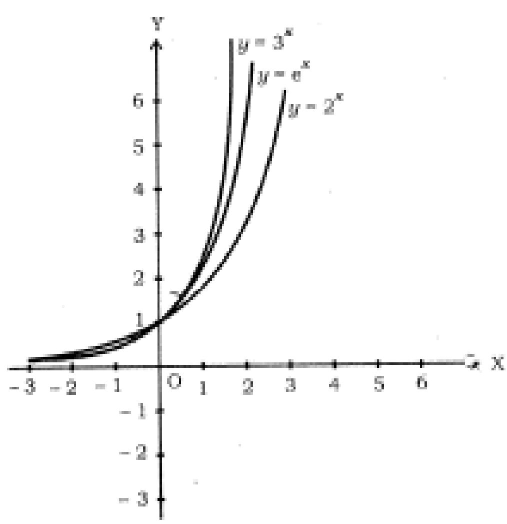


Fig. 2.31

2.5.9 Logarithmic Functions

Now Consider the function $y = e^x$ .....(3)

We write it equivalently as $x = \log_e y$ Thus, $y = \log_e x$ .....(4)

is the inverse function of $y = e^x$

The base of the logarithm is not written if it is e and so $\log_e x$ is usually written as log x.

As $y = e^x$ and $y = \log x$ are inverse functions, their graphs are also symmetric w.r.t. the line $y = x$. The graph of the function $y = \log x$ can be obtained from that of $y = e^x$ by reflecting it in the line y = x.

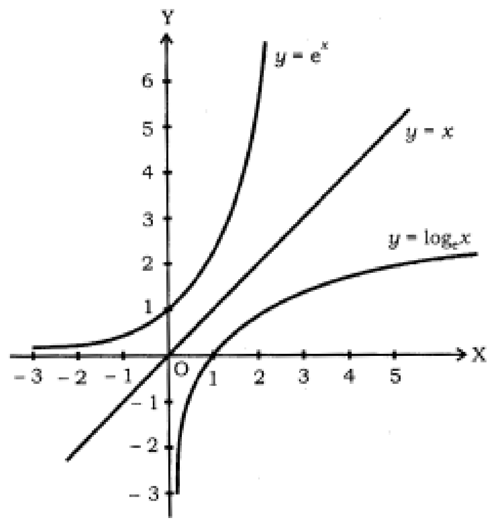


Fig. 2.32

Note

(i) The learner may recall the laws of indices which you have already studied in the Secondary Mathematics : If a > 0, and m and n are any rational numbers, then

$$a^m \cdot a^n = a^{m+n},\ a^m \div a^n = a^{m-n},\ (a^m)^n = a^{mn},\ a^0 = 1$$

(ii) The corresponding laws of logarithms are

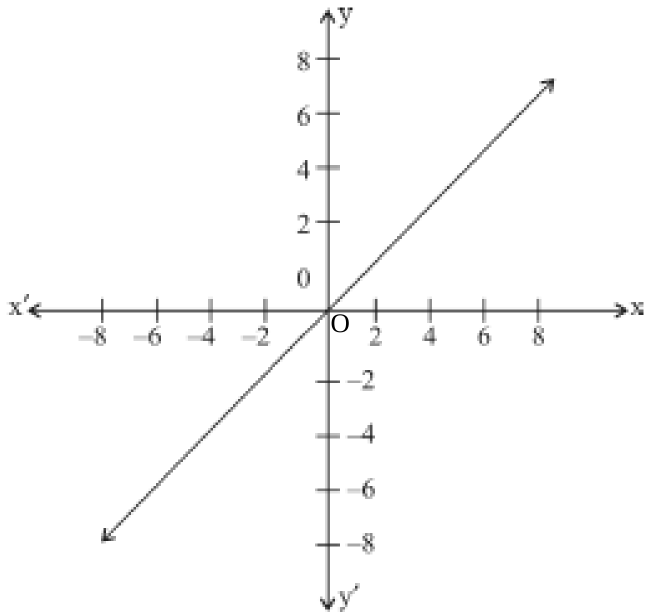
$$\log_a(mn) = \log_a m + \log_a n,\ \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a(m^n) = n\log_a m,\ \log_b m = \frac{\log_a m}{\log_a b} \text{ or } \log_b m = \log_a m \log_b a$$

Here a, b > 0, $a \neq 1$, $b \neq 1$.

2.5.10 Identity Function

Let R be the set of real numbers. Define the real valued function f:R→R by $y = f(x) = x$ for each $x \in$R. Such a function is called the identity function. Here the domain and range of f are R. The graph is a straight line. It passes through the origin.

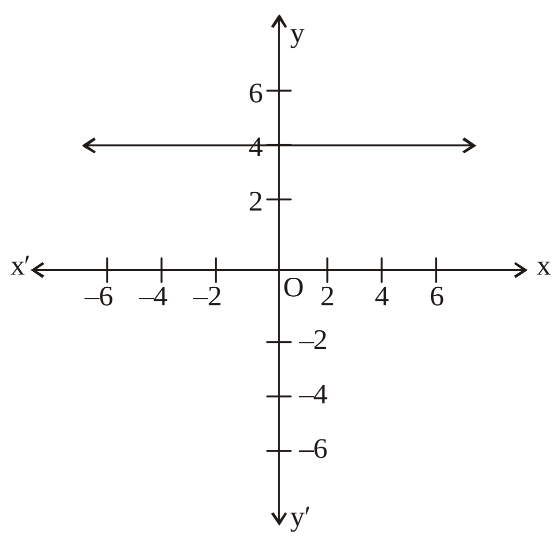


$f(x) = x$ **Fig. 2.33**

2.5.11 Constant Function

Define the function f : R → R by $y = f(x) = c$, $x \in$R where c is a constant and each $x \in$R. Here domain of f is R and its range is $\{c\}$.

The graph is a line parallel to x-axis. For example, $f(x) = 4$ for each $x \in$R, then its graph will be shown as



$f(x) = 4$ **Fig. 2.34**

2.5.12 Signum Function

The function $f : \mathrm{R} \to \mathrm{R}$ defined by $f(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{if } x < 0 \end{cases}$ is called a signum function.

The domain of the signum function is R and the range is the set {–1, 0, 1}.

The graph of the signum function is given as under :

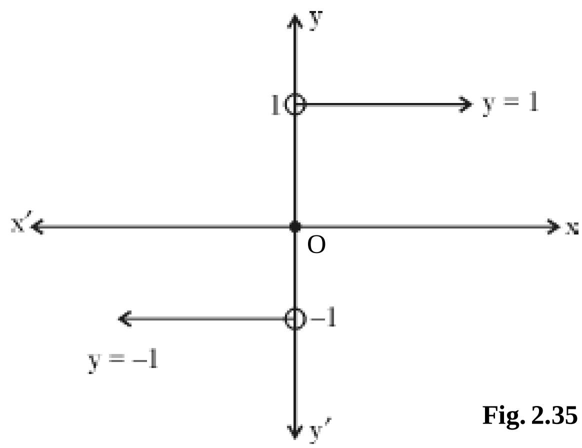


Fig. 2.35

CHECK YOUR PROGRESS 2.5

1. Which of the following statements are true or false.

   (i) function $f(x) = 2x^4 + 7x^2 + 9x$ is an even function.

   (ii) Odd function is symmetrical about y-axis.

   (iii) $f(x) = x^{1/2} - x^3 + x^5$ is a polynomial function.

   (iv) $f(x) = \dfrac{x-3}{3+x}$ is a rational function for all $x \in \mathrm{R}$.

   (v) $f(x) = \dfrac{\sqrt{5}}{3}$ is a constant function.

   (vi) Domain of the function defined by $f(x) = \dfrac{1}{x}$ is the set of real numbers except 0.

   (vii) Greatest integer function is neither even nor odd.

2. Which of the following functions are even or odd functions ?

   (a) $f(x) = \dfrac{x^2 - 1}{x + 1}$ (b) $f(x) = \dfrac{x^2}{5 + x^2}$ (c) $f(x) = \dfrac{1}{x^2 + 5}$

(d) $f(x) = \frac{2}{x^3}$ (e) $f(x) = \frac{x}{x^2+1}$ (f) $f(x) = \frac{5}{x-5}$

(g) $f(x) = \frac{x-3}{3+x}$ (h) $f(x) = x - x^3$

3. Draw the graph of the function $y = [x] - 2$.

4. Specify the following functions as polynomial function, rational function, reciprocal function or constant function.

(a) $y = 3x^8 - 5x^7 + 8x^5$ (b) $y = \frac{x^2+2x}{x^3-2x+3},\ x^3 - 2x + 3 \neq 0$

(c) $y = \frac{3}{x^2},\ x \neq 0$ (d) $y = 3 + \frac{2x+1}{x},\ x \neq 0$

(e) $y = 1 - \frac{1}{x},\ x \neq 0$ (f) $y = \frac{x^2-5x+6}{x-2},\ x \neq 2$

(g) $y = \frac{1}{9}$.

2.6 Sum, difference, product and quotient of functions

(i) Addition of two real functions :

Let $f : X \to R$ and $g : X \to R$ be any two functions, where $X \subset R$. Then, we define $(f + g) : X \to R$ by $(f + g)(x) = f(x) + g(x)$, for all $x \in X$

Let $f(x) = x^2$, $g(x) = 2x + 1$

Then $(f + g)(x) = f(x) + g(x) = x^2 + 2x + 1$

(ii) **Subtraction of a real function from another :**

Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$. Then, we define $(f - g) : X \to R$ by $(f - g)x = f(x) - g(x)$, for all $x \in X$

Let $f(x) = x^2$, $g(x) = 2x + 1$

then $(f - g)(x) = f(x) - g(x) = x^2 - (2x + 1) = x^2 - 2x - 1$

(iii) **Multiplication of two real functions :**

The product of two real functions $f : X \to R$ and $g : X \to R$ is a function $f g : X \to R$ defined by $(f g)(x) = f(x) . g(x)$, for all $x \in X$

Let $f(x) = x^2$, $g(x) = 2x + 1$

Then $f g(x) = f(x) . g(x) = x^2 . (2x + 1) = 2x^3 + x^2$

(iv) **Quotient of two real functions :**

Let f and g be two real functions defined from $X \to R$ where $X \subset R$. The real quotient of f by g denoted by $\frac{f}{g}$ is a function defined by

$$\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}, \text{ provided } g(x) \neq 0,\ x \in X$$

Let $f(x) = x^2,\ g(x) = 2x + 1$

Then $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}, x \neq \frac{-1}{2}$

Example 2. 17 Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f + g)(x)$, $(f - g)(x)$, $(f\,g)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution : We have $f(x) = \sqrt{x}, g(x) = x$

Then $(f + g)(x) = f(x) + g(x) = \sqrt{x} + x$

$(f - g)(x) = f(x) - g(x) = \sqrt{x} - x$

$(f\,g)(x) = f(x) \,.\, g(x) = \sqrt{x}\,.\,x = x^{3/2}$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = x^{\frac{-1}{2}}, x \neq 0$

CHECK YOUR PROGRESS 2.6

1. A function f is defined as $f(x) = 3x + 4$. Write down the values of

   (*i*) $f(0)$ (*ii*) $f(7)$ (*iii*) $f(-3)$

2. Let $f, g : R \to R$ be defined, respectively by $f(x) = x + 1$, $g(x) = 2x - 3$. Find $(f + g)$, $(f - g)$ $(f\,g)$ and $\left(\frac{f}{g}\right)$.

LET US SUM UP

- Cartesian product of two sets A and B is the set of all ordered pairs of the elements of A and B. It is denoted by $A \times B$ i.e $A \times B = \{(a,b): a \in A \text{ and } b \in B\}$

- Relation is a sub set of A×B where *A* and *B* are sets.

  i.e. $R \subseteq A \times B = \{(a, b) : a \in A \text{ and } b \in B \text{ and } aRb\}$

- Function is a special type of relation.

- Functrions $f : A \rightarrow B$ is a rule of correspondence from *A* to *B* such that to every element of *A* $\exists$ a unique element in *B*.

- Functions can be described as a set of ordered pairs.

- Let *f* be a function from *A* to *B*.

  Domain : Set of all first elements of ordered pairs belonging to *f*.

  Range : Set of all second elements of ordered pairs belonging to *f*.

- Functions can be written in the form of equations such as $y = f(x)$

  where x is independent variable, y is dependent variable.

  Domain : Set of independent variable.

  Range : Set of dependent variable.

  Every equation does not represent a function.

- **Vertical line test :** To check whether a graph is a function or not, we draw a line parallel to y-axis. If this line cuts the graph in more than one point, we say that graph does not represent a function.

- A function is said to be monotonic on an interval if it is either increasing or decreasing on that interval.

- A function is called even function if $f(x) = f(-x)$, and odd function if $f(-x) = -f(x),\ x, -x \in D_f$

- $f, g : X \rightarrow R$ and $X \subset R$, then

  $(f + g)(x) = f(x) + g(x),\ (f - g)(x) = f(x) - g(x)$

  $(f \cdot g)x = f(x) \cdot g(x),\ \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided $g(x) \neq 0$.

- A real function has the set of real number or one of its subsets both as its domain and as its range.

SUPPORTIVE WEB SITES

http://www.bbc.co.uk/education/asguru/maths/13pure/02functions/06composite/ index.shtml

http://en.wikipedia.org/wiki/functions

http://en.wikipedia.org/wiki/relations

TERMINAL EXERCISE

1. Given $A = \{a, b, c,\}$, $B = \{2, 3\}$. Find the number of relations from A to B.

2. Given that $A = \{7, 8, 9\}$, $B = \{9, 10, 11\}$, $C = \{11, 12\}$ verify that

   (i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

   (ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

3. Which of the following equations represent functions? In each of the case $x \in R$

   (a) $y = \dfrac{2x+3}{4-5x}, x \neq \dfrac{4}{5}$ (b) $y = \dfrac{3}{x}, x \neq 0$ (c) $y = \dfrac{3}{x^2 - 16}, x \neq 4, -4$

   (d) $y = \sqrt{x-1},\ x \geq 1$ (e) $y = \dfrac{1}{x^2+1}$ (f) $x^2 + y^2 = 25$

4. Write domain and range of the following functions :

   $f_1 : \{(0,1), (2,3), (4,5)(6,7), ... (100, 101)\}$

   $f_2 : \{(-2,4), (-4,16), (-6,36) ....\}$

   $f_3 : \left\{(1,1), \left(\dfrac{1}{2}, -1\right), \left(\dfrac{1}{3}, 1\right)\left(\dfrac{1}{4}, -1\right), ...\right\}$

   $f_4 : \{........ (3,0), (-1,2), (4,-1)\}$

   $f_5 : \{........ (-3,3), (-2,2), (-1,1)(0,0)(1,1), (2,2), ...\}$

5. Write domain of the following functions :

   (a) $f(x) = x^3$ (b) $f(x) = \dfrac{1}{x^2 - 1}$ (c) $f(x) = \sqrt{3x+1}$

   (d) $f(x) = \dfrac{1}{\sqrt{(x+1)(x+6)}}$ (e) $f(x) = \dfrac{1}{\sqrt{(x-1)(2x-5)}}$

6. Write range of each of the following functions :

   (a) $y = 3x + 2,\ x \in R$ (b) $y = \dfrac{1}{x-2,}, x \in R - \{2\}$

   (c) $y = \dfrac{x-1}{x+1}, x \in \{0, 2, 3, 5, 7, 9\}$ (d) $y = \dfrac{2}{\sqrt{x}},\ x \in R^+$

   (All non-negative real values)

7. Draw the graph of each of the following functions :

(a) $y = x^2 + 3,\ x \in R$ (b) $y = \dfrac{1}{x-2}, x \in R - \{2\}$

(c) $y = \dfrac{x-1}{x+1}, x \in \{0, 2, 3, 5, 7, 9\}$ (d) $y = \dfrac{1}{\sqrt{x}}, x \in R^+.$

8. Which of the following graphs represent a function ?

(a)

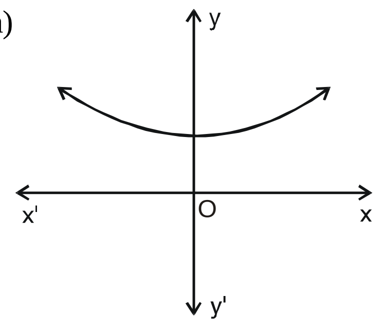


Fig. 2.36

(b)

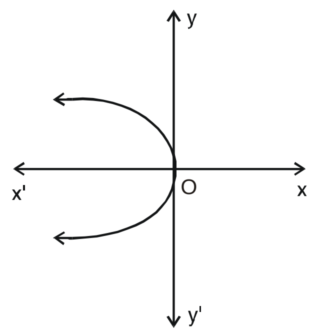


Fig. 2.37

(c)

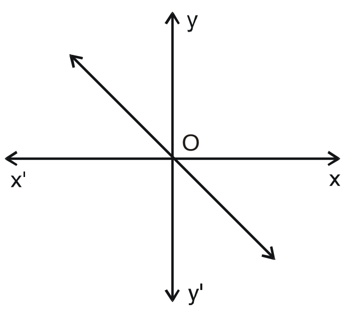


Fig. 2.38

(d)

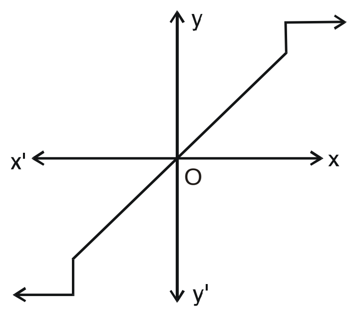


Fig. 2.39

(e)

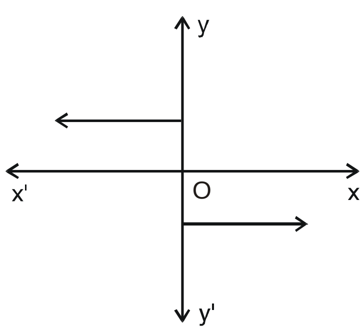


Fig. 2.40

(f)

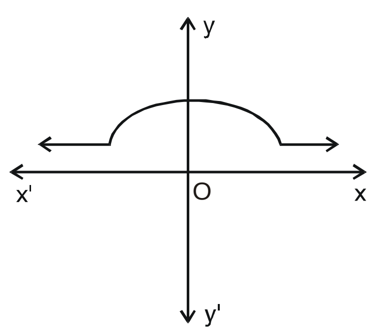


Fig. 2.41

(g)

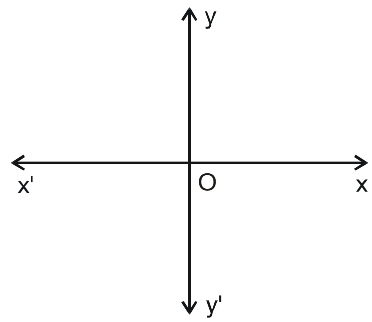


Fig. 2.42

(h)

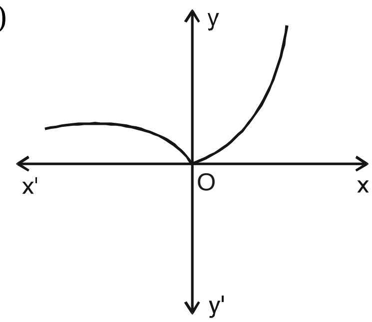


Fig. 2.43

9. Which of the following functions are rational functions ?

(a) $f(x) = \dfrac{2x-3}{x+2}, x \in R - \{-2\}$ (b) $f(x) = \dfrac{x}{\sqrt{x}}, x \in R^+$

(c) $f(x) = \dfrac{x+2}{x^2+4x+4}, x \in R - \{-2\}$ (d) $y = x, x \in R$

10. Which of the following functions are polynomial functions ?

(a) $f(x) = x^2 + \sqrt{3}\,x + 2$ (b) $f(x) = (x+2)^2$

(c) $f(x) = 3 - x + 2x^3 - x^4$ (d) $f(x) = \sqrt{x} + x - 5, \ x \ge 0$

(e) $f(x) = \sqrt{x^2 - 4}, x \notin (-2, 2)$

11. Which of the following functions are even or odd functions ?

(a) $f(x) = \sqrt{9 - x^2} \quad x \in [-3, 3]$ (b) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

(c) $f(x) = |x|$ (d) $f(x) = x - x^5$

(e)

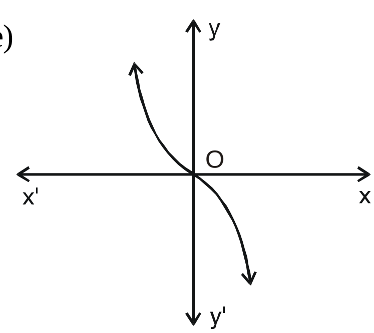


Fig. 2.44

(f)

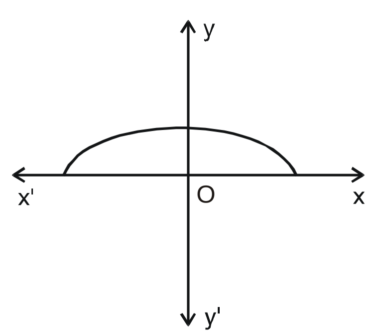


Fig. 2.45

(g)

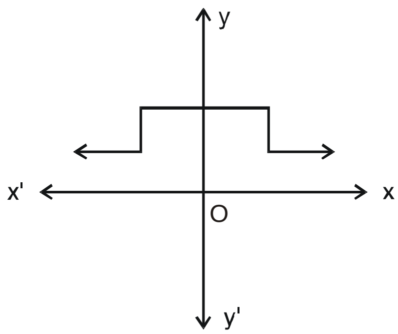


Fig. 2.46

12. Let 'f' be a function defined by $f(x) = 5x^2 + 2$, $x \in R$.
    (i) Find the image of 3 under f.
    (ii) Find $f(3) \times f(2)$
    (iii) Find x such that $f(x) = 22$
13. Let $f(x) = x + 2$ and $g(x) = 2x - 3$ be two real functions. Find the following functions
    (i) $f + g$ (ii) $f - g$
    (iii) $f \cdot g$ (iv) $\frac{f}{g}$
14. If $f(x) = (2x + 5)$, $g(x) = x^2 - 1$ are two real valued functions, find the following functions
    (i) $f + g$ (ii) $f - g$ (iii) $f\,g$
    (iv) $\frac{f}{g}$ (v) $\frac{g}{f}$

ANSWERS

CHECK YOUR PROGRESS 2.1

2. $\{(2,1),(4,1),(2,4),(4,4)\}$.

3. (i) $R=\{(4,2),(4,4),(6,2),(6,3),(8,2),(8,4),(10,2),(10,5)\}$.

   (ii) Domain of $R=\{4,6,8,10\}$. (iii) Range of $R=\{2,3,4,5\}$.

4. (i) $R=\{(1,8),(2,4)\}$. (ii) Domain of $R=\{1,2\}$.

   (iii) Range of $R=\{1,2\}\{8,4\}$

5. (i) $R=\{(2,4),(3,9),(5,25),(7,49),(11,121),(13,169)\}$

   Domain of $R=\{2,3,5,7,11,13\}$, Range of $R=\{4,9,25,49,121,169\}$.

6. (i) Domain of $R=\phi$ (ii) Domain of $R=\phi$ (iii) Range of $R=\phi$

7. x = 2, y = 3

8. {(-1,-1,-1), (-1,-1,1), (-1,1,-1), (-1,1,1), (1,-1,-1), (1,-1,1), (1,1,-1), (1,1,1)

9. A={a,b}, B={x,y}.

10. 18

CHECK YOUR PROGRESS 2.2

1. (a), (c), (f) 2. (a), (b) 3. (a), (d)

4. (a) Domain $=\left\{\sqrt{2},\sqrt{5},\sqrt{3}\right\}$, Range $=\{2,-1,5\}$

   (b) Domain $=\{-3,-2,-1\}$, Range $=\left\{\frac{1}{2}\right\}$

   (c) Domain $=\{1,0,2,-1\}$, Range $=\{1,0,2,-1\}$

   (d) Domain = {Deepak, Sandeep, Rajan}, Range $=\{16,28,24\}$.

5. (a) Domain $=\{1,2,3\}$, Range $=\{4,5,6\}$

   (b) Domain $=\{1,2,3\}$, Range $=\{4\}$

   (c) Domain $=\{1,2,3\}$, Range $=\{1,2,3\}$

(d) Domain $= \{\text{Gagan, Ram, Salil}\}$, Range $= \{8, 9, 5\}$

(e) Domain $= \{a.b, c, d\}$, Range $= \{2, 4\}$

CHECK YOUR PROGRESS 2.3

1. (a) (i) Domain = Set of real numbers. (ii) Domain = Set of real numbers.
(iii) Domain = Set of a real numbers.

(b) (i) Domain $= R - \left\{\frac{1}{3}\right\}$ (ii) Domain $= R - \left\{-\frac{1}{4}, 5\right\}$

(iii) Domain $= R - \{3, 5\}$ (iv) Domain $= R - \{3, 5\}$

(c) (i) Domain $= \{x \in R : x \le 6\}$ (ii) Domain $= \{x \in R : x \ge -7\}$

(iii) Domain $= \left\{x : x \in R, x \ge -\frac{5}{3}\right\}$

(d) (i) Domain $= \{x : x \in R \text{ and } 3 \le x \le 5\}$

(ii) Domain $= \{x : x \in R \;\; x \ge 3, x \le -5\}$

(iii) Domain $= \{x : x \in R \;\; x \ge -3, x \le -7\}$

(iv) Domain $= \{x : x \in R \;\; x \ge 3, x \le -7\}$

2. (a) (i) Range $= \{13, 25, 31, 7, 4\}$ (ii) Range $= \{19, 9, 33, 1\}$

(iii) Range $= \{2, 4, 8, 14, 22\}$

(b) (i) Range $= \{f(x) : -2 \le f(x) \le 2\}$ (ii) Range $= \{f(x) : 1 \le f(x) \le 10\}$

(c) (i) Range $= \{f(x) : 1 \le f(x) \le 25\}$ (ii) Range $= \{f(x) : -6 \le f(x) \le 6\}$

(iii) Range $= \{f(x) : 1 \le f(x) \le 5\}$ (iv) Range $= \{f(x) : 0 \le f(x) \le 5\}$

(d) (i) Range = R (ii) Range = R (iii) Range = R

(iv) Range $= \{f(x) : f(x) < 0\}$

(v) Range $= \{f(x) : -1 \le f(x) < 0\}$

(vi) Range $= \{f(x) : 0.5 \le f(x) < 0\}$ (vii) Range $= \{f(x) : f(x) > 0\}$

(viii) Range : All values of $f(x)$ except values at $x = -5$.

CHECK YOUR PROGRESS 2.4

1. (i) No. (ii) Yes

2. (a)

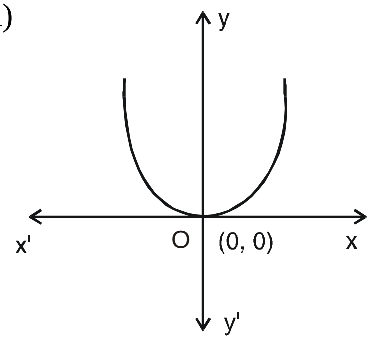


Fig. 2.47

(b)

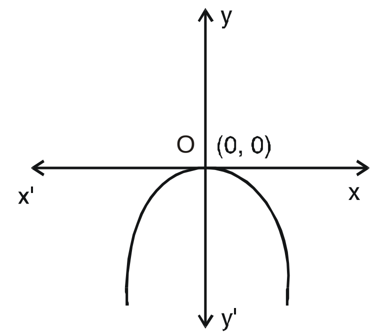


Fig. 2.48

(c)

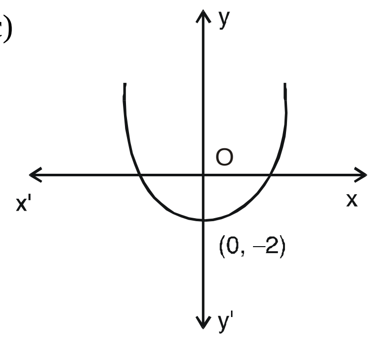


Fig. 2.49

(d)

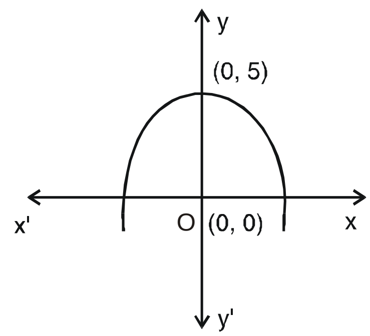


Fig. 2.50

(e)

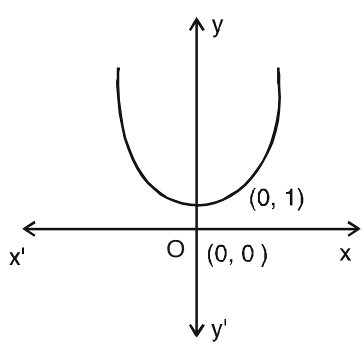


Fig. 2.51

(f)

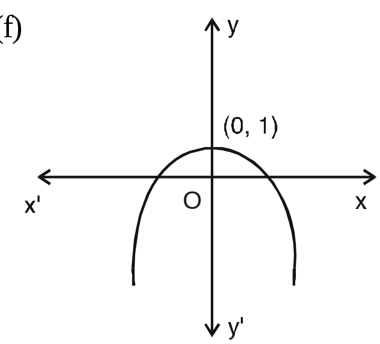


Fig. 2.52

3. (c), (d) and (e).

CHECK YOUR PROGRESS 2.5

1. v, vi, vii are true statements.
(i), (ii), (iii), (iv) are false statement.

2. (b) (c) are even functions. (d) (e) (h) are odd functions.

3.

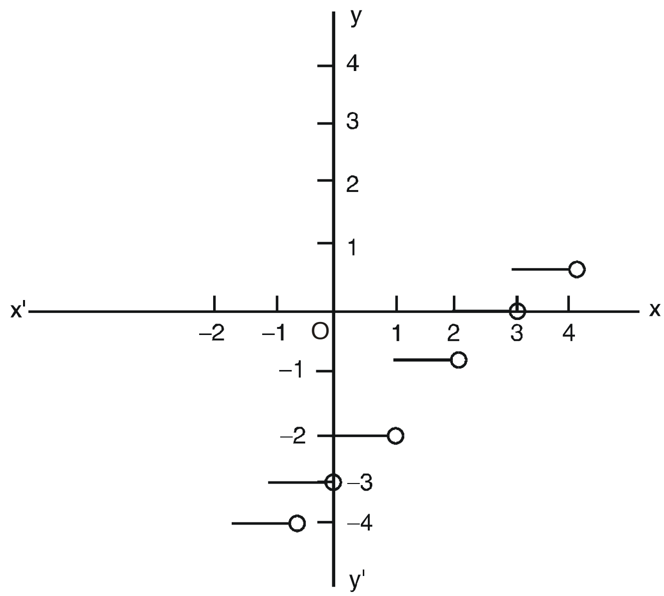


Fig. 2.53

4. (a) Polynomial function (b) Rational function. (c) Rational function.
(d) Rational function. (e) Rational function. (f) Rational function.
(g) Constant function.

CHECK YOUR PROGRESS 2.6

1. (i) 4 (ii) 25 (iii) -5

2. (f+g) x=3x-2, (f-g) x=4-x , (f.g) x= $2x^2 - x - 3$, $\left(\frac{f}{g}\right)x = \frac{x-1}{2x-3},\ x \neq \frac{3}{2}$

TERMINAL EXERCISE

2. 2^6 i.e., 64.

3. (a), (b), (c), (d), (e) are functions.

4. f_1 – Domain = $\{0, 2, 4, 6, ...100\}$ Range = $\{1, 3, 5, 7, ...101\}$.

f_2 – Domain = $\{-2, -4, -6, ...\}$. Range = $\{4, 16, 36, ...\}$.

f_3 – Domain = $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$. Range = $\{1, -1\}$.

f_4 – Domain $= \{3, -1, 4\}$. Range $= \{0, 2, -1\}$.

f_5 – Domain $= \{...... -3, -2, -1, 0, 1, 2, 3, ...\}$. Range $= \{0, 1, 2, 3, ...\}$.

5. (a) Domain $= R$.

(b) Domain $= R - \{-1, 1\}$.

(c) Domain $= x \geq -\frac{1}{3} \quad \forall x \in R$.

(d) Domain $= x \geq -1, x \leq -6$.

(e) Domain $= x \geq \frac{5}{2}, x \leq 1$.

6. (a) Range $= R$

(b) Range = All values of y except at x = 2.

(c) Range $= \left\{-1, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$

(d) Range = All values of y for $x > 0$

8. (a), (c), (e), (f), (h). Use hint given in check your progress 15.7, Q. No. 7 for the solution.

9. (a), (c)

10. (a), (b), (c)

11. Even functions : (a), (b), (c), (f), (g)

Odd functions : (d), (e)

12. (i) $f(3) = 47$ (ii) $f(3) \times f(2) = 1034$ (iii) $x = 2, -2$

13. (i) $f + g = 3x - 1$ (ii) $f - g = -x + 5$ (iii) $fg = 2x^2 + x - 6$

(iv) $f/g = \frac{x+2}{2x-3}, x \neq \frac{3}{2}$

14. (i) $f + g = x^2 + 2x + 4$ (ii) $f - g = -x^2 + 2x + 6$

(iii) $f \cdot g = 2x^3 + 5x^2 - 2x - 5$ (iv) $f/g = \frac{2x+5}{x^2-1}, x \neq \pm 1$

(v) $g/f = \frac{x^2-1}{2x+5}, x \neq \frac{-5}{2}$

3

TRIGONOMETRIC FUNCTIONS-I

We have read about trigonometric ratios in our earlier classes.

Recall that we defined the ratios of the sides of a right triangle as follows :

$\sin\theta = \frac{c}{b}$, $\cos\theta = \frac{a}{b}$, $\tan\theta = \frac{c}{a}$

and $\operatorname{cosec}\theta = \frac{b}{c}$, $\sec\theta = \frac{b}{a}$, $\cot\theta = \frac{a}{c}$

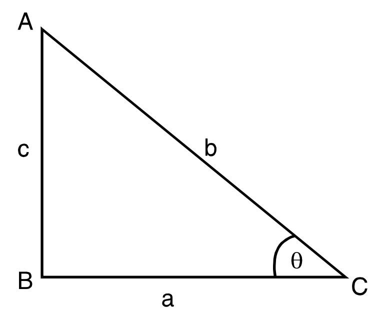


Fig.3.1

We also developed relationships between these trigonometric ratios as $\sin^2\theta + \cos^2\theta = 1$,

$\sec^2\theta = 1 + \tan^2\theta$, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

We shall try to describe this knowledge gained so far in terms of functions, and try to develop this lesson using functional approach.

In this lesson, we shall develop the science of trigonometry using functional approach. We shall develop the concept of trigonometric functions using a unit circle. We shall discuss the radian measure of an angle and also define trigonometric functions of the type

y = sin x, y = cos x, y = tan x, y = cot x, y = sec x, y = cosec x, y = a sin x, y = b cos x, etc., where *x, y* are real numbers. We shall draw the graphs of functions of the type

y = sin x, y = cos x, y = tan x, y = cotx, y = secx, and y = cosecx y = a sin x, y = a cos x.

OBJECTIVES

After studying this lesson, you will be able to :

- define positive and negative angles;
- define degree and radian as a measure of an angle;
- convert measure of an angle from degrees to radians and vice-versa;
- state the formula $\ell = r\theta$ where r and θ have their usual meanings;
- solve problems using the relation $\ell = r\theta$;
- define trigonometric functions of a real number;
- draw the graphs of trigonometric functions; and
- interpret the graphs of trigonometric functions.

EXPECTED BACKGROUND KNOWLEDGE

- Definition of an angle.
- Concepts of a straight angle, right angle and complete angle.
- Circle and its allied concepts.
- Special products : $(a \pm b)^2 = a^2 + b^2 \pm 2ab$, $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$
- Knowledge of Pythagoras Theorem and Py thagorean numbers.

3.1 CIRCULAR MEASURE OF ANGLE

An angle is a union of two rays with the common end point. An angle is formed by the rotation of a ray as well. Negative and positive angles are formed according as the rotation is clock-wise or anticlock-wise.

3.1.1 A Unit Circle

It can be seen easily that when a line segment makes one complete rotation, its end point describes a circle. In case the length of the rotating line be one unit then the circle described will be a circle of unit radius. Such a circle is termed as ***unit circle.***

3.1.2 A Radian

A radian is another unit of measurement of an angle other than degree.

A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius (r) of the circle. In a unit circle one radian will be the angle subtended at the centre of the circle by an arc of unit length.

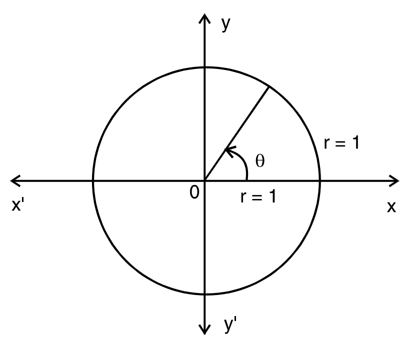


Fig. 3.2

Note : A radian is a constant angle; implying that the measure of the angle subtended by an are of a circle, with length equal to the radius is always the same irrespective of the radius of the circle.

3.1.3 Relation between Degree and Radian

An arc of unit length subtends an angle of 1 radian. The circumference 2π $(\because r = 1)$ subtend an angle of 2π radians.

Hence 2π radians $= 360°$, $\Rightarrow$ π radians $= 180°$, $\Rightarrow$ $\frac{\pi}{2}$ radians $= 90°$

$$\Rightarrow \quad \frac{\pi}{4} \text{ radians} = 45° \quad \Rightarrow \quad 1 \text{ radian} = \left(\frac{360}{2\pi}\right)^{\circ} = \left(\frac{180}{\pi}\right)^{\circ}$$

$$\text{or } 1° = \frac{2\pi}{360} \text{ radians} = \frac{\pi}{180} \text{ radians}$$

Example 3.1 Convert

(i) 90° into radians (ii) 15° into radians

(iii) $\frac{\pi}{6}$ radians into degrees. (iv) $\frac{\pi}{10}$ radians into degrees.

Solution :

(i) $1° = \frac{2\pi}{360}$ radians

$\Rightarrow \quad 90° = \frac{2\pi}{360} \times 90$ radians or $90° = \frac{\pi}{2}$ radians

(ii) $15° = \frac{2\pi}{360} \times 15$ radians or $15° = \frac{\pi}{12}$ radians

(iii) $1 \text{ radian} = \left(\frac{360}{2\pi}\right)^{\circ}$, $\frac{\pi}{6}$ radians $= \left(\frac{360}{2\pi} \times \frac{\pi}{6}\right)^{\circ}$

$\frac{\pi}{6}$ radians = 30°

(iv) $\frac{\pi}{10}$ radians $= \left(\frac{360}{2\pi} \times \frac{\pi}{10}\right)^{\circ}$, $\frac{\pi}{10}$ radians = 18°

CHECK YOUR PROGRESS 3.1

1. Convert the following angles (in degrees) into radians :

   (i) 60° (ii) 15° (iii) 75° (iv) 105° (v) 270°

2. Convert the following angles into degrees:

   (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{\pi}{20}$ (iv) $\frac{\pi}{60}$ (v) $\frac{2\pi}{3}$

3. The angles of a triangle are 45°, 65° and 70°. Express these angles in radians

4. The three angles of a quadrilateral are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$. Find the fourth angle in radians.

5. Find the angle complementary to $\frac{\pi}{6}$.

3.1.4 Relation Between Length of an Arc and Radius of the Circle

An angle of 1 radian is subtended by an arc whose length is equal to the radius of the circle. An angle of 2 radians will be substened if arc is double the radius.

An angle of 2½ radians willbe subtended if arc is 2½ times the radius.

All this can be read from the following table :

Length of the arc (*l*)	**Angle subtended at the centre of the circle θ (in radians)**
r	1
2r	2
(2½)r	2½
4r	4

Therefore, $\theta = \frac{\ell}{r}$ or $\ell = r\theta$, where r = radius of the circle,

θ = angle substended at the centre in radians, and ℓ = length of the arc.

The angle subtended by an arc of a circle at the centre of the circle is given by the ratio of the length of the arc and the radius of the circle.

Note : In arriving at the above relation, we have used the radian measure of the angle and not the degree measure. Thus the relation $\theta = \frac{\ell}{r}$ is valid only when the angle is measured in radians.

Example 3.2 Find the angle in radians subtended by an arc of length 10 cm at the centre of a circle of radius 35 cm.

Solution : $\ell = 10cm$ and r = 35 cm.

$$\theta = \frac{\ell}{r} \text{ radians} \quad \text{or} \quad \theta = \frac{10}{35} \text{ radians , or} \quad \theta = \frac{2}{7} \text{ radians}$$

Example 3.3 A railroad curve is to be laid out on a circle. What should be the radius of a circular track if the railroad is to turn through an angle of 45° in a distance of 500m?

Solution : Angle θ is given in degrees. To apply the formula $\ell = r\theta$, θ must be changed to radians.

$$\theta = 45° = 45 \times \frac{\pi}{180} \text{ radians} \quad \text{....(1)} \qquad = \frac{\pi}{4} \text{ radians}$$

$$\ell = 500 \text{ m} \quad \text{....(2)}$$

$\ell = r\ \theta$ gives $r = \dfrac{\ell}{\theta}$ $\quad\therefore\quad$ $r = \dfrac{500}{\frac{\pi}{4}}$ m $\quad$ [using (1) and (2)]

$$= 500 \times \frac{4}{\pi}\ \text{m}\ ,\ = 2000 \times 0.32\ \text{m}\ \left(\frac{1}{\pi} = 0.32\right),\ = 640\ \text{m}$$

Example 3.4 A train is travelling at the rate of 60 km per hour on a circular track. Through what angle will it turn in 15 seconds if the radius of the track is $\frac{5}{6}$ km.

Solution : The speed of the train is 60 km per hour. In 15 seconds, it will cover

$$\frac{60 \times 15}{60 \times 60}\ \text{km} = \frac{1}{4}\ \text{km}$$

$\therefore$ We have $\ell = \dfrac{1}{4}$ km and $r = \dfrac{5}{6}$ km

$$\therefore \quad \theta = \frac{\ell}{r} = \frac{\frac{1}{4}}{\frac{5}{6}}\ \text{radians} = \frac{3}{10}\ \text{radians}$$

CHECK YOUR PROGRESS 3.2

1. Express the following angles in radians :

   (a) 30° (b) 60° (c) 150°

2. Express the following angles in degrees :

   (a) $\frac{\pi}{5}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{9}$

3. Find the angle in radians and in degrees subtended by an arc of length 2.5 cm at the centre of a circle of radius 15 cm.

4. A train is travelling at the rate of 20 km per hour on a circular track. Through what angle will it turn in 3 seconds if the radius of the track is $\frac{1}{12}$ of a km?.

5. A railroad curve is to be laid out on a circle. What should be the radius of the circular track if the railroad is to turn through an angle of 60° in a distance of 100 m?

6. Complete the following table for l, r, θ having their usual meanings.

	l	r	θ
(a)	1.25m	....	135°
(b)	30 cm	....	$\frac{\pi}{4}$
(c)	0.5 cm	2.5 m	....
(d)	....	6 m	120°
(e)	....	150 cm	$\frac{\pi}{15}$
(f)	150 cm	40 m	....
(g)	....	12 m	$\frac{\pi}{6}$
(h)	1.5 m	0.75 m	....
(i)	25 m	....	75°

3.2 TRIGONOMETRIC FUNCTIONS

While considering, a unit circle you must have noticed that for every real number between *0* and 2π, there exists a ordered pair of numbers x and y. This ordered pair (x, y) represents the coordinates of the point *P*.

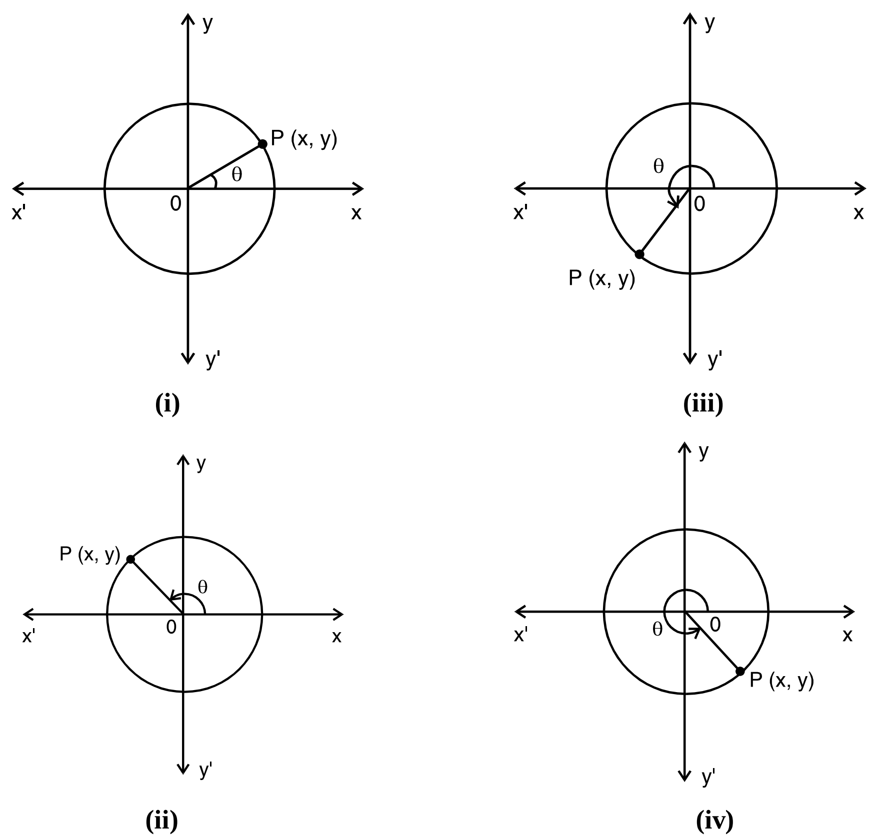


Fig. 3.3

If we consider $\theta = 0$ on the unit circle, we will have a point whose coordinates are (1,0).

If $\theta = \frac{\pi}{2}$, then the corresponding point on the unit circle will have its coordinates (0,1).

In the above figures you can easily observe that no matter what the position of the point, corresponding to every real number θ we have a unique set of coordinates (*x*, *y*). The values of *x* and *y* will be negative or positive depending on the quadrant in which we are considering the point.

Considering a point *P* (on the unit circle) and the corresponding coordinates (*x*, *y*), we define trigonometric functions as :

$$\sin\theta = y, \ \cos\theta = x$$

$$\tan\theta = \frac{y}{x} \ (\text{for } x \neq 0), \ \cot\theta = \frac{x}{y} \ (\text{for } y \neq 0)$$

$$\sec\theta = \frac{1}{x} \ (\text{for } x \neq 0), \ \text{cosec}\,\theta = \frac{1}{y} \ (\text{for } y \neq 0)$$

Now let the point *P* moves from its original position in anti-clockwise direction. For various positions of this point in the four quadrants, various real numbers θ will be generated. We summarise, the above discussion as follows. For values of θ in the :

I quadrant, both *x* and *y* are positve.

II quadrant, *x* will be negative and *y* will be positive.

III quadrant, *x* as well as *y* will be negative.

IV quadrant, *x* will be positive and *y* will be negative.

or	I quadrant	II quadrant	III quadrant	IV quadrant
	All positive	sin positive	tan positive	cos positive
		cosec positive	cot positive	sec positive

Where what is positive can be rememebred by :

	All	sin	tan	cos
Quardrant	I	II	III	IV

If (x, y) are the coordinates of a point P on a unit circle and θ, the real number generated by the position of the point, then sin θ = y and cos θ = x. This means the coordinates of the point P can also be written as (cos θ, sin θ)

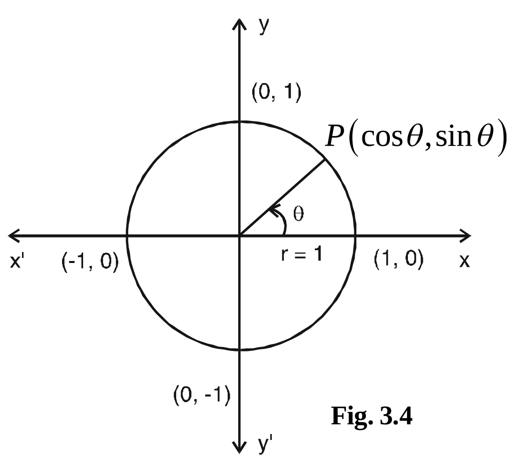


Fig. 3.4

From Fig. 3.4, you can easily see that the values of x will be between –1 and + 1 as *P* moves on the unit circle. Same will be true for *y* also.

Thus, for all P on the unit circle

$$-1 \le x \le 1 \qquad \text{and } -1 \le y \le 1$$

Thereby, we conclude that for all real numbers θ

$$-1 \le \cos\theta \le 1 \quad \text{and} \qquad -1 \le \sin\theta \le 1$$

In other words, $\sin\theta$ and $\cos\theta$ can not be numerically greater than 1

Example 3.5 What will be sign of the following ?

(i) $\sin\frac{7\pi}{18}$ (ii) $\cos\frac{4\pi}{9}$ (iii) $\tan\frac{5\pi}{9}$

Solution :

(i) Since $\frac{7\pi}{18}$ lies in the first quadrant, the sign of $\sin\frac{7\pi}{18}$ will be posilive.

(ii) Since $\frac{4\pi}{9}$ lies in the first quadrant, the sign of $\cos\frac{4\pi}{9}$ will be positive.

(iii) Since $\frac{5\pi}{9}$ lies in the second quadrant, the sign of $\tan\frac{5\pi}{9}$ will be negative.

Example 3.6 Write the values of (i) $\sin\frac{\pi}{2}$ (ii) $\cos 0$ (iii) $\tan\frac{\pi}{2}$

Solution : (i) From Fig. 3.5, we can see that the coordinates of the point A are (0,1)

$\therefore \sin\frac{\pi}{2} = 1$, as $\sin\theta = y$

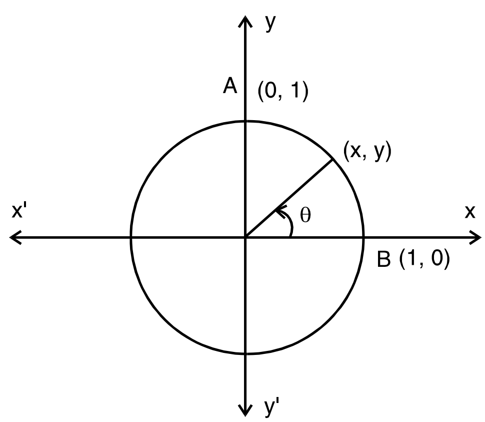


Fig. 3.5

(ii) Coordinates of the point B are (1, 0) $\quad \therefore \cos 0 = 1$, as $\cos\theta = x$

(iii) $\tan\frac{\pi}{2} = \dfrac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \frac{1}{0}$ which is not defined , Thus $\tan\frac{\pi}{2}$ is not defined.

Example 3.7 Write the minimum and maximum values of $\cos\theta$.

Solution : We know that $-1 \le \cos\theta \le 1$

$\therefore$ The maximum value of $\cos\theta$ is 1 and the minimum value of $\cos\theta$ is –1.

CHECK YOUR PROGRESS 3.3

1. What will be the sign of the following ?

(i) $\cos\frac{2\pi}{3}$ (ii) $\tan\frac{5\pi}{6}$ (iii) $\sec\frac{2\pi}{3}$

(iv) $\sec\frac{35\pi}{18}$ (v) $\tan\frac{25\pi}{18}$ (vi) $\cot\frac{3\pi}{4}$

(vii) $\operatorname{cosec}\frac{8\pi}{3}$ (viii) $\cot\frac{7\pi}{8}$

2. Write the value of each of the following :

(i) $\cos\frac{\pi}{2}$ (ii) $\sin 0$ (iii) $\cos\frac{2\pi}{3}$ (iv) $\tan\frac{3\pi}{4}$

(v) $\sec 0$ (vi) $\tan\frac{\pi}{2}$ (vii) $\tan\frac{3\pi}{2}$ (viii) $\cos 2\pi$

3.2.1 Relation Between Trigonometric Functions

By definition $x = \cos\theta$, $y = \sin\theta$

As $\tan\theta = \frac{y}{x}, \ (x \ne 0)$, $= \frac{\sin\theta}{\cos\theta}, \ \theta \ne \frac{n\pi}{2}$

and $\cot\theta = \frac{x}{y}, \ (y \ne 0)$,

i.e., $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}, \ (\theta \ne n\pi)$

Similarly, $\sec\theta = \frac{1}{\cos\theta}$ $\left(\theta \ne \frac{n\pi}{2}\right)$

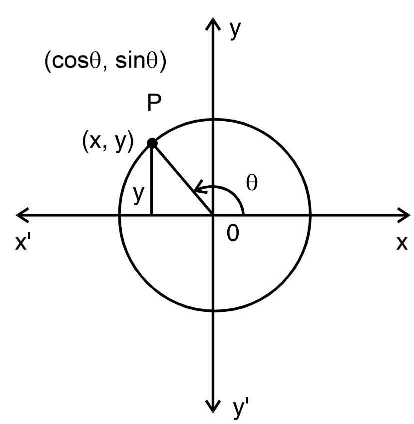


Fig. 3.6

and $\operatorname{cosec}\,\theta = \dfrac{1}{\sin\theta}$ $\qquad (\theta \neq n\pi)$

Using Pythagoras theorem we have, $x^2 + y^2 = 1$, i.e., $(\cos\theta)^2 + (\sin\theta)^2 = 1$

or, $\mathbf{\cos^2\theta + \sin^2\theta = 1}$

Note : $(\cos\theta)^2$ is written as $\cos^2\theta$ and $(\sin\theta)^2$ as $\sin^2\theta$

Again $x^2 + y^2 = 1$ or $1 + \left(\dfrac{y}{x}\right)^2 = \left(\dfrac{1}{x}\right)^2$, for $x \neq 0$

or, $1 + (\tan\theta)^2 = (\sec\theta)^2$, i.e. $\mathbf{\sec^2\theta = 1 + \tan^2\theta}$

Similarly, $\mathbf{\operatorname{cosec}^2\theta = 1 + \cot^2\theta}$

Example 3.8 Prove that $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$

Solution : L.H.S. $= \sin^4\theta + \cos^4\theta$

$$= \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta$$

$$= \left(\sin^2\theta + \cos^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta$$

$$= 1 - 2\sin^2\theta\cos^2\theta \quad \left(\because \sin^2\theta + \cos^2\theta = 1\right), = \text{R.H.S.}$$

Example 3.9 Prove that $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

Solution : L.H.S. $= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\dfrac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}} = \sqrt{\dfrac{(1-\sin\theta)^2}{1-\sin^2\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}} = \frac{1-\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{R.H.S.}$$

Example 3.10 If $\sin\theta = \dfrac{21}{29}$, prove that $\sec\theta + \tan\theta = -2\dfrac{1}{2}$, given that θ lies in the second quadrant.

Solution : $\sin\theta = \dfrac{21}{29}$ Also, $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \quad \cos^2\theta = 1-\sin^2\theta = 1-\frac{441}{841} = \frac{400}{841} = \left(\frac{20}{29}\right)^2$$

$$\Rightarrow \quad \cos\theta = \frac{-20}{29} \quad (\cos\theta \text{ is negative as } \theta \text{ lies in the second quardrant})$$

$$\therefore \quad \tan\theta = \frac{-21}{20} \quad (\tan\theta \text{ is negative as } \theta \text{ lies in the second qudrant})$$

$$\therefore \quad \sec\theta + \tan\theta = \frac{-29}{20} + \frac{-21}{20} = \frac{-29-21}{20}, \ = \frac{-5}{2} = -2\frac{1}{2} = \text{R.H.S.}$$

CHECK YOUR PROGRESS 3.4

1. Prove that $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

2. If $\tan\theta = \frac{1}{2}$, find the other five trigonometric functions. where θ lies in the first quardrant)

3. If $\operatorname{cosec}\theta = \frac{b}{a}$, find the other five trigonometric functions, if θ lies in the first quardrant.

4. Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

5. If $\cot\theta + \operatorname{cosec}\theta = 1.5$, show that $\cos\theta = \frac{5}{13}$

6. If $\tan\theta + \sec\theta = m$, find the value of $\cos\theta$

7. Prove that $(\tan A + 2)(2\tan A + 1) = 5\tan A + 2\sec^2 A$

8. Prove that $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

9. Prove that $\frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta} = \cos\theta + \sin\theta$

10. Prove that $\frac{\tan\theta}{1+\cos\theta} + \frac{\sin\theta}{1-\cos\theta} = \cot\theta + \operatorname{cosec}\theta \cdot \sec\theta$

11. If $\sec x = \frac{13}{5}$ and x lies in the fourth quadrant, Find other five trigonometric ratios.

3.3 TRIGONOMETRIC FUNCTIONS OF SOME SPECIFIC REAL NUMBERS

The values of the trigonometric functions of 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$ are summarised below in the form of a table :

Function ↓ / Real Numbers → (θ)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

As an aid to memory, we may think of the following pattern for above mentioned values of sin function : , $\sqrt{\frac{0}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{4}}$

On simplification, we get the values as given in the table. The values for cosines occur in the reverse order.

Example 3.11 Find the value of the following :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3} - \cos\frac{\pi}{4}\cos\frac{\pi}{3}$ (b) $4\tan^2\frac{\pi}{4} - \operatorname{cosec}^2\frac{\pi}{6} - \cos^2\frac{\pi}{3}$

Solution :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3} - \cos\frac{\pi}{4}\cos\frac{\pi}{3} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(b) $4\tan^2\frac{\pi}{4} - \operatorname{cosec}^2\frac{\pi}{6} - \cos^2\frac{\pi}{3}, = 4(1)^2 - (2)^2 - \left(\frac{1}{2}\right)^2, = 4 - 4 - \frac{1}{4} = -\frac{1}{4}$

Example 3.12 If $A = \frac{\pi}{3}$ and $B = \frac{\pi}{6}$, verify that $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Solution : L.H.S. $= \cos(A+B) = \cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \cos\frac{\pi}{2} = 0$

R.H.S. $= \cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$

$\therefore$ L.H.S. = 0 = R.H.S.

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

CHECK YOUR PROGRESS 3.5

1. Find the value of

(i) $\sin^2\frac{\pi}{6}+\tan^2\frac{\pi}{4}+\tan^2\frac{\pi}{3}$ (ii) $\sin^2\frac{\pi}{3}+\operatorname{cosec}^2\frac{\pi}{6}+\sec^2\frac{\pi}{4}-\cos^2\frac{\pi}{3}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{3}-\sin\frac{2\pi}{3}\sin\frac{\pi}{3}$ (iv) $4\cot^2\frac{\pi}{3}+\operatorname{cosec}^2\frac{\pi}{4}+\sec^2\frac{\pi}{3}\tan^2\frac{\pi}{4}$

(v) $\left(\sin\frac{\pi}{6}+\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3}-\cos\frac{\pi}{4}\right)+\frac{1}{4}$

2. Show that

$$\left(1+\tan\frac{\pi}{6}\tan\frac{\pi}{3}\right)+\left(\tan\frac{\pi}{6}-\tan\frac{\pi}{3}\right)=\sec^2\frac{\pi}{6}\sec^2\frac{\pi}{3}$$

3. Taking $A=\frac{\pi}{3}, B=\frac{\pi}{6}$, verify that

(i) $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$ (ii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

4. If $\theta=\frac{\pi}{4}$, verify: (i) $\sin 2\theta=2\sin\theta\cos\theta$

(ii) $\cos 2\theta=\cos^2\theta-\sin^2\theta \quad =2\cos^2\theta-1 \quad =1-2\sin^2\theta$

5. If $A=\frac{\pi}{6}$, verify that, (i) $\cos 2A=2\cos^2 A-1$

(ii) $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$ (iii) $\sin 2A=2\sin A\cos A$

3.4 GRAPHS OF TRIGONOMETRIC FUNCTIONS

Given any function, a pictorial or a graphical representation makes a lasting impression on the minds of learners and viewers. The importance of the graph of functions stems from the fact that this is a convenient way of presenting many properties of the functions. By observing the graph we can examine several characteristic properties of the functions such as (i) periodicity, (ii) intervals in which the function is increasing or decreasing (iii) symmetry about axes, (iv) maximum and minimum points of the graph in the given interval. It also helps to compute the areas enclosed by the curves of the graph.

3.4.1 Variations of sinθ as θ Varies Continuously From 0 to 2π.

Let $X'OX$ and $Y'OY$ be the axes of coordinates. With centre O and radius OP = unity, draw a circle. Let OP starting from OX and moving in anticlockwise direction make an angle θ with the x-axis, i.e. $\angle XOP = \theta$. Draw $PM \perp X'OX$, then $\sin\theta = MP$ as $OP=1$.

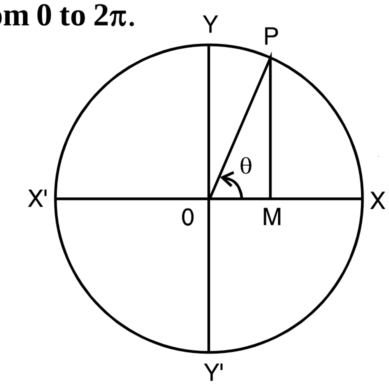


Fig. 3.7

$\therefore$ The variations of sin θ are the same as those of MP.

I Quadrant :

As θ increases continuously from 0 to $\frac{\pi}{2}$

PM is positive and increases from 0 to 1.

$\therefore$ sin θ is positive.

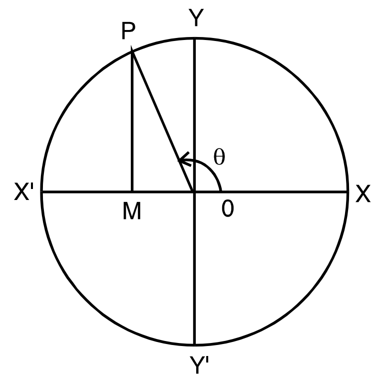


Fig. 3.8

II Quadrant $\left[\frac{\pi}{2}, \pi\right]$

In this interval, θ lies in the second quadrant.

Therefore, point P is in the second quadrant. Here PM = y is positive, but decreases from 1 to 0 as θ varies from $\frac{\pi}{2}$ to π. Thus sin θ is positive.

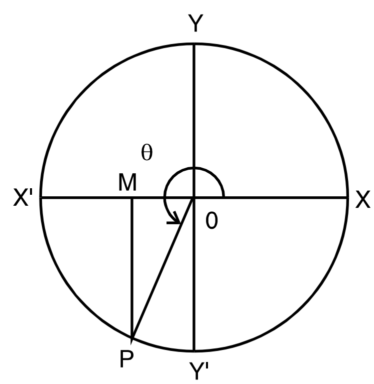


Fig. 3.9

III Quadrant $\left[\pi, \frac{3\pi}{2}\right]$

In this interval, θ lies in the third quandrant. Therefore, point P can move in the third quadrant only. Hence $PM = y$ is negative and decreases from 0 to -1 as θ varies from π to $\frac{3\pi}{2}$. In this interval $\sin\theta$ decreases from 0 to -1. In this interval sin θ is negative.

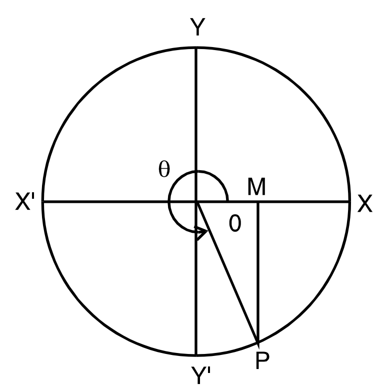


Fig. 3.10

IV Quadrant $\left[\frac{3\pi}{2}, 2\pi\right]$

In this interval, θ lies in the fourth quadrant. Therefore, point P can move in the fourth quadrant only. Here again PM = y is negative but increases from -1 to 0 as θ varies from $\frac{3\pi}{2}$ to 2π. Thus sin θ is negative in this interval.

3.4.2 Graph of sin θ as θ varies from 0 to 2π.

Let *X'OX* and *Y'OY* be the two coordinate axes of reference. The values of θ are to be measured along x-axis and the values of sin θ are to be measured along y-axis.

(Approximate value of $\sqrt{2} = 1.41, \frac{1}{\sqrt{2}} = .707, \frac{\sqrt{3}}{2} = .87$)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin θ	0	.5	.87	1	.87	.5	0	–.5	–.87	–1	–.87	–.5	0

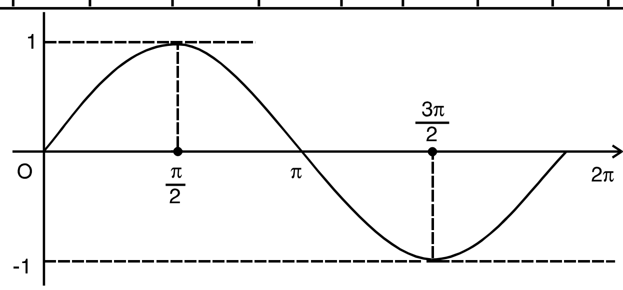


Fig. 3.11

Some Observations

(i) Maximum value of sin θ is 1. (ii) Minimum value of sin θ is –1.

(iii) It is continuous everywhere. (iv) It is increasing from 0 to $\frac{\pi}{2}$ and from $\frac{3\pi}{2}$ to 2π.

It is decreasing from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. With the help of the graph drawn in Fig. 6.11 we can always draw another graph y = sin θ in the interval of $[2\pi, 4\pi]$ (see Fig. 3.12)

What do you observe ?

The graph of y = sin θ in the interval $[2\pi, 4\pi]$ is the same as that in 0 to 2π. Therefore, this graph can be drawn by using the property $\sin(2\pi + \theta) = \sin\theta$. Thus, sin θ repeats itself when θ is increased by 2π. This is known as the periodicity of sin θ.

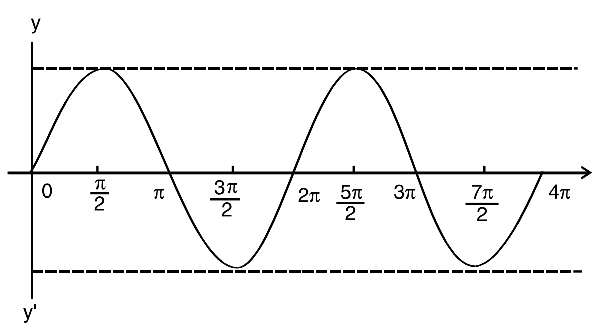


Fig. 3.12

We shall discuss in details the periodicity later in this lesson.

Example 3.13 Draw the graph of $y = \sin 2\theta$ in the interval 0 to π.

Solution :

θ :	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ :	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\sin 2\theta$:	0	.87	1	.87	0	–.87	–1	–.87	0

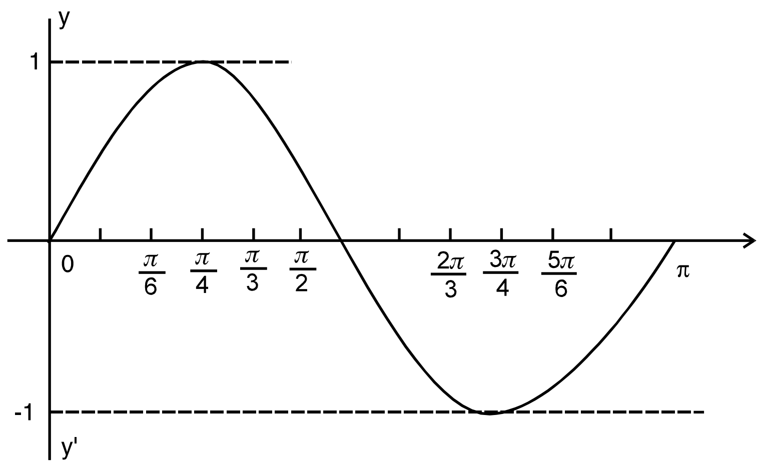


Fig. 3.13

The graph is similar to that of $y = \sin\theta$

Some Observations

1. The other graphs of $\sin\theta$, like $a \sin\theta$, $3\sin 2\theta$ can be drawn applying the same method.
2. Graph of $\sin\theta$, in other intervals namely $[4\pi, 6\pi]$, $[-2\pi, 0]$, $[-4\pi, -2\pi]$, can also be drawn easily. This can be done with the help of properties of allied angles: $\sin(\theta + 2\pi) = \sin\theta$, $\sin(\theta - 2\pi) = \sin\theta$. i.e., θ repeats itself when increased or decreased by 2π.

CHECK YOUR PROGRESS 3.6

1. What are the maximum and minimum values of $\sin\theta$ in $[0, 2\pi]$?
2. Explain the symmetry in the graph of $\sin\theta$ in $[0, 2\pi]$
3. Sketch the graph of $y = 2\sin\theta$, in the interval $[0, \pi]$

4. For what values of θ in $[\pi, 2\pi]$, sin θ becomes, (a) $\frac{-1}{2}$ (b) $\frac{-\sqrt{3}}{2}$

5. Sketch the graph of $y = \sin x$ in the interval of $[-\pi, \pi]$

3.4.3 Graph of cosθ as θ Varies From 0 to 2π

As in the case of $\sin\theta$, we shall also discuss the changes in the values of $\cos\theta$ when θ assumes values in the intervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

I Quadrant : In the interval $\left[0, \frac{\pi}{2}\right]$, point *P* lies in the first quadrant, therefore, $OM = x$ is positive but decreases from 1 to 0 as θ increases from 0 to $\frac{\pi}{2}$.

Thus in this interval $\cos\theta$ decreases from 1 to 0.

$\therefore$ $\cos\theta$ is positive in this quadrant.

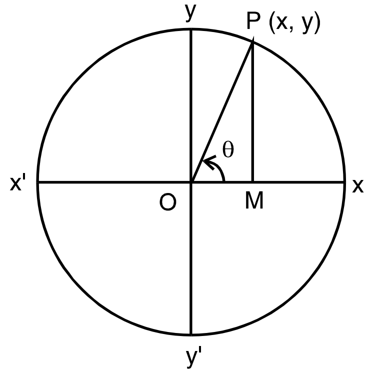


Fig. 3.14

II Quadrant : In the interval $\left[\frac{\pi}{2}, \pi\right]$, point *P* lies in the second quadrant and therefore point M lies on the negative side of x-axis. So in this case $OM = x$ is negative and decreases from 0 to -1 as θ increases from $\frac{\pi}{2}$ to π. Hence in this inverval $\cos\theta$ decreases from 0 to -1.

$\therefore$ $\cos\theta$ is negative.

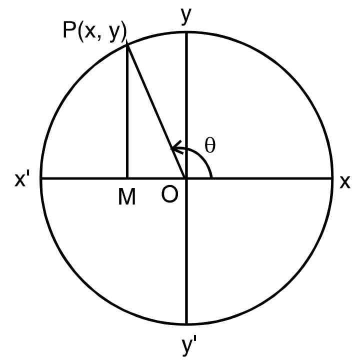


Fig. 3.15

III Quadrant : In the interval $\left[\pi, \frac{3\pi}{2}\right]$, point *P* lies in the third quadrant and therefore, $OM = x$ remains negative as it is on the negative side of x-axis. Therefore $OM = x$ is negative but increases from–1 to 0 as θ increases from π to $\frac{3\pi}{2}$. Hence in this interval $\cos\theta$ increases from -1 to 0.

$\therefore$ $\cos\theta$ is negative.

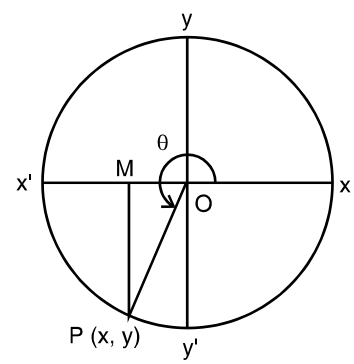


Fig. 3.16

IV Quadrant : In the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, point P lies in the fourth quadrant and M moves on the positive side of x-axis.Therefore $OM = x$ is positive. Also it increases from 0 to 1 as θ increases from $\frac{3\pi}{2}$ to 2π.

Thus in this interval $\cos\theta$ increases from 0 to 1.

$\therefore$ $\cos\theta$ is positive.

Let us tabulate the values of cosines of some suitable values of θ.

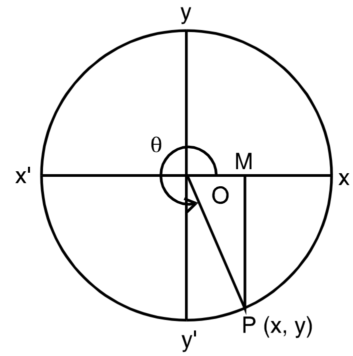


Fig. 3.17

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	.87	.5	0	0.5	–.87	–1	–.87	–.5	0	0.5	.87	1

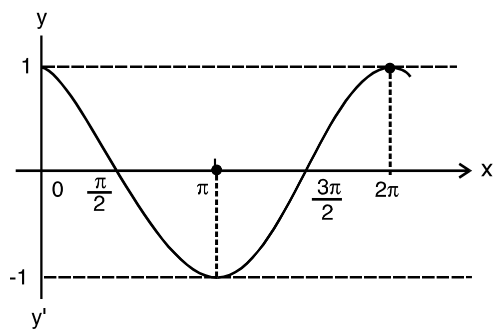


Fig. 3.18

Let $X'OX$ and $Y'OY$ be the axes. Values of θ are measured along x-axis and those of $\cos\theta$ along y-axis.

Some observations

(i) Maximum value of $\cos\theta = 1$. (ii) Minimum value of $\cos\theta = -1$.

(iii) It is continuous everywhere.

(iv) $\cos(\theta + 2\pi) = \cos\theta$. Also $\cos(\theta - 2\pi) = \cos\theta$. Cos θ repeats itself when θ is increased or decreased by 2π. It is called periodicity of $\cos\theta$. We shall discuss in details about this in the later part of this lesson.

(v) Graph of $\cos\theta$ in the intervals $[2\pi, 4\pi]$ $[4\pi, 6\pi]$ $[-2\pi, 0]$, will be the same as in $[0, 2\pi]$.

Example 3.14 Draw the graph of $\cos\theta$ as θ varies from $-\pi$ to π. From the graph read the values of θ when $\cos\theta = \pm 0.5$.

Solution :

θ :	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
cos θ:	–1.0	–0.87	–0.5	0	.50	–.87	1.0	0.87	0.5	0	–0.5	–0.87	–1

$$\cos\theta = 0.5$$

when $\theta = \frac{\pi}{3}, \frac{-\pi}{3}$

$$\cos\theta = -0.5$$

when $\theta = \frac{2\pi}{3}, \frac{-2\pi}{3}$

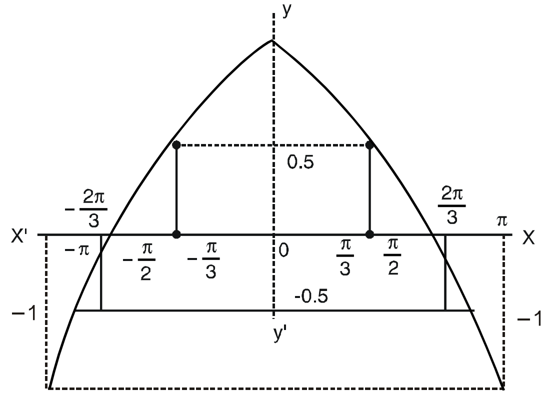


Fig. 3.19

Example 3.15 Draw the graph of $\cos 2\theta$ in the interval 0 to π.

Solution :

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
cos 2θ	1	0.5	0	–0.5	–1	–0.5	0	0.5	1

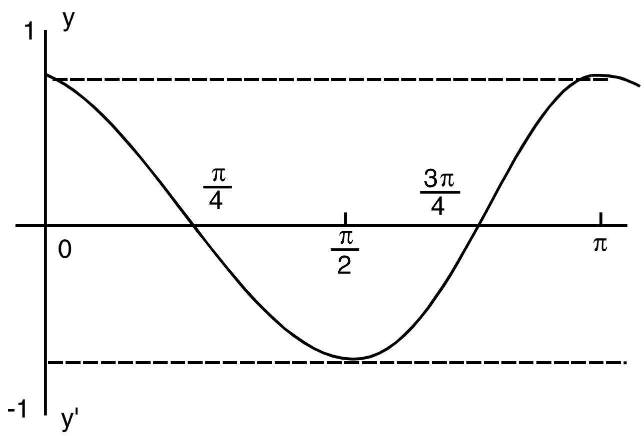


Fig. 3.20

CHECK YOUR PROGRESS 3.7

1. (a) Sketch the graph of $y = \cos\theta$ as θ varies from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$.

(b) Draw the graph of $y = 3\cos\theta$ as θ varies from 0 to 2π.

(c) Draw the graph of $y = \cos 3\theta$ from $-\pi$ to π and read the values of θ when $\cos\theta = 0.87$ and $\cos\theta = -0.87$.

(d) Does the graph of $y = \cos\theta$ in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ lie above x-axis or below x-axis?

(e) Draw the graph of $y = \cos\theta$ in $[2\pi, 4\pi]$

3.4.4 Graph of tan θ as θ Varies from 0 to 2π

In I Quadrant : $\tan\theta$ can be written as $\frac{\sin\theta}{\cos\theta}$

Behaviour of $\tan\theta$ depends upon the behaviour of $\sin\theta$ and $\frac{1}{\cos\theta}$

In I quadrant, $\sin\theta$ increases from 0 to 1, $\cos\theta$ decreases from 1 to 0

But $\frac{1}{\cos\theta}$ increases from 1 indefintely (and write it as increasses from 1 to ∞) $\tan\theta > 0$

$\therefore$ $\tan\theta$ increases from 0 to ∞. (See the table and graph of $\tan\theta$).

In II Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$ decreases from 1 to 0.

$\cos\theta$ decreases from 0 to -1.

$\tan\theta$ is negative and increases from $-\infty$ to 0

In III Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$ decreases from 0 to -1

$\cos\theta$ increases from -1 to 0

$\therefore$ $\tan\theta$ is positive and increases from 0 to ∞

In IV Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$ increases from -1 to 0

$\cos\theta$ increases from 0 to 1

$\tan\theta$ is negative and increases form $-\infty$ to 0

Graph of tan θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0°$	$\frac{\pi}{2}+0°$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0°$	$\frac{3\pi}{2}+0°$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\tan\theta$	0	.58	1.73	$+\infty$	-1.73	-.58	0	.58	1.73	$+\infty$	$-\infty$	-1.73	-.58	0	0

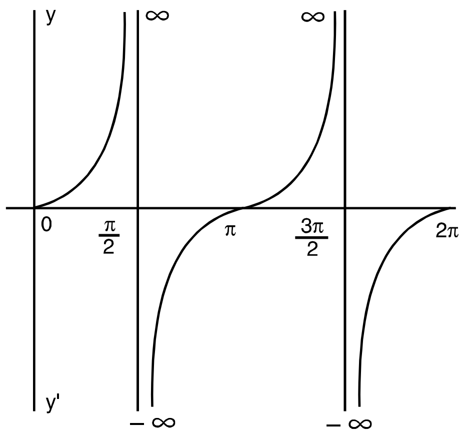


Fig. 3.21

Observations

(i) $\tan(180° + \theta) = \tan\theta$. Therefore, the complete graph of $\tan\theta$ consists of infinitely many repetitions of the same to the left as well as to the right.

(ii) Since $\tan(-\theta) = -\tan\theta$, therefore, if $(\theta, \tan\theta)$ is any point on the graph then $(-\theta, -\tan\theta)$ will also be a point on the graph.

(iii) By above results, it can be said that the graph of $y = \tan\theta$ is symmetrical in opposite quadrants.

(iv) $\tan\theta$ may have any numerical value, positve or negative.

(v) The graph of $\tan\theta$ is discontinuous (has a break) at the points $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

(vi) As θ passes through these values, $\tan\theta$ suddenly changes from $+\infty$ to $-\infty$.

3.4.5 Graph of cot θ as θ Varies From 0 to 2π

The behaviour of $\cot\theta$ depends upon the behaviour of $\cos\theta$ and $\frac{1}{\sin\theta}$ as $\cot\theta = \cos\theta \frac{1}{\sin\theta}$

We discuss it in each quadrant.

I Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

$\cos\theta$ decreases from 1 to 0

$\sin\theta$ increases from 0 to 1

$\therefore$ $\cot\theta$ also decreases from $+\infty$ to 0 but $\cot\theta > 0$.

II Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

$\cos\theta$ decreases from 0 to -1

$\sin\theta$ decreases from 1 to 0

$\Rightarrow \quad \cot\theta < 0$ or $\cot\theta$ decreases from 0 to $-\infty$

III Quadrant : $\cot\theta = \cos\theta \times \dfrac{1}{\sin\theta}$

$\cos\theta$ increases from -1 to 0

$\sin\theta$ decreases from 0 to -1

$\therefore$ $\cot\theta$ decreases from $+\infty$ to 0.

IV Quadrant : $\cot\theta = \cos\theta \times \dfrac{1}{\sin\theta}$

$\cos\theta$ increases from 0 to 1

$\sin\theta$ increases from -1 to 0

$\therefore \cot\theta < 0$

$\cot\theta$ decreases from 0 to $-\infty$

Graph of $\cot\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cot\theta$	∞	1.73	.58	0	-.58	-1.73	$-\infty$	$+\infty$	1.73	.58	0	-.58	-1.73	$-\infty$

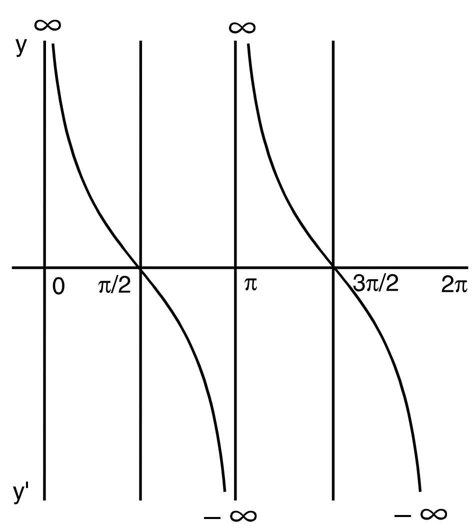


Fig. 3.22

Observations

(i) Since $\cot(\pi+\theta) = \cot\theta$, the complete graph of $\cot\theta$ consists of the portion from $\theta = 0$ to $\theta = \pi$ or $\theta = \dfrac{\pi}{2}$ to $\theta = \dfrac{3\pi}{2}$.

(ii) $\cot\theta$ can have any numerical value - positive or negative.

(iii) The graph of $\cot\theta$ is discontinuous, i.e. it breaks at 0, π, 2π, .

(iv) As θ takes values 0, π, 2π, $\cot\theta$ suddently changes from $-\infty$ to $+\infty$

CHECK YOUR PROGRESS 3.8

1. (a) What is the maximum value of $\tan\theta$?

   (b) What changes do you observe in $\tan\theta$ at $\frac{\pi}{2}, \frac{3\pi}{2}$?

   (c) Draw the graph of $y = \tan\theta$ from $-\pi$ to π. Find from the graph the value of θ for which $\tan\theta = 1.7$.

2. (a) What is the maximum value of $\cot\theta$?

   (b) Find the value of θ when $\cot\theta = -1$, from the graph.

3.4.6 To Find the Variations And Draw The Graph of sec θ As θ Varies From 0 to 2π.

Let *X'OX* and *Y'OY* be the axes of coordinates. With centre *O*, draw a circle of unit radius.

Let *P* be any point on the circle. Join *OP* and draw $PM \perp X'OX$.

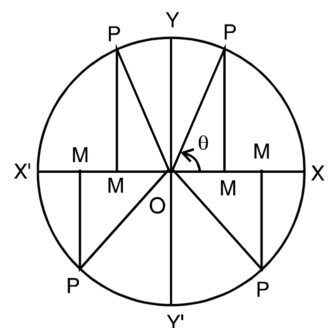


Fig. 3.23

$$\sec\theta = \frac{OP}{OM} = \frac{1}{OM}$$

$\therefore$ Variations will depend upon *OM*.

I Quadrant : sec θ is positive as *OM* is positive.

Also $\sec 0 = 1$ and $\sec\frac{\pi}{2} = \infty$ when we approach $\frac{\pi}{2}$ from the right.

$\therefore$ As θ varies from 0 to $\frac{\pi}{2}$, $\sec\theta$ increases from 1 to ∞.

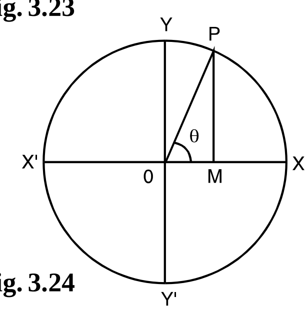


Fig. 3.24

II Quadrant : $\sec\theta$ is negative as *OM* is negative.

$\sec\frac{\pi}{2} = -\infty$ when we approach $\frac{\pi}{2}$ from the left. Also $\sec\pi = -1$.

$\therefore$ As θ varies from $\frac{\pi}{2}$ to π, $\sec\theta$ changes from $-\infty$ to -1.

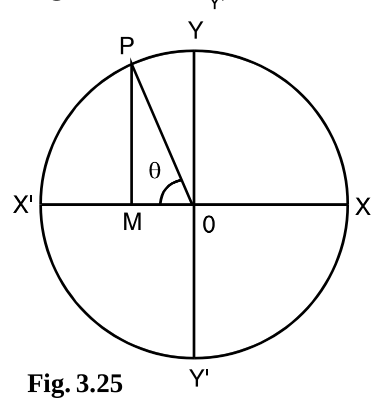


Fig. 3.25

It is observed that as θ passes through $\frac{\pi}{2}$, $\sec\theta$ changes from $+\infty$ to $-\infty$.

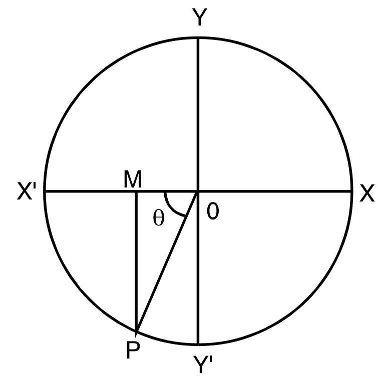


Fig. 3.26

III Quadrant : $\sec\theta$ is negative as *OM* is negative. $\sec\pi = -1$ and $\sec\frac{3\pi}{2} = -\infty$ when the angle approaches $\frac{3\pi}{2}$ in the counter clockwise direction. As θ varies from π to $\frac{3\pi}{2}$, $\sec\theta$ decreases from -1 to $-\infty$.

IV Quadrant : $\sec\theta$ is positive as OM is positive. when θ is slightly greater than $\frac{3\pi}{2}$, $\sec\theta$ is positive and very large. Also $\sec 2\pi = 1$. Hence $\sec\theta$ decreases from ∞ to 1 as θ varies from $\frac{3\pi}{2}$ to 2π.

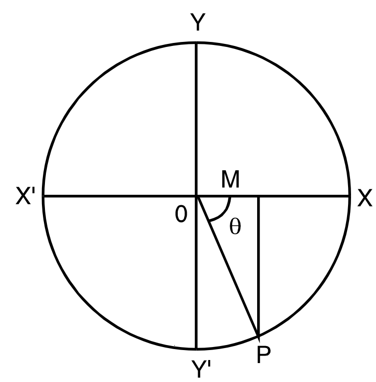


Fig. 3.27

It may be observed that as θ passes through $\frac{3\pi}{2}$; $\sec\theta$ changes from $-\infty$ to $+\infty$.

Graph of sec θ as θ varies from 0 to 2π

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0$	$\frac{\pi}{2}+0$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0$	$\frac{3\pi}{2}+0$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cot θ	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$	$+\infty$	2	1.15	

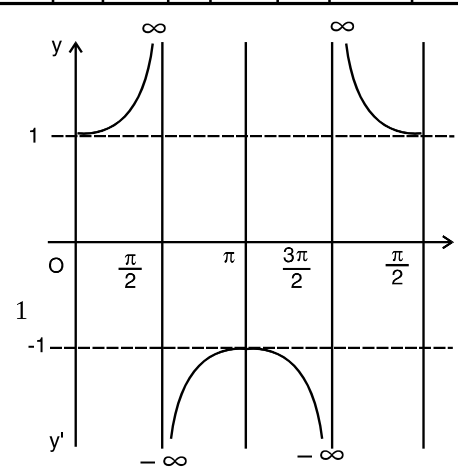


Fig. 3.28

Observations

(a) $\sec\theta$ cannot be numerically less than 1.

(b) Graph of $\sec\theta$ is discontinuous, discontinuities (breaks) occuring at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

(c) As θ passes through $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, see θ changes abruptly from $+\infty$ to $-\infty$ and then from $-\infty$ to $+\infty$ respectively.

3.4.7 Graph of cosec θ as θ Varies From 0 to 2π

Let *X'OX* and *Y'OY* be the axes of coordinates. With centre *O* draw a circle of unit radius. Let *P* be any point on the circle. Join *OP* and draw *PM* perpendicular to *X'OX*.

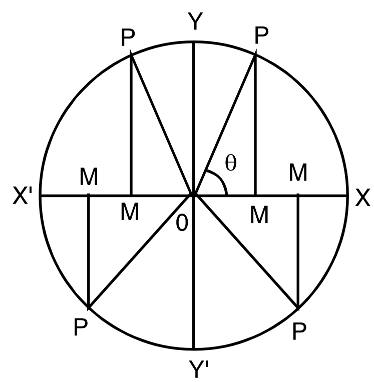


Fig. 3.29

$$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{1}{MP}$$

$\therefore$ The variation of $\operatorname{cosec}\theta$ will depend upon *MP*.

I Quadrant : cosec θ is positive as *MP* is positive.

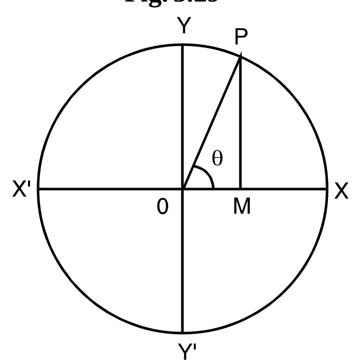


Fig. 3.30

cosec $\frac{\pi}{2} = 1$ when θ is very small, *MP* is also small and therefore, the value of cosec θ is very large.

$\therefore$ As θ varies from 0 to $\frac{\pi}{2}$, cosec θ decreases from ∞ to 1.

II Quadrant : PM is positive. Therefore, cosec θ is positive. cosec $\frac{\pi}{2} = 1$ and cosec $\pi = \infty$ when the revolving line approaches π in the counter clockwise direction.

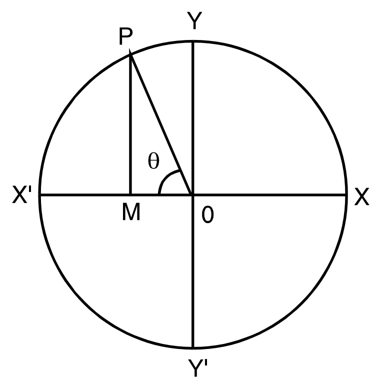


Fig. 3.31

$\therefore$ As θ varies from $\frac{\pi}{2}$ to π, cosec θ increases from 1 to ∞.

III Quadrant : *PM* is negative

$\therefore$ cosec θ is negative. When θ is slightly greater than π, cosec θ is very large and negative.

Also cosec $\frac{3\pi}{2} = -1$.

$\therefore$ As θ varies from π to $\frac{3\pi}{2}$, $\text{cosec}\,\theta$ changes from $-\infty$ to -1.

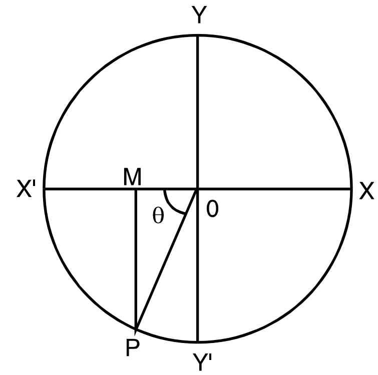


Fig. 3.32

It may be observed that as θ passes through π, cosec θ changes from $+\infty$ to $-\infty$.

IV Quadrant :

PM is negative.

Therefore, cosec $\theta = -\infty$ as θ approaches 2π.

$\therefore$ as θ varies from $\frac{3\pi}{2}$ to 2π, $\text{cosec}\,\theta$ varies from -1 to $-\infty$.

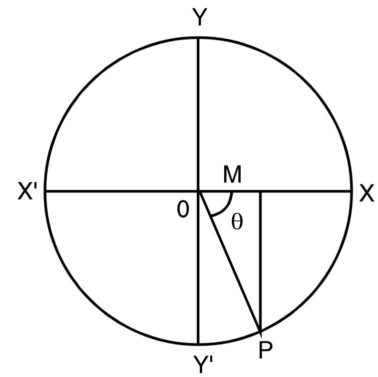


Fig. 3.33

Graph of cosec θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cosec θ	∞	2	1.15	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$

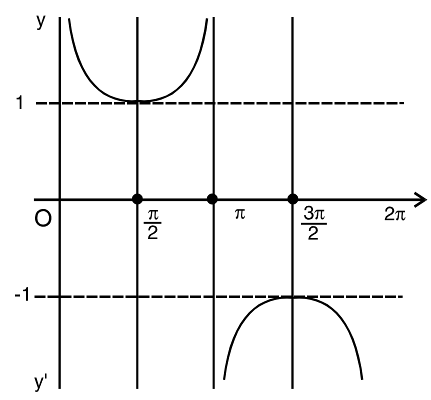


Fig. 3.34

Observations

(a) $\text{cosec}\,\theta$ cannot be numerically less than 1.

(b) Graph of $\text{cosec}\,\theta$ is discountinous and it has breaks at $\theta = 0, \pi, 2\pi$.

(c) As θ passes through π, cosec θ changes from $+\infty$ to $-\infty$. The values at 0 and 2π are $+\infty$ and $-\infty$ respectively.

Example 3.16 Trace the changes in the values of sec θ as θ lies in $-\pi$ to π.

Soluton :

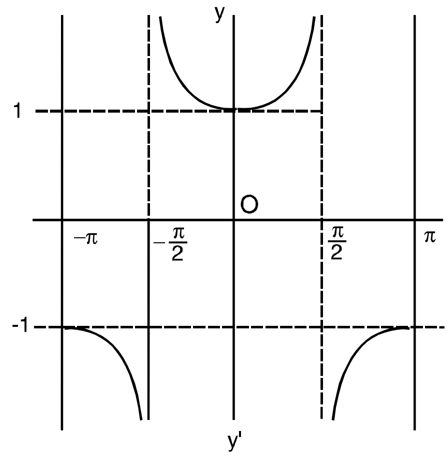


Fig. 3.35

CHECK YOUR PROGRESS 3.9

1. (a) Trace the changes in the values of sec θ when θ lies between -2π and 2π and draw the graph between these limits.

   (b) Trace the graph of cosec θ ,when θ lies between -2π and 2π.

3.5 PERIODICITY OF THE TRIGONOMETRIC FUNCTIONS

From your daily experience you must have observed things repeating themselves after regular intervals of time. For example, days of a week are repeated regularly after 7 days and months of a year are repeated regularly after 12 months. Position of a particle on a moving wheel is another example of the type. The property of repeated occurence of things over regular intervals is known as ***periodicity.***

Definition : A function $f(x)$ is said to be periodic if its value is unchanged when the value of the variable is increased by a constant, that is if $f(x + p) = f(x)$ for all x.

If p is smallest positive constant of this type, then p is called the period of the function $f(x)$.

If $f(x)$ is a periodic function with period p, then $\frac{1}{f(x)}$ is also a periodic function with period p.

3.5.1 Periods of Trigonometric Functions

(i) $\sin x = \sin(x + 2n\pi)$; $n = 0, \pm 1, \pm 2, \ldots..$

(ii) $\cos x = \cos(x + 2n\pi)$; $n = 0, \pm 1, \pm 2, \ldots..$

Also there is no p, lying in 0 to 2π, for which

$$\sin x = \sin(x + p)$$

$$\cos x = \cos(x + p), \text{ for all } x$$

$\therefore$ 2π is the smallest positive value for which

$\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$

$\Rightarrow$ sin x and cos x each have the period 2π.

(iii) The period of cosec x is also 2π because $\text{cosec } x = \dfrac{1}{\sin x}$.

(iv) The period of sec x is also 2π as $\sec x = \dfrac{1}{\cos x}$.

(v) Also $\tan(x + \pi) = \tan x$. Suppose $p\ (0 < p < \pi)$ is the period of tan x, then

$\tan(x + p) = \tan x$, for all x. Put x = 0, then tan p = 0, i.e., p = 0 or π.

$\Rightarrow$ the period of tan x is π.

$\therefore$ p can not have values between 0 and π for which $\tan x = \tan(x + p)$

$\therefore$ The period of $\tan x$ is π

(vi) Since $\cot x = \dfrac{1}{\tan x}$, therefore, the period of cot x is also π.

Example 3.17 Find the period of each the following functions :

(a) $y = 3 \sin 2x$ (b) $y = \cos \dfrac{x}{2}$ (c) $y = \tan \dfrac{x}{4}$

Solution :

(a) Period is $\dfrac{2\pi}{2}$, i.e., π.

(b) $y = \cos \dfrac{1}{2} x$, therefore period $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$

(c) Period of $y = \tan \dfrac{x}{4} = \dfrac{\pi}{\frac{1}{4}} = 4\pi$

CHECK YOUR PROGRESS 3.10

1. Find the period of each of the following functions :

(a) $y = 2 \sin 3x$ (b) $y = 3 \cos 2x$

(c) $y = \tan 3x$ (d) $y = \sin^2 2x$

LET US SUM UP

- An angle is generated by the rotation of a ray.
- The angle can be negative or positive according as rotation of the ray is clockwise or anticlockwise.
- A degree is one of the measures of an angle and one complete rotation generates an angle of 360°.
- An angle can be measured in radians, 360° being equivalent to 2π radians.
- If an arc of length l subtends an angle of θ radians at the centre of the circle with radius r, we have $l = r\theta$.
- If the coordinates of a point P of a unit circle are (x, y) then the six trigonometric functions are defined as $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$, $\sec\theta = \frac{1}{x}$ and $\operatorname{cosec}\theta = \frac{1}{y}$.

  The coordinates (x, y) of a point P can also be written as $(\cos\theta, \sin\theta)$.

  Here θ is the angle which the line joining centre to the point P makes with the positive direction of x-axis.
- The values of the trigonometric functions $\sin\theta$ and $\cos\theta$ when θ takes values 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ are given by

Real numbers θ → / Functions	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

- Graphs of $\sin\theta$, $\cos\theta$ are continuous every where
  - — Maximum value of both $\sin\theta$ and $\cos\theta$ is 1.
  - — Minimum value of both $\sin\theta$ and $\cos\theta$ is -1.
  - — Period of these functions is 2π.

- $\tan\theta$ and $\cot\theta$ can have any value between $-\infty$ and $+\infty$.

— The function $\tan\theta$ has discontinuities (breaks) at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ in $(0, 2\pi)$.

— Its period is π.

— The graph of $\cot\theta$ has discontinuities (breaks) at $0, \pi, 2\pi$. Its period is π.

- $\sec\theta$ cannot have any value numerically less than 1.

(i) It has breaks at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. It repeats itself after 2π.

(ii) $\operatorname{cosec}\theta$ cannot have any value between -1 and $+1$.

It has discontinuities (breaks) at $0, \pi, 2\pi$. It repeats itself after 2π.

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Trigonometric_functions

http://mathworld.wolfram.com/ Trigonometric_functions.html

TERMINAL EXERCISE

1. A train is moving at the rate of 75 km/hour along a circular path of radius 2500 m. Through how many radians does it turn in one minute ?

2. Find the number of degrees subtended at the centre of the circle by an arc whose length is 0.357 times the radius.

3. The minute hand of a clock is 30 cm long. Find the distance covered by the tip of the hand in 15 minutes.

4. Prove that

(a) $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

(b) $\frac{1}{\sec\theta + \tan\theta} = \sec\theta - \tan\theta$

(c) $\frac{\tan\theta}{1+\tan^2\theta} - \frac{\cot\theta}{1+\cot^2\theta} = 2\sin\theta\cos\theta$

(d) $\frac{1+\sin\theta}{1-\sin\theta} = (\tan\theta + \sec\theta)^2$

(e) $\sin^8\theta - \cos^8\theta = \left(\sin^2\theta - \cos^2\theta\right)\left(1 - 2\sin^2\theta\cos^2\theta\right)$

(f) $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$

5. If $\theta = \frac{\pi}{4}$, verify that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

6. Evaluate :

(a) $\sin \frac{25\pi}{6}$ (b) $\sin \frac{21\pi}{4}$

(c) $\tan\left(\frac{3\pi}{4}\right)$ (d) $\sin \frac{17}{4}\pi$

(e) $\cos \frac{19}{3}\pi$

7. Draw the graph of cos x from $x = -\frac{\pi}{2}$ to $x = \frac{3\pi}{2}$.

8. Define a periodic function of x and show graphically that the period of tan x is π, i.e. the position of the graph from x = π to 2π is repetition of the portion from x = 0 to π.

ANSWERS

CHECK YOUR PROGRESS 3.1

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{5\pi}{12}$ (iv) $\frac{7\pi}{12}$ (v) $\frac{3\pi}{2}$

2. (i) 45° (ii) 15° (iii) 9° (iv) 3° (v) 120°

3. $\frac{\pi}{4}, \frac{13\pi}{36}, \frac{14\pi}{36}$ 4. $\frac{5\pi}{6}$ 5. $\frac{\pi}{3}$

CHECK YOUR PROGRESS 3.2

1. (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{5\pi}{6}$

2. (a) 36° (b) 30° (c) 20°

3. $\frac{1}{6}$ radian; 9.55° 4. $\frac{1}{5}$ radian 5. 95.54 m

6. (a) 0.53 m (b) 38.22 cm (c) 0.002 radian
   (d) 12.56 m (e) 31.4 cm (f) 3.75 radian
   (g) 6.28 m (h) 2 radian (i) 19.11 m.

CHECK YOUR PROGRESS 3.3

1. (i) – ve (ii) – ve (iii) – ve (iv) + ve
   (v) + ve (vi) – ve (vii) + ve (viii) – ve

2. (i) zero (ii) zero (iii) $-\frac{1}{2}$ (iv) – 1
   (v) 1 (vi) Not defined (vii) Not defined (viii) 1

CHECK YOUR PROGRESS 3.4

2. $\sin\theta = \frac{1}{\sqrt{5}}, \ \cos\theta = \frac{2}{\sqrt{5}}, \cot\theta = 2, \ \text{cosec}\ \theta = \sqrt{5}, \ \sec\ \theta = \frac{\sqrt{5}}{2}$

3. $\sin\theta = \frac{a}{b}, \quad \cos\theta = \frac{\sqrt{b^2 - a^2}}{b}, \quad \sec\theta = \frac{b}{\sqrt{b^2 - a^2}},$

   $\tan\theta = \frac{a}{\sqrt{b^2 - a^2}}, \quad \cot\theta = \frac{\sqrt{b^2 - a^2}}{a}$ 6. $\frac{2m}{1 + m^2}$

11. $\cos x = \frac{5}{13}, \sin x = \frac{-12}{13}, \cos ec = \frac{-13}{12}, \tan x = \frac{-12}{5}, \cot x = \frac{-5}{12}$

CHECK YOUR PROGRESS 3.5

1. (i) $4\frac{1}{4}$ (ii) $6\frac{1}{2}$ (iii) -1 (iv) $\frac{22}{3}$ (v) Zero

CHECK YOUR PROGRESS 3.6

1. 1, –1

3. Graph of $y = 2 \sin \theta$, $[0, \pi]$

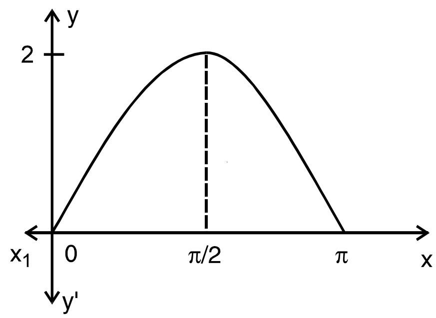


Fig. 3.36

4. (a) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{4\pi}{3}, \frac{5\pi}{3}$

5. $y = \sin x$ from $-\pi$ to π

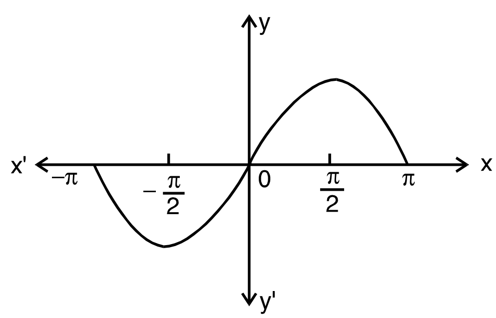


Fig. 3.37

CHECK YOUR PROGRESS 3.7

1. (a) $y = \cos\theta,\ -\frac{\pi}{4}$ to $\frac{\pi}{4}$

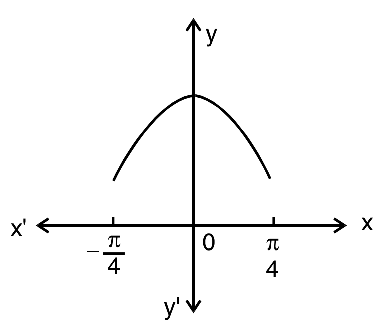


Fig. 3.38

(b) $y = 3\cos\theta;\ 0 \text{ to } 2\pi$

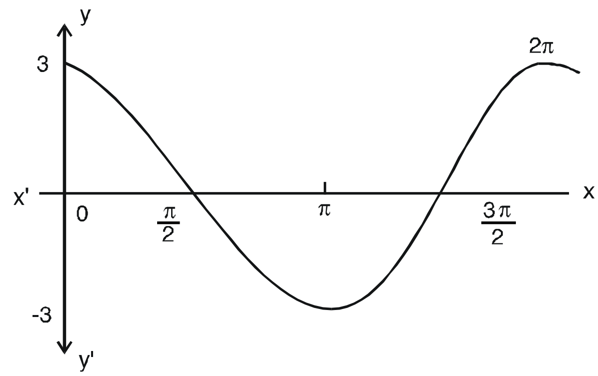


Fig. 3.39

(c) $y = \cos 3\theta, -\pi \text{ to } \pi$

$$\cos\theta = 0.87$$

$$\theta = \frac{\pi}{6}, -\frac{\pi}{6}$$

$$\cos\theta = -0.87$$

$$\theta = \frac{5\pi}{6}, -\frac{5\pi}{6}$$

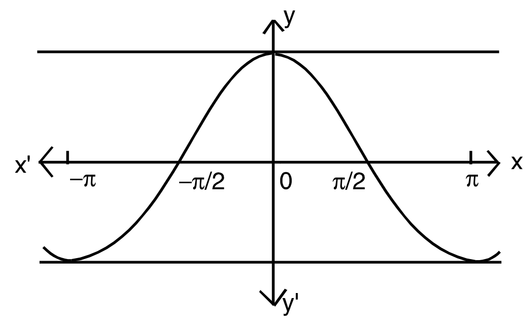


Fig. 3.40

(d) Graph of $y = \cos\theta$ in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ lies below the x-axis.

(e) $y = \cos\theta$

θ lies in 2π to 4π

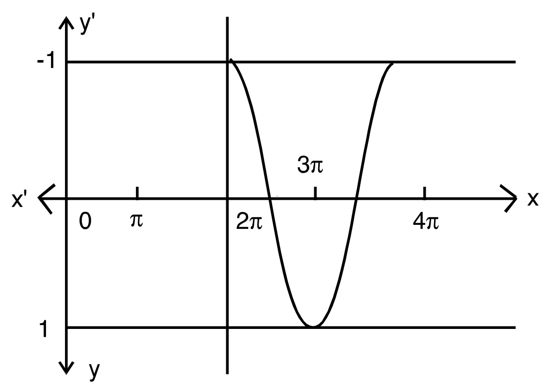


Fig. 3.41

CHECK YOUR PROGRESS 3.8

1. (a) Infinite (b) At $\frac{\pi}{2}, \frac{3\pi}{2}$ there are breaks in graphs.

(c) $y = \tan 2\theta, \quad -\pi$ to π

$$\text{At } \theta = \frac{\pi}{3}, \tan\theta = 1.7$$

2. (a) Infinite (b) $\cot\theta = -1$ at $\theta = \frac{3\pi}{4}$

CHECK YOUR PROGRESS 3.9

1. (a) $y = \sec\theta$

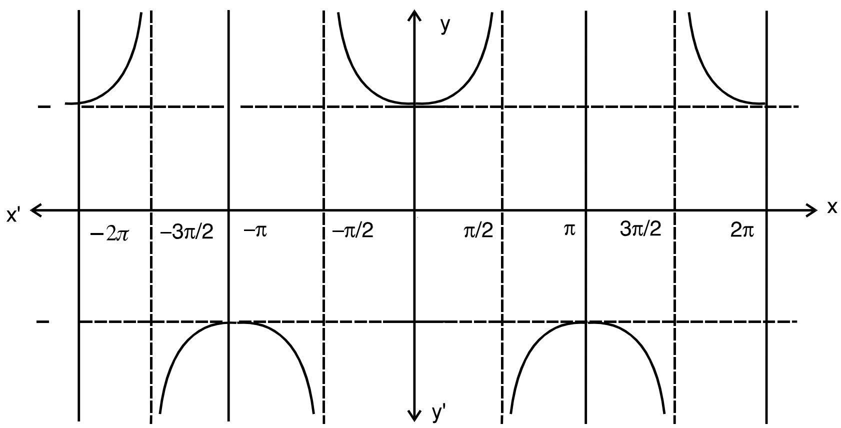


Fig. 3.42

Points of discontinuity of $\sec 2\theta$ are at $\frac{\pi}{4}, \frac{3\pi}{4}$ in the interval $[0, 2\pi]$.

(b) In tracing the graph from 0 to -2π, use $\text{cosec}(-\theta) = -\text{cosec}\,\theta$.

CHECK YOUR PROGRESS 3.10

1. (a) Period is $\frac{2\pi}{3}$ (b) Period is $\frac{2\pi}{2} = \pi$ (c) Period of y is $\frac{\pi}{3}$

(d) $y = \sin^2 2x = \frac{1 - \cos 4x}{2} = \frac{1}{2} - \frac{1}{2}\cos 4x$; Period of y is $\frac{2\pi}{4}$ i.e $\frac{\pi}{2}$

(e) $y = 3\cot\left(\frac{x+1}{3}\right)$, Period of y is $\frac{\pi}{\frac{1}{3}} = 3\pi$

TERMINAL EXERCISE

1. $\frac{1}{2}$ radian 2. 20.45° 3. 15π cm

6. (a) $\frac{1}{2}$ (b) $-\frac{1}{\sqrt{2}}$ (c) -1 (d) $\frac{1}{\sqrt{2}}$ (e) $\frac{1}{2}$

7.

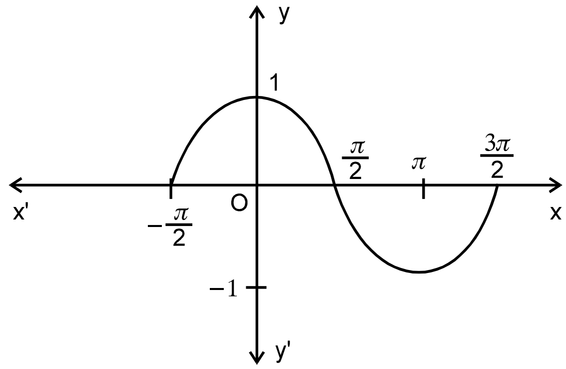
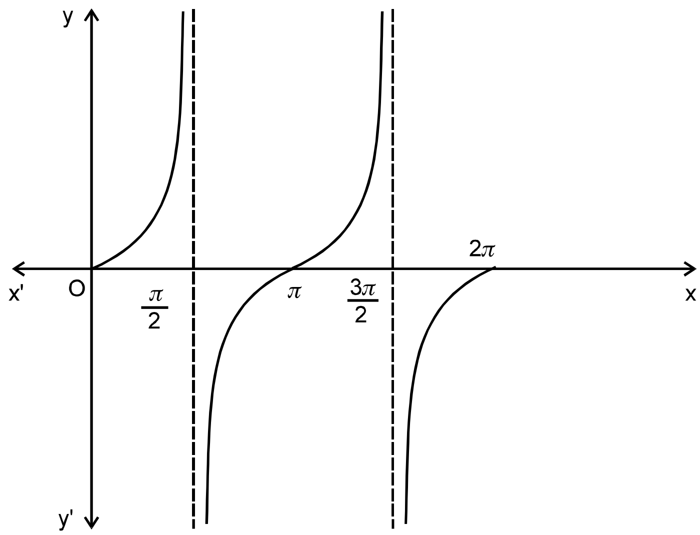


Fig. 3.43

8.



$y = \sec\theta$

Fig. 3.44

4

TRIGONOMETRIC FUNCTIONS-II

In the previous lesson, you have learnt trigonometric functions of real numbers, drawn and interpretd the graphs of trigonometric functions. In this lesson we will establish addition and subtraction formulae for $\cos(A \pm B)$, $\sin(A \pm B)$ and $\tan(A \pm B)$. We will also state the formulae for the multiple and sub multiples of angles and solve examples thereof. The general solutions of simple trigonometric functions will also be discussed in the lesson.

OBJECTIVES

After studying this lesson, you will be able to :

- write trigonometric functions of $-x, \frac{x}{2}, x \pm y, \frac{\pi}{2} \pm x, \pi \pm x$ where x, y are real nunbers;
- establish the addition and subtraction formulae for :

  $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$,

  $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- solve problems using the addition and subtraction formulae;
- state the formulae for the multiples and sub-multiples of angles such as cos2A, sin 2A, tan 2A, cos 3A, sin 3A, tan 3A, $\sin\frac{A}{2}, \cos\frac{A}{2}$ and $\tan\frac{A}{2}$; and
- solve simple trigonometric equations of the type :

  $\sin x = \sin\alpha$, $\cos x = \cos\alpha$, $\tan x = \tan\alpha$

EXPECTED BACKGROUND KNOWLEDGE

- Definition of trigonometric functions.
- Trigonometric functions of complementary and supplementary angles.
- Trigonometric identities.

MODULE - I
Sets, Relations and Functions

Notes

4.1 ADDITION AND MULTIPLICATION OF TRIGONOMETRIC FUNCTIONS

In earlier sections we have learnt about circular measure of angles, trigonometric functions, values of trigonometric functions of specific numbers and of allied numbers.

You may now be interested to know whether with the given values of trigonometric functions of any two numbers *A* and *B*, it is possible to find trigonometric functions of sums or differences.

You will see how trigonometric functions of sum or difference of numbers are connected with those of individual numbers. This will help you, for instance, to find the value of trigonometric functions of $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ etc.

$\frac{\pi}{12}$ can be expressed as $\frac{\pi}{4}-\frac{\pi}{6}$ and $\frac{5\pi}{12}$ can be expressed as $\frac{\pi}{4}+\frac{\pi}{6}$

How can we express $\frac{7\pi}{12}$ in the form of addition or subtraction?

In this section we propose to study such type of trigonometric functions.

4.1.1 Addition Formulae

For any two numbers A and B,

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

In given figure trace out

$\angle SOP = A$

$\angle POQ = B$

$\angle SOR = -B$

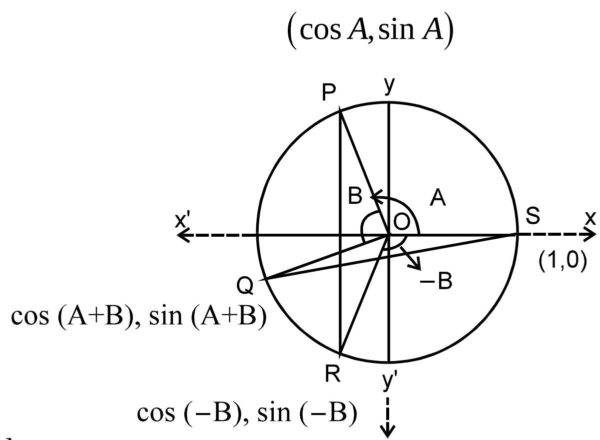


Fig. 4.1

where points P, Q, R, S lie on the unit circle.

Coordinates of P, Q, R, S will be (cos A, sin A),

[cos (A + B), sin (A + B)],

$[\cos(-B), \sin(-B)]$, and (1, 0).

From the given figure, we have

side OP = side OQ, $\angle$POR = $\angle$QOS (each angle = $\angle$B + $\angle$ QOR), side OR = side OS

$\Delta POR \cong \Delta QOS$ (by SAS) $\therefore$ PR = QS

$$PR=\sqrt{(\cos A-\cos(-B))^2+(\sin A-\sin(-B))^2}$$

$$QS=\sqrt{(\cos(A+B)-1)^2+(\sin(A+B)-0)^2}$$

Since $PR^2 = QS^2$ $\therefore$ $\cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B + 2\sin A \sin B$

$$= \cos^2(A+B) + 1 - 2\cos(A+B) + \sin^2(A+B)$$

$\Rightarrow$ $1 + 1 - 2(\cos A \cos B - \sin A \sin B) = 1 + 1 - 2\cos(A+B)$

$\Rightarrow$ $\cos A \cos B - \sin A \sin B = \cos(A+B)$ (I)

Corollary 1

For any two numbers A and B, cos (A – B) = cos A cos B + sin A sin B

Proof : Replace B by $-B$ in (I)

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

[$\because$ $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$]

Corollary 2

For any two numbers A and B, $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Proof : We know that $\cos\left(\frac{\pi}{2} - A\right) = \sin A$ and $\sin\left(\frac{\pi}{2} - A\right) = \cos A$

$\therefore$ $\sin(A+B) = \cos\left[\left(\frac{\pi}{2} - (A+B)\right)\right] = \cos\left[\left(\frac{\pi}{2} - A\right) - B\right]$

$$= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

or $\sin(A+B) = \sin A \cos B + \cos A \sin B$ .....(II)

Corollary 3

For any two numbers A and B, $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Proof : Replacing B by $-B$ in (2), we have

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

or $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Example 4.1

(a) Find the value of each of the following :

(i) $\sin\frac{5\pi}{12}$ (ii) $\cos\frac{\pi}{12}$ (iii) $\cos\frac{7\pi}{12}$

(b) If $\sin A = \frac{1}{\sqrt{10}}, \sin B = \frac{1}{\sqrt{5}}$ show that $A + B = \frac{\pi}{4}$

Solution :

(a) (i) $\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$= \cos\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Observe that $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}$

(iii) $\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$$= \cos\frac{\pi}{3}\cdot\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

(b) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\cos A = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}} \text{ and } \cos B = \sqrt{1-\frac{1}{5}} = \frac{2}{\sqrt{5}}$$

Substituting all these values in the above formula, we get

$$\sin(A+B) = \frac{1}{\sqrt{10}}\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}}\frac{1}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{10}\sqrt{5}} + \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \text{ or } A + B = \frac{\pi}{4}$$

CHECK YOUR PROGRESS 4.1

1. (a) Find the values of each of the following :

(i) $\sin\frac{\pi}{12}$ (ii) $\sin\frac{\pi}{9}\cdot\cos\frac{2\pi}{9}+\cos\frac{\pi}{9}\cdot\sin\frac{2\pi}{9}$

(b) Prove the following :

(i) $\sin\left(\frac{\pi}{6}+A\right)=\frac{1}{2}\left(\cos A+\sqrt{3}\sin A\right)$ (ii) $\sin\left(\frac{\pi}{4}-A\right)=\frac{1}{\sqrt{2}}(\cos A-\sin A)$

(c) If $\sin A=\frac{8}{17}$ and $\sin B=\frac{5}{13}$, find $\sin(A-B)$

2. (a) Find the value of $\cos\frac{5\pi}{12}$.

(b) Prove that:

(i) $\cos\theta+\sin\theta=\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)$ (ii) $\sqrt{3}\sin\theta-\cos\theta=2\sin\left(\theta-\frac{\pi}{6}\right)$

(iii) $\cos(n+1)A\cos(n-1)A+\sin(n+1)A\sin(n-1)A=\cos 2A$

(iv) $\cos\left(\frac{\pi}{4}+A\right)\cos\left(\frac{\pi}{4}-B\right)+\sin\left(\frac{\pi}{4}+A\right)\sin\left(\frac{\pi}{4}-B\right)=\cos(A+B)$

Corollary 4 : $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

Proof : $\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$

Dividing by cos A cos B, we have

$$\tan(A+B)=\frac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}}$$

or $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \quad ......(III)$$

Corollary 5 : $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

Proof : Replacing B by $-B$ in (III), we get the required result.

Corollary 6 : $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

Proof : $\cot(A+B) = \dfrac{\cos(A+B)}{\sin(A+B)} = \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

Dividing by sin A sin B, we have ......(IV)

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

Corollary 7 : $\tan\left(\dfrac{\pi}{4}+A\right) = \dfrac{1+\tan A}{1-\tan A}$

Proof : $\tan\left(\dfrac{\pi}{4}+A\right) = \dfrac{\tan\dfrac{\pi}{4}+\tan A}{1-\tan\dfrac{\pi}{4}\cdot\tan A} = \dfrac{1+\tan A}{1-\tan A}$ as $\tan\dfrac{\pi}{4}=1$

Similarly, it can be proved that $\tan\left(\dfrac{\pi}{4}-A\right) = \dfrac{1-\tan A}{1+\tan A}$

Example 4.2 Find $\tan\dfrac{\pi}{12}$

Solution : $\tan\dfrac{\pi}{12} = \tan\left(\dfrac{\pi}{4}-\dfrac{\pi}{6}\right) = \dfrac{\tan\dfrac{\pi}{4}-\tan\dfrac{\pi}{6}}{1+\tan\dfrac{\pi}{4}\cdot\tan\dfrac{\pi}{6}} = \dfrac{1-\dfrac{1}{\sqrt{3}}}{1+1.\dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$$= \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

$$\therefore \quad \tan\frac{\pi}{12} = 2-\sqrt{3}$$

Example 4.3 Prove the following :

(a) $\dfrac{\cos\dfrac{7\pi}{36}+\sin\dfrac{7\pi}{36}}{\cos\dfrac{7\pi}{36}-\sin\dfrac{7\pi}{36}} = \tan\dfrac{4\pi}{9}$

(b) $\tan 7A - \tan 4A - \tan 3A = \tan 7A \tan 4A \cdot \tan 3A$

Solution : (a) Dividing numerator and denominator by $\cos\frac{7\pi}{36}$, we get

$$\text{L.H.S.} = \frac{\cos\frac{7\pi}{36}+\sin\frac{7\pi}{36}}{\cos\frac{7\pi}{36}-\sin\frac{7\pi}{36}} = \frac{1+\tan\frac{7\pi}{36}}{1-\tan\frac{7\pi}{36}} = \frac{\tan\frac{\pi}{4}+\tan\frac{7\pi}{36}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{7\pi}{36}}$$

$$= \tan\left(\frac{\pi}{4}+\frac{7\pi}{36}\right) = \tan\frac{16\pi}{36} = \tan\frac{4\pi}{9} = \text{R.H.S.}$$

(b) $\tan 7A = \tan(4A+3A) = \dfrac{\tan 4A + \tan 3A}{1-\tan 4A\ \tan 3A}$

or $\quad \tan 7A - \tan 7A\ \tan 4A\ \tan 3A = \tan 4A + \tan 3A$

or $\quad \tan 7A - \tan 4A - \tan 3A = \tan 7A \tan 4A \tan 3A$

CHECK YOUR PROGRESS 4.2

1. Fill in the blanks :

(i) $\sin\left(\frac{\pi}{4}+A\right)\sin\left(\frac{\pi}{4}-A\right) = \ldots\ldots\ldots$

(ii) $\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \ldots\ldots\ldots$

2. (a) Prove that :

(i) $\tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{\pi}{4}-\theta\right) = 1.$

(ii) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(iii) $\tan\frac{\pi}{12}+\tan\frac{\pi}{6}+\tan\frac{\pi}{12}\cdot\tan\frac{\pi}{6} = 1$

(b) If $\tan A = \frac{a}{b}$; $\tan B = \frac{c}{d}$, Prove that $\tan(A+B) = \dfrac{ad+bc}{bd-ac}$.

(c) Find the value of $\cos\frac{11\pi}{12}$.

3. (a) Prove that : (i) $\tan\left(\frac{\pi}{4}+A\right)\tan\left(\frac{3\pi}{4}+A\right)=-1$

(ii) $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan\left(\frac{\pi}{4}+\theta\right)$ (iii) $\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\tan\left(\frac{\pi}{4}-\theta\right)$

4.2 TRANSFORMATION OF PRODUCTS INTO SUMS AND VICE VERSA

4.2.1 Transformation of Products into Sums or Differences

We know that $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

By adding and subtracting the first two formulae, we get respectively

$$2\sin A\cos B=\sin(A+B)+\sin(A-B) \quad .....(1)$$

and $$2\cos A\sin B=\sin(A+B)-\sin(A-B) \quad .....(2)$$

Similarly, by adding and subtracting the other two formulae, we get

$$2\cos A\cos B=\cos(A+B)+\cos(A-B) \quad ....(3)$$

and $$2\sin A\sin B=\cos(A-B)-\cos(A+B) \quad ....(4)$$

We can also quote these as

$$2\sin A\cos B=\sin(\text{sum})+\sin(\text{difference})$$

$$2\cos A\sin B=\sin(\text{sum})-\sin(\text{difference})$$

$$2\cos A\cos B=\cos(\text{sum})+\cos(\text{difference})$$

$$2\sin A\sin B=\cos(\text{difference})-\cos(\text{sum})$$

4.2.2 Transformation of Sums or Differences into Products

In the above results put

$$A+B=C \text{ and } A-B=D$$

Then $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$ and (1), (2), (3) and (4) become

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$$

4.2.3 Further Applications of Addition and Subtraction Formulae

We shall prove that (i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$ or $\cos^2 B - \sin^2 A$

Proof : (i) $\sin(A+B)\sin(A-B)$

$$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$$

$$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$$

$$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$$

$$= \sin^2 A - \sin^2 B$$

(ii) $\cos(A+B)\cos(A-B)$

$$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$$

$$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$$

$$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B = \cos^2 A - \sin^2 B$$

$$= (1-\sin^2 A) - (1-\cos^2 B) = \cos^2 B - \sin^2 A$$

Example 4.4 Express the following products as a sum or difference

(i) $2\sin 3\theta\cos 2\theta$ (ii) $\cos 6\theta\cos\theta$ (iii) $\sin\frac{5\pi}{12}\sin\frac{\pi}{12}$

Solution :

(i) $2\sin 3\theta\cos 2\theta = \sin(3\theta+2\theta) + \sin(3\theta-2\theta) = \sin 5\theta + \sin\theta$

(ii) $\cos 6\theta\cos\theta = \frac{1}{2}(2\cos 6\theta\cos\theta) = \frac{1}{2}[\cos(6\theta+\theta) + \cos(6\theta-\theta)]$

$$= \frac{1}{2}(\cos 7\theta + \cos 5\theta)$$

(iii) $$\sin\frac{5\pi}{12}\sin\frac{\pi}{12} = \frac{1}{2}\left[2\sin\frac{5\pi}{12}\sin\frac{\pi}{12}\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{5\pi-\pi}{12}\right) - \cos\left(\frac{5\pi+\pi}{12}\right)\right] = \frac{1}{2}\left[\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\right]$$

Example 4.5 Express the following sums as products.

(i) $\cos\frac{5\pi}{9} + \cos\frac{7\pi}{9}$ (ii) $\sin\frac{5\pi}{36} + \cos\frac{7\pi}{36}$

Solution :

(i) $$\cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = 2\cos\frac{5\pi+7\pi}{9\times 2}\cos\frac{5\pi-7\pi}{9\times 2}$$

$$= 2\cos\frac{2\pi}{3}\cos\frac{\pi}{9} \quad \left[\because \cos\left(-\frac{\pi}{9}\right) = \cos\frac{\pi}{9}\right]$$

$$= 2\cos\left(\pi - \frac{\pi}{3}\right)\cos\frac{\pi}{9} = -2\cos\frac{\pi}{3}\cos\frac{\pi}{9}$$

$$= -\cos\frac{\pi}{9} \quad \left[\because \cos\frac{\pi}{3} = \frac{1}{2}\right]$$

(ii) $$\sin\frac{5\pi}{36} + \cos\frac{7\pi}{36} = \sin\left(\frac{\pi}{2} - \frac{13\pi}{36}\right) + \cos\frac{7\pi}{36}$$

$$= \cos\frac{13\pi}{36} + \cos\frac{7\pi}{36}$$

$$= 2\cos\frac{13\pi+7\pi}{36\times 2}\cos\frac{13\pi-7\pi}{36\times 2} = 2\cos\frac{5\pi}{18}\cos\frac{\pi}{12}$$

Example 4.6 Prove that $\dfrac{\cos 7A - \cos 9A}{\sin 9A - \sin 7A} = \tan 8A$

Solution :

$$\text{L.H.S.} = \frac{2\sin\frac{7A+9A}{2}\sin\frac{9A-7A}{2}}{2\cos\frac{9A+7A}{2}\sin\frac{9A-7A}{2}}$$

$$= \frac{\sin 8A\ \sin A}{\cos 8A\ \sin A} = \frac{\sin 8A}{\cos 8A} = \tan 8A = \text{R.H.S.}$$

Example 4.7 Prove the following :

(i) $$\cos^2\left(\frac{\pi}{4}-A\right)-\sin^2\left(\frac{\pi}{4}-B\right)=\sin(A+B)\cos(A-B)$$

(ii) $$\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)=\frac{1}{\sqrt{2}}\sin A$$

Solution :

(i) Applying the formula

$\cos^2 A-\sin^2 B=\cos(A+B)\cos(A-B)$, we have

$$\text{L.H.S.} =\cos\left[\frac{\pi}{4}-A+\frac{\pi}{4}-B\right]\cos\left[\frac{\pi}{4}-A-\frac{\pi}{4}+B\right]$$

$$=\cos\left[\frac{\pi}{2}-(A+B)\right]\cos\left[-(A-B)\right]=\sin(A+B)\cos(A-B)=\text{R.H.S.}$$

(ii) Applying the formula

$\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$, we have

$$\text{L.H.S.} =\sin\left(\frac{\pi}{8}+\frac{A}{2}+\frac{\pi}{8}-\frac{A}{2}\right)\sin\left(\frac{\pi}{8}+\frac{A}{2}-\frac{\pi}{8}+\frac{A}{2}\right)$$

$$=\sin\frac{\pi}{4}\sin A=\frac{1}{\sqrt{2}}\sin A=\text{R.H.S.}$$

Example 4.8 Prove that

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{3}\cos\frac{4\pi}{9}=\frac{1}{16}$$

Solution : $$\text{L.H.S. } \cos\frac{\pi}{3}\left[\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}$$

$$=\frac{1}{2}\cdot\frac{1}{2}\left[2\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}\qquad\left[\because \cos\frac{\pi}{3}=\frac{1}{2}\right]$$

$$=\frac{1}{4}\left[\cos\frac{\pi}{3}+\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}=\frac{1}{8}\cos\frac{4\pi}{9}+\frac{1}{8}\left[2\cos\frac{4\pi}{9}\cos\frac{\pi}{9}\right]$$

$$=\frac{1}{8}\cos\frac{4\pi}{9}+\frac{1}{8}\left[\cos\frac{5\pi}{9}+\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{8}\cos\frac{4\pi}{9} + \frac{1}{8}\cos\frac{5\pi}{9} + \frac{1}{16} \qquad .....(1)$$

Now $$\cos\frac{5\pi}{9} = \cos\left[\pi - \frac{4\pi}{9}\right] = -\cos\frac{4\pi}{9} \qquad .....(2)$$

From (1) and (2), we get L.H.S. $= \frac{1}{16} =$ R.H.S.

CHECK YOUR PROGRESS 4.3

1. Express each of the following as sums or differences :

   (a) $2\cos 3\theta \sin 2\theta$ (b) $2\sin 4\theta \sin 2\theta$

   (c) $2\cos\frac{\pi}{4}\cos\frac{\pi}{12}$ (d) $2\sin\frac{\pi}{3}\cos\frac{\pi}{6}$

2. Express each of the following as a product :

   (a) $\sin 6\theta + \sin 4\theta$ (b) $\sin 7\theta - \sin 3\theta$

   (c) $\cos 2\theta - \cos 4\theta$ (d) $\cos 7\theta + \cos 5\theta$

3. Prove that :

   (a) $\sin\frac{5\pi}{18} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$ (b) $\dfrac{\cos\frac{\pi}{9} - \cos\frac{7\pi}{18}}{\sin\frac{7\pi}{18} - \sin\frac{\pi}{9}} = 1$

   (c) $\sin\frac{5\pi}{18} - \sin\frac{7\pi}{18} + \sin\frac{\pi}{18} = 0$ (d) $\cos\frac{\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = 0$

4. Prove that :

   (a) $\sin^2(n+1)\theta - \sin^2 n\theta = \sin(2n+1)\theta \cdot \sin\theta$

   (b) $\cos\beta\cos(2\alpha - \beta) = \cos^2\alpha - \sin^2(\alpha - \beta)$

   (c) $\cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{12} = \frac{\sqrt{3}}{4}$

5. Show that $\cos^2\left(\frac{\pi}{4} + \theta\right) - \sin^2\left(\frac{\pi}{4} - \theta\right)$ is independent of θ.

6. Prove that :

(a) $$\frac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$$

(b) $$\sin\frac{\pi}{18}\sin\frac{5\pi}{6}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18} = \frac{1}{16}$$

(c) $$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\frac{\alpha-\beta}{2}$$

4.3 TRIGONOMETRIC FUNCTIONS OF MULTIPLES OF ANGLES

(a) To express sin 2A in terms of sin A, cos A and tan A.

We know that $\sin(A+B) = \sin A\cos B + \cos A\sin B$

By putting $B = A$, we get $\sin 2A = \sin A\cos A + \cos A\sin A = 2\sin A\cos A$

$\therefore$ sin 2A can also be written as

$$\sin 2A = \frac{2\sin A\cos A}{\cos^2 A + \sin^2 A} \qquad (\because 1 = \cos^2 A + \sin^2 A)$$

Dividing numerator and denomunator by $\cos^2 A$, we get

$$\sin 2A = \frac{2\left(\dfrac{\sin A\cos A}{\cos^2 A}\right)}{\dfrac{\cos^2 A}{\cos^2 A} + \dfrac{\sin^2 A}{\cos^2 A}} = \frac{2\tan A}{1+\tan^2 A}$$

(b) To express cos 2A in terms of sin A, cos A and tan A.

We know that $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Putting $B = A$, we have $\cos 2A = \cos A\cos A - \sin A\sin A$

or $\quad \cos 2A = \cos^2 A - \sin^2 A$

Also $\cos 2A = \cos^2 A - (1-\cos^2 A) = \cos^2 A - 1 + \cos^2 A$

i.e, $\quad \cos 2A = 2\cos^2 A - 1 \quad \Rightarrow \quad \cos^2 A = \dfrac{1+\cos 2A}{2}$

Also $\cos 2A = \cos^2 A - \sin^2 A = 1 - \sin^2 A - \sin^2 A$

i.e., $\quad \cos 2A = 1 - 2\sin^2 A \quad \Rightarrow \quad \sin^2 A = \dfrac{1-\cos 2A}{2}$

$$\therefore \quad \cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

Dividing the numerator and denominator of R.H.S. by $\cos^2 A$, we have

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

(c) To express tan 2A in terms of tan A.

$$\tan 2A = \tan(A+A) = \frac{\tan A + \tan A}{1-\tan A \tan A} = \frac{2\tan A}{1-\tan^2 A}$$

Thus we have derived the following formulae :

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}, \quad \cos^2 A = \frac{1+\cos 2A}{2}, \quad \sin^2 A = \frac{1-\cos 2A}{2}$$

Example 4.9 Prove that $\frac{\sin 2A}{1+\cos 2A} = \tan A$

Solution : $$\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin A}{\cos A} = \tan A$$

Example 4.10 Prove that $\cot A - \tan A = 2\cot 2A$.

Solution : $$\cot A - \tan A = \frac{1}{\tan A} - \tan A = \frac{1-\tan^2 A}{\tan A}$$

$$= \frac{2(1-\tan^2 A)}{2\tan A}$$

$$= \frac{2}{\left(\frac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{2}{\tan 2A} = 2\cot 2A.$$

Example 4.11 Evaluate $\cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8}$.

Solution : $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}=\frac{1+\cos\frac{\pi}{4}}{2}+\frac{1+\cos\frac{3\pi}{4}}{2}$

$$=\frac{1+\frac{1}{\sqrt{2}}}{2}+\frac{1-\frac{1}{\sqrt{2}}}{2}=\frac{\left(\sqrt{2}+1\right)+\left(\sqrt{2}-1\right)}{2\sqrt{2}}=1$$

Example 4.12 Prove that $\frac{\cos A}{1-\sin A}=\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)$.

Solution : R.H.S. $=\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)=\frac{\tan\frac{\pi}{4}+\tan\frac{A}{2}}{1-\tan\frac{\pi}{4}\tan\frac{A}{2}}$

$$=\frac{1+\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}{1-\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}=\frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}$$

$$=\frac{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)}{\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)^2}$$

[Multiplying Numerator and Denominator by $\left(\frac{\cos A}{2}-\frac{\sin A}{2}\right)$]

$$=\frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}-2\cos\frac{A}{2}\sin\frac{A}{2}}=\frac{\cos A}{1-\sin A}=\text{L.H.S}.$$

CHECK YOUR PROGRESS 4.4

1. If $A=\frac{\pi}{3}$, verify that

   (a) $\sin 2A=2\sin A\cos A=\frac{2\tan A}{1+\tan^2 A}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

2. Find the value of sin 2A when (assuming $0 < A < \frac{\pi}{2}$)

(a) $\cos A = \frac{3}{5}$ (b) $\sin A = \frac{12}{13}$ (c) $\tan A = \frac{16}{63}$.

3. Find the value of cos 2A when

(a) $\cos A = \frac{15}{17}$ (b) $\sin A = \frac{4}{5}$ (c) $\tan A = \frac{5}{12}$

4. Find the value of tan 2A when

(a) $\tan A = \frac{3}{4}$ (b) $\tan A = \frac{a}{b}$

5. Evaluate $\sin^2 \frac{\pi}{8} + \sin^2 \frac{3\pi}{8}$.

6. Prove the following :

(a) $\dfrac{1+\sin 2A}{1-\sin 2A} = \tan^2\left(\dfrac{\pi}{4} + A\right)$ (b) $\dfrac{\cot^2 A + 1}{\cot^2 A - 1} = \sec 2A$

7. (a) Prove that $\dfrac{\sin 2A}{1-\cos 2A} = \cos A$ (b) Prove that $\tan A + \cot A = 2\,\mathrm{cosec}\, 2A$.

8. (a) Prove that $\dfrac{\cos A}{1+\sin A} = \tan\left(\dfrac{\pi}{4} - \dfrac{A}{2}\right)$

(b) Prove that $(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2\dfrac{\alpha-\beta}{2}$

4.3.1 Trigonometric Functions of 3A in Terms of A

(a) sin 3A in terms of sin A

Substituting 2A for B in the formula

$\sin(A+B) = \sin A\cos B + \cos A\sin B$, we get

$$\sin(A+2A) = \sin A\cos 2A + \cos A\sin 2A$$

$$= \sin A\left(1 - 2\sin^2 A\right) + (\cos A \times 2\sin A\cos A)$$

$$= \sin A - 2\sin^3 A + 2\sin A\left(1 - \sin^2 A\right)$$

$$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$$

$\therefore$ $\sin 3A = 3\sin A - 4\sin^3 A$ ....(1)

(b) **cos 3A in terms of cos A**

Substituting 2A for B in the formula

$\cos(A+B) = \cos A\cos B - \sin A\sin B$, we get

$\cos(A+2A) = \cos A\cos 2A - \sin A\sin 2A$

$= \cos A\left(2\cos^2 A - 1\right) - (\sin A) \times 2\sin A\cos A$

$= 2\cos^3 A - \cos A - 2\cos A\left(1-\cos^2 A\right)$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$\therefore \quad \cos 3A = 4\cos^3 A - 3\cos A$ ....(2)

(c) **tan 3A in terms of tan A**

Putting B = 2A in the formula $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$, we get

$$\tan(A+2A) = \frac{\tan A + \tan 2A}{1 - \tan A\tan 2A} = \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \times \dfrac{2\tan A}{1-\tan^2 A}}$$

$$= \frac{\dfrac{\tan A - \tan^3 A + 2\tan A}{1-\tan^2 A}}{\dfrac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \quad \text{....(3)}$$

(d) **Formulae for** $\sin^3 A$ **and** $\cos^3 A$

$\because \quad \sin 3A = 3\sin A - 4\sin^3 A$

$\therefore \quad 4\sin^3 A = 3\sin A - \sin 3A$ or $\sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

Similarly, $\cos 3A = 4\cos^3 A - 3\cos A$

$\therefore \quad 3\cos A + \cos 3A = 4\cos^3 A$ or $\cos^3 A = \dfrac{3\cos A + \cos 3A}{4}$

Example 4.13 Prove that

$$\sin\alpha\sin\left(\frac{\pi}{3}+\alpha\right)\sin\left(\frac{\pi}{3}-\alpha\right) = \frac{1}{4}\sin 3\alpha$$

Solution : $\sin\alpha\sin\left(\dfrac{\pi}{3}+\alpha\right)\sin\left(\dfrac{\pi}{3}-\alpha\right)$

$$= \frac{1}{2}\sin\alpha\left[\cos 2\alpha - \cos\frac{2\pi}{3}\right] = \frac{1}{2}\sin\alpha\left[1-2\sin^2\alpha - \left(1-2\sin^2\frac{\pi}{3}\right)\right]$$

$$= 2\frac{1}{2}\sin\alpha\left[\sin^2\frac{\pi}{3} - \sin^2\alpha\right]$$

$$= \sin\alpha\left[\frac{3}{4} - \sin^2\alpha\right] = \frac{3\sin\alpha - 4\sin^3\alpha}{4} = \frac{1}{4}\sin 3\alpha$$

Example 4.14 Prove that $\cos^3 A \sin 3A + \sin^3 A \cos 3A = \frac{3}{4}\sin 4A$

Solution : $\cos^3 A \sin 3A + \sin^3 A \cos 3A$

$$= \cos^3 A\left(3\sin A - 4\sin^3 A\right) + \sin^3 A\left(4\cos^3 A - 3\cos A\right)$$

$$= 3\sin A\cos^3 A - 4\sin^3 A\cos^3 A + 4\sin^3 A\cos^3 A - 3\sin^3 A\cos A$$

$$= 3\sin A\cos^3 A - 3\sin^3 A\cos A$$

$$= 3\sin A\cos A\left(\cos^2 A - \sin^2 A\right) = \left(3\sin A\cos A\right)\cos 2A$$

$$= \frac{3\sin 2A}{2}\times\cos 2A = \frac{3}{2}\frac{\sin 4A}{2} = \frac{3}{4}\sin 4A.$$

Example 4.15 Prove that $\cos^3\frac{\pi}{9} + \sin^3\frac{\pi}{18} = \frac{3}{4}\left(\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right)$

Solution : L.H.S. $= \frac{1}{4}\left[3\cos\frac{\pi}{9} + \cos\frac{\pi}{3}\right] + \frac{1}{4}\left(3\sin\frac{\pi}{18} - \sin\frac{\pi}{6}\right)$

$$= \frac{3}{4}\left[\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right] + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\right) = \frac{3}{4}\left[\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right] = \text{R.H.S.}$$

CHECK YOUR PROGRESS 4.5

1. If $A = \frac{\pi}{3}$, verify that (a) $\sin 3A = 3\sin A - 4\sin^3 A$

   (b) $\cos 3A = 4\cos^3 A - 3\cos A$ (c) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

2. Find the value of sin 3A when (a) $\sin A = \frac{2}{3}$ (b) $\sin A = \frac{p}{q}$.

3. Find the value of cos 3A when (a) $\cos A = -\frac{1}{3}$ (b) $\cos A = \frac{c}{d}$.

4. Prove that $\cos\alpha \cos\left(\frac{\pi}{3} - \alpha\right)\cos\left(\frac{\pi}{3} + \alpha\right) = \frac{1}{4}\cos 3\alpha$.

5. (a) Prove that $\sin^3\frac{2\pi}{9} - \sin^3\frac{\pi}{9} = \frac{3}{4}\left(\sin\frac{2\pi}{9} - \sin\frac{\pi}{9}\right)$

(b) Prove that $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A}$ is constant.

6. (a) Prove that $\cot 3A = \frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

(b) Prove that

$$\cos 10A + \cos 8A + 3\cos 4A + 3\cos 2A = 8\cos A\cos^3 3A$$

4.4 TRIGONOMETRIC FUNCTIONS OF SUBMULTIPLES OF ANGLES

$\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$ are called submultiples of A.

It has been proved that

$$\sin^2 A = \frac{1 - \cos 2A}{2}, \quad \cos^2 A = \frac{1 + \cos 2A}{2}, \quad \tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$$

Replacing A by $\frac{A}{2}$, we easily get the following formulae for the sub-multiple $\frac{A}{2}$:

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}, \quad \cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}} \quad \text{and} \quad \tan\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

We will choose either the positive or the negative sign depending on whether corresponding value of the function is positive or negative for the value of $\frac{A}{2}$. This will be clear from the following examples

Example 4.16 Find the values of $\sin\left(-\frac{\pi}{8}\right)$ and $\cos\left(-\frac{\pi}{8}\right)$.

Solution : We use the formula $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

and take the lower sign, i.e., negative sign, because $\sin\left(-\frac{\pi}{8}\right)$ is negative.

$$\sin\left(-\frac{\pi}{8}\right) = -\sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}} = -\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$

$$= -\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$$

Similarly, $$\cos\left(-\frac{\pi}{8}\right) = +\sqrt{\frac{1+\cos\left(-\frac{\pi}{4}\right)}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Example 4.17 If $\cos A = \frac{7}{25}$ and $\frac{3\pi}{2} < A < 2\pi$, find the values of

(i) $\sin\frac{A}{2}$ (ii) $\cos\frac{A}{2}$ (iii) $\tan\frac{A}{2}$

Solution : $\because$ A lies in the 4th-quardrant, $\frac{3\pi}{2} < A < 2\pi$

$$\Rightarrow \quad 3\frac{\pi}{4} < \frac{A}{2} < \pi$$

$$\therefore \quad \sin\frac{A}{2} > 0,\ \cos\frac{A}{2} < 0,\ \tan\frac{A}{2} < 0.$$

$$\therefore \quad \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-\frac{7}{25}}{2}} = \sqrt{\frac{18}{50}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}} = -\sqrt{\frac{1+\frac{7}{25}}{2}} = -\sqrt{\frac{32}{50}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

and $$\tan\frac{A}{2} = -\sqrt{\frac{1-\cos A}{1+\cos A}} = -\sqrt{\frac{1-\frac{7}{25}}{1+\frac{7}{25}}} = -\sqrt{\frac{18}{32}} = -\sqrt{\frac{9}{16}} = -\frac{3}{4}$$

CHECK YOUR PROGRESS 4.6

1. If $A = \frac{\pi}{3}$, verify that (a) $\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$

   (b) $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$ (c) $\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$

2. Find the values of $\sin\frac{\pi}{12}$ and $\sin\frac{\pi}{24}$.

3. Determine the values of

   (a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$ (c) $\tan\frac{\pi}{8}$.

4.5 TRIGONOMETRIC EQUATIONS

You are familiar with the equations like simple linear equations, quadratic equations in algebra. You have also learnt how to solve the same.

Thus, (i) $x - 3 = 0$ gives one value of *x* as a solution.

(ii) $x^2 - 9 = 0$ gives two values of *x*.

You must have noticed, the number of values depends upon the degree of the equation.

Now we need to consider as to what will happen in case *x*'s and *y*'s are replaced by trigonometric functions.

Thus solution of the equation $\sin\theta - 1 = 0$, will give

$$\sin\theta = 1 \text{ and } \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots.$$

Clearly, the solution of simple equations with only finite number of values does not necessarily hold good in case of trigonometric equations.

So, we will try to find the ways of finding solutions of such equations.

4.5.1 To find the general solution of the equation sin θ = sin α

It is given that $\sin\theta = \sin\alpha, \Rightarrow \quad \sin\theta - \sin\alpha = 0$

or $$2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$\therefore$ Either $\cos\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$$\Rightarrow \quad \frac{\theta+\alpha}{2} = (2p+1)\frac{\pi}{2} \quad \text{or} \quad \frac{\theta-\alpha}{2} = q\pi,\ p,\ q \in Z$$

$$\Rightarrow \quad \theta = (2p+1)\pi - \alpha \quad \text{or} \quad \theta = 2q\pi + \alpha \quad \text{....(1)}$$

From (1), we get

$\theta = n\pi + (-1)^n \alpha,\ n \in Z$ as the geeneral solution of the equation $\sin\theta = \sin\alpha$

4.5.2 To find the general solution of the equation cos θ = cos α

It is given that, $\cos\theta = \cos\alpha$, $\Rightarrow \quad \cos\theta - \cos\alpha = 0$

$$\Rightarrow \quad -2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

$$\therefore \quad \text{Either, } \sin\frac{\theta+\alpha}{2} = 0 \quad \text{or } \sin\frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \quad \frac{\theta+\alpha}{2} = p\pi \ \text{ or } \ \frac{\theta-\alpha}{2} = q\pi,\ p, q \in Z$$

$$\Rightarrow \quad \theta = 2p\pi - \alpha \quad \text{or } \theta = 2z\pi + \alpha \qquad \text{....(1)}$$

From (1), we have

$\theta = 2n\pi \pm \alpha, n \in Z$ as the general solution of the equation $\cos\theta = \cos\alpha$

4.5.3 To find the general solution of the equation tan θ = tan α

It is given that, $\tan\theta = \tan\alpha$, $\Rightarrow \quad \dfrac{\sin\theta}{\cos\theta} - \dfrac{\sin\alpha}{\cos\alpha} = 0$

$$\Rightarrow \quad \sin\theta\cos\alpha - \sin\alpha\cos\theta = 0, \ \Rightarrow \quad \sin(\theta-\alpha) = 0$$

$$\Rightarrow \quad \theta - \alpha = n\pi,\ n \in Z, \ \Rightarrow \quad \theta = n\pi + \alpha,\ n \in Z$$

Similarly, for $\quad \operatorname{cosec}\theta = \operatorname{cosec}\alpha$, the general solution is $\theta = n\pi + (-1)^n \alpha$

and, for $\quad \sec\theta = \sec\alpha$, the general solution is $\theta = 2n\pi \pm \alpha$

and for $\quad \cot\theta = \cot\alpha$, $\theta = n\pi + \alpha$ is its general solution

Example 4.18 Find the general solution of the following equations :

(a) (i) $\sin\theta = \dfrac{1}{2}$ (ii) $\sin\theta = -\dfrac{\sqrt{3}}{2}$ (b) (i) $\cos\theta = \dfrac{\sqrt{3}}{2}$ (ii) $\cos\theta = -\dfrac{1}{2}$

(c) $\cot\theta = -\sqrt{3}$ (d) $4\sin^2\theta = 1$

Solution : (a) (i) $\sin\theta = \dfrac{1}{2} = \sin\dfrac{\pi}{6}$

$\therefore \quad \theta = n\pi + (-1)^n \frac{\pi}{6}, \quad n \in Z$

(ii) $\sin\theta = \frac{-\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin\frac{4\pi}{3}$

$\therefore \quad \theta = n\pi + (-1)^n \frac{4\pi}{3}, \quad n \in Z$

(b) (i) $\cos\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}, \ \therefore \quad \theta = 2n\pi \pm \frac{\pi}{6}, \quad n \in Z$

(ii) $\cos\theta = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\therefore \quad \theta = 2n\pi \pm \frac{2\pi}{3}, \quad n \in Z$

(c) $\cot\theta = -\sqrt{3}, \ \tan\theta = -\frac{1}{\sqrt{3}} = -\tan\frac{\pi}{6} = \tan\left(\pi - \frac{\pi}{6}\right) = \tan\frac{5\pi}{6}$

$\therefore \quad \theta = n\pi + \frac{5\pi}{6}, \quad n \in Z$

(d) $4\sin^2\theta = 1 \quad \Rightarrow \quad \sin^2\theta = \frac{1}{4} = \left(\frac{1}{2}\right)^2 = \sin^2\frac{\pi}{6}$

$\Rightarrow \quad \sin\theta = \sin\left(\pm\frac{\pi}{6}\right) \ \therefore \ \theta = n\pi \pm \frac{\pi}{6}, \quad n \in Z$

Example 4.19 Solve the following to find general solution :

(a) $2\cos^2\theta + 3\sin\theta = 0$ (b) $\cos 4x = \cos 2x$

(c) $\cos 3x = \sin 2x$ (d) $\sin 2x + \sin 4x + \sin 6x = 0$

Solution :

(a) $2\cos^2\theta + 3\sin\theta = 0, \quad \Rightarrow \quad 2(1 - \sin^2\theta) + 3\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta - 3\sin\theta - 2 = 0, \quad \Rightarrow \quad (2\sin\theta + 1)(\sin\theta - 2) = 0$

$\Rightarrow \quad \sin\theta = -\frac{1}{2}$ or $\sin\theta = 2,$ Since $\sin\theta = 2$ is not possible.

$\therefore \quad \sin\theta = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$

$\therefore \quad \theta = n\pi + (-1)^n \cdot \frac{7\pi}{6}, \ n \in Z$

(b) $\cos 4x = \cos 2x$ i.e., $\cos 4x - \cos 2x = 0$

$\Rightarrow \quad -2\sin 3x \sin x = 0$

$\Rightarrow \quad \sin 3x = 0 \quad$ or $\quad \sin x = 0$

$\Rightarrow \quad 3x = n\pi \quad$ or $\quad x = n\pi$

$\Rightarrow \quad x = \dfrac{n\pi}{3} \quad$ or $\quad x = n\pi \qquad n \in Z$

(c) $\cos 3x = \sin 2x \Rightarrow \cos 3x = \cos\left(\dfrac{\pi}{2} - 2x\right)$

$\Rightarrow \quad 3x = 2n\pi \pm \left(\dfrac{\pi}{2} - 2x\right) \qquad n \in Z$

Taking positive sign only, we have $3x = 2n\pi + \dfrac{\pi}{2} - 2x$

$\Rightarrow \quad 5x = 2n\pi + \dfrac{\pi}{2} \qquad \Rightarrow \qquad x = \dfrac{2n\pi}{5} + \dfrac{\pi}{10}$

Now taking negative sign, we have

$$3x = 2n\pi - \frac{\pi}{2} + 2x \quad \Rightarrow \quad x = 2n\pi - \frac{\pi}{2} \quad n \in Z$$

(d) $\sin 2x + \sin 4x + \sin 6x = 0$

or $\quad (\sin 6x + \sin 2x) + \sin 4x = 0$

or $\quad 2\sin 4x \cos 2x + \sin 4x = 0$

or $\quad \sin 4x\,[2\cos 2x + 1] = 0$

$\therefore \quad \sin 4x = 0 \quad$ or $\cos 2x = -\dfrac{1}{2} = \cos\dfrac{2\pi}{3}$

$\Rightarrow \quad 4x = n\pi \quad$ or $2x = 2n\pi \pm \dfrac{2\pi}{3},\ n \in Z$

$x = \dfrac{n\pi}{4} \quad$ or $\quad x = n\pi \pm \dfrac{\pi}{3} \qquad n \in Z$

CHECK YOUR PROGRESS 4.7

1. Find the general value of θ satisfying :

(i) $\sin\theta = \dfrac{\sqrt{3}}{2}$ (ii) $\operatorname{cosec}\theta = \sqrt{2}$

(iii) $\sin\theta = -\frac{\sqrt{3}}{2}$ (iv) $\sin\theta = -\frac{1}{\sqrt{2}}$

2. Find the general value of θ satisfying :

(i) $\cos\theta = -\frac{1}{2}$ (ii) $\sec\theta = -\frac{2}{\sqrt{3}}$

(iii) $\cos\theta = \frac{\sqrt{3}}{2}$ (iv) $\sec\theta = -\sqrt{2}$

3. Find the general value of θ satisfying :

(i) $\tan\theta = -1$ (ii) $\tan\theta = \sqrt{3}$ (iii) $\cot\theta = -1$

4. Find the general value of θ satisfying :

(i) $\sin 2\theta = \frac{1}{2}$ (ii) $\cos 2\theta = \frac{1}{2}$ (iii) $\tan 3\theta = \frac{1}{\sqrt{3}}$

(iv) $\cos 3\theta = -\frac{\sqrt{3}}{2}$ (v) $\sin^2\theta = \frac{3}{4}$ (vi) $\sin^2 2\theta = \frac{1}{4}$

(vii) $4\cos^2\theta = 1$ (viii) $\cos^2 2\theta = \frac{3}{4}$

5. Find the general solution of the following :

(i) $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$ (ii) $4\cos^2\theta - 4\sin\theta = 1$

(iii) $\cot\theta + \tan\theta = 2\,\mathrm{cosec}\,\theta$

LET US SUM UP

- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$,

  $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

  $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$, $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

  $\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A}$, $\cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}$

- $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

  $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

  $2\cos A\cos B = \cos(A+B) - \cos(A-B)$

  $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

- $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

  $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

  $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

  $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

- $\sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B$

  $\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B$

- $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$

- $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

- $\sin^2 A = \frac{1-\cos 2A}{2}$, $\cos^2 A = \frac{1+\cos 2A}{2}$, $\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

- $\sin 3A = 3\sin A - 4\sin^3 A$, $\cos 3A = 4\cos^3 A - 3\cos A$

  $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

- $\sin^3 A = \frac{3\sin A - \sin 3A}{4}$, $\cos^3 A = \frac{3\cos A + \cos 3A}{4}$

- $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

  $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

- $\sin\theta = \sin\alpha \quad \Rightarrow \quad \theta = n\pi + (-1)^n\alpha, \quad n \in Z$

- $\cos\theta = \cos\alpha \quad \Rightarrow \quad \theta = 2n\pi \pm \alpha, \quad n \in Z$

- $\tan\theta = \tan\alpha \quad \Rightarrow \quad \theta = n\pi + \alpha, \quad n \in Z$

SUPPORTIVE WEB SITES

http://mathworld.wolfram.com/ Trigonometric_functions.html
http://en.wikipedia.org/wiki/Trigonometric_functions

TERMINAL EXERCISE

1. Prove that $\tan(A+B)\times\tan(A-B)=\dfrac{\cos^2 B-\cos^2 A}{\cos^2 B-\sin^2 A}$

2. Prove that $\cos\theta-\sqrt{3}\sin\theta=2\cos\left(\theta+\dfrac{\pi}{3}\right)$

3. If $A+B=\dfrac{\pi}{4}$

   Prove that $(1+\tan A)(1+\tan B)=2$ and $(\cot A-1)(\cos B-1)=2$

4. Prove each of the following :

   (i) $\dfrac{\sin(A-B)}{\cos A\cos B}+\dfrac{\sin(B-C)}{\cos B\cos C}+\dfrac{\sin(C-A)}{\cos C\cos A}=0$

   (ii) $\cos\left(\dfrac{\pi}{10}-A\right)\cdot\cos\left(\dfrac{\pi}{10}+A\right)+\cos\left(\dfrac{2\pi}{5}-A\right)\cdot\cos\left(\dfrac{2\pi}{5}+A\right)=\cos 2A$

   (iii) $\cos\dfrac{2\pi}{9}\cdot\cos\dfrac{4\pi}{9}\cdot\cos\dfrac{9\pi}{9}=-\dfrac{1}{8}$

   (iv) $\cos\dfrac{13\pi}{45}+\cos\dfrac{17\pi}{45}+\cos\dfrac{43\pi}{45}=0$

   (v) $\tan\left(A+\dfrac{\pi}{6}\right)+\cot\left(A-\dfrac{\pi}{6}\right)=\dfrac{1}{\sin 2A-\sin\dfrac{\pi}{3}}$

   (vi) $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta$ (vii) $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan 2\theta+\sec 2\theta$

   (viii) $\left(\dfrac{1-\sin\theta}{1+\sin\theta}\right)^2=\tan^2\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$

   (ix) $\cos^2 A+\cos^2\left(A+\dfrac{\pi}{3}\right)+\cos^2\left(A-\dfrac{\pi}{3}\right)=\dfrac{3}{2}$

(x) $\dfrac{\sec 8A - 1}{\sec 4A - 1} = \dfrac{\tan 8A}{\tan 2A}$ (xi) $\cos\dfrac{\pi}{30}\cos\dfrac{7\pi}{30}\cos\dfrac{11\pi}{30}\cos\dfrac{13\pi}{30} = \dfrac{11}{16}$

(xii) $\sin\dfrac{\pi}{10} + \sin\dfrac{13\pi}{10} = -\dfrac{1}{2}$

5. Find the general value of 'θ' satisfying

(a) $\sin\theta = \dfrac{1}{\sqrt{2}}$ (b) $\sin\theta = \dfrac{\sqrt{3}}{2}$

(c) $\sin\theta = -\dfrac{1}{\sqrt{2}}$ (d) $\text{cosec}\,\theta = \sqrt{2}$

6. Find the general value of 'θ' satisfying

(a) $\cos\theta = \dfrac{1}{2}$ (b) $\sec\theta = \dfrac{2}{\sqrt{3}}$

(c) $\cos\theta = \dfrac{-\sqrt{3}}{2}$ (d) $\sec\theta = -2$

7. Find the general value of 'θ' satisfying

(a) $\tan\theta = 1$ (b) $\tan\theta = -1$ (c) $\cot\theta = -\dfrac{1}{\sqrt{3}}$

8. Find the general value of 'θ' satisfying

(a) $\sin^2\theta = \dfrac{1}{2}$ (b) $4\cos^2\theta = 1$ (c) $2\cot^2\theta = \text{cosec}^2\,\theta$

9. Solve the following for θ :

(a) $\cos p\theta = \cos q\theta$ (b) $\sin 9\theta = \sin\theta$ (c) $\tan 5\theta = \cot\theta$

10. Solve the following for θ :

(a) $\sin m\theta + \sin n\theta = 0$ (b) $\tan m\theta + \cot n\theta = 0$

(c) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ (d) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

ANSWERS

CHECK YOUR PROGRESS 4.1

1. (a) (i) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\dfrac{\sqrt{3}}{2}$ (c) $\dfrac{21}{221}$

2. (a) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$

CHECK YOUR PROGRESS 4.2

1. (i) $\dfrac{\cos^2 A - \sin^2 A}{2}$ (ii) $-\dfrac{1}{4}$

2. (c) $-\dfrac{\left(\sqrt{3}+1\right)}{2\sqrt{2}}$

CHECK YOUR PROGRESS 4.3

1. (a) $\sin 5\theta - \sin\theta$; (b) $\cos 2\theta - \cos 6\theta$

   (c) $\cos\dfrac{\pi}{3} + \cos\dfrac{\pi}{6}$ (d) $\sin\dfrac{\pi}{2} + \sin\dfrac{\pi}{6}$

2. (a) $2\sin 5\theta\cos\theta$ (b) $2\cos 5\theta \cdot \sin 2\theta$

   (c) $2\sin 3\theta \cdot \sin\theta$ (d) $2\cos 6\theta \cdot \cos\theta$

CHECK YOUR PROGRESS 4.4

2. (a) $\dfrac{24}{25}$ (b) $\dfrac{120}{169}$ (c) $\dfrac{2016}{4225}$

3. (a) $\dfrac{161}{289}$ (b) $\dfrac{-7}{25}$ (c) $\dfrac{119}{169}$

4. (a) $\dfrac{24}{7}$ (b) $\dfrac{2ab}{b^2 - a^2}$

5. 1

CHECK YOUR PROGRESS 4.5

2. (a) $\dfrac{22}{27}$ (b) $\dfrac{\left(3pq^2 - 4p^3\right)}{q^3}$

3. (a) $\frac{23}{27}$ (b) $\frac{4c^3 - 3cd^2}{d^3}$

CHECK YOUR PROGRESS 4.6

2. (a) $\frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{\sqrt{\left(4-\sqrt{2}-\sqrt{6}\right)}}{2\sqrt{2}}$

3. (a) $\frac{\sqrt{2-\sqrt{2}}}{2}$ (b) $\frac{\sqrt{2+\sqrt{2}}}{2}$ (c) $\sqrt{2}-1$

CHECK YOUR PROGRESS 4.7

1. (i) $\theta = n\pi + (-1)^n \frac{\pi}{3}, n \in Z$ (ii) $\theta = n\pi + (-1)^n \frac{\pi}{4}, n \in Z$

   (iii) $\theta = n\pi + (-1)^n \frac{4\pi}{3}, n \in Z$ (iv) $\theta = n\pi + (-1)^n \frac{5\pi}{4}, n \in Z$

2. (i) $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$ (ii) $\theta = 2n\pi \pm \frac{5\pi}{6}, n \in Z$

   (iii) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z$ (iv) $\theta = 2n\pi \pm \frac{3\pi}{4}, n \in Z$

3. (i) $\theta = n\pi + \frac{3\pi}{4}, n \in Z$ (ii) $\theta = n\pi + \frac{\pi}{3}, n \in Z$

   (iii) $\theta = n\pi - \frac{\pi}{4}, n \in Z$

4. (i) $\theta = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}, n \in Z$ (ii) $\theta = n\pi \pm \frac{\pi}{6}, n \in Z$

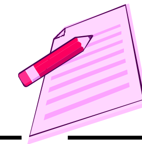
   (iii) $\theta = \frac{n\pi}{3} + \frac{\pi}{18}, n \in Z$ (iv) $\theta = \frac{2n\pi}{3} \pm \frac{5\pi}{18}, n \in Z$

   (v) $\theta = n\pi \pm \frac{\pi}{3}, n \in Z$ (vi) $\theta = \frac{n\pi}{2} \pm \frac{\pi}{12}, n \in Z$

   (vii) $\theta = n\pi \pm \frac{\pi}{3}, n \in Z$ (viii) $\theta = \frac{n\pi}{2} \pm \frac{\pi}{12}, n \in Z$

5. (i) $\theta = 2n\pi \pm \frac{5\pi}{6}, n \in Z$ (ii) $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in Z$

   (iii) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$



TERMINAL EXERCISE

5. (a) $\theta = n\pi + (-1)^n \frac{\pi}{4}, n \in Z$

(b) $\theta = n\pi + (-1)^n \frac{\pi}{3}, n \in Z$

(c) $\theta = n\pi + (-1)^n \frac{5\pi}{4}, n \in Z$

(d) $\theta = n\pi + (-1)^n \frac{\pi}{4}, n \in Z$

6. (a) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

(b) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z$

(c) $\theta = 2n\pi \pm \frac{5\pi}{6}, n \in Z$

(d) $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

7. (a) $\theta = n\pi + \frac{\pi}{4}, n \in Z$

(b) $\theta = n\pi + \frac{3\pi}{4}, n \in Z$

(c) $\theta = n\pi + \frac{2\pi}{3}, n \in Z$

8. (a) $\theta = n\pi \pm \frac{\pi}{4}, n \in Z$

(b) $\theta = n\pi \pm \frac{\pi}{3}, n \in Z$

(c) $\theta = n\pi \pm \frac{\pi}{4}, n \in Z$

9. (a) $\theta = \frac{2n\pi}{p \mp q}, n \in Z$

(b) $\theta = \frac{n\pi}{4}$ or $(2n+1)\frac{\pi}{10}, n \in Z$

(c) $\theta = (2n+1)\frac{\pi}{12}, n \in Z$

10. (a) $\theta = \frac{(2k+1)\pi}{m-n}$ or $\frac{2k\pi}{m+n}, k \in I$

(b) $\theta = \frac{(2k+1)\pi}{2(m-n)}, k \in Z$

(c) $\theta = (2n+1)\frac{\pi}{4}$ or $2n\pi \pm \frac{2\pi}{3}, n \in Z$

(d) $\theta = \frac{2n\pi}{5}$ or $\theta = n\pi \pm \frac{\pi}{2}, n \in Z$ or $\theta = (2n-1)\pi, n \in Z$

5

RELATIONS BETWEEN SIDES AND ANGLES OF A TRIANGLE

In earlier lesson, we have learnt about trigonometric functions of real numbers, relations between them, drawn the graphs of trigonometric functions, studied the characteristics from their graphs, studied about trigonometric functions of sum and difference of real numbers, and deduced trigonometric functions of multiple and sub-multiples of real numbers.

In this lesson, we shall try to establish some results which will give the relationship between sides and angles of a triangle and will help in finding unknown parts of a triangle.

OBJECTIVES

After studying this lesson, you will be able to :

- derive sine formula, cosine formula and projection formula
- apply these formulae to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Trigonometric functions.
- Formulae for sum and difference of trigonometric functions of real numbers.
- Trigonometric functions of multiples and sub-multiples of real numbers.

5.1 SINE FORMULA

In a ΔABC, the angles corresponding to the vertices A, B, and C are denoted by A, B, and C and the sides opposite to these vertices are denoted by a, b and c respectively. These angles and sides are called six elements of the triangle.

Prove that in any triangle, the lengths of the sides are proportional to the sines of the angles opposite to the sides,

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof : In ΔABC, in Fig. 5.1 [(i), (ii) and (iii)], BC = a, CA = b and AB = c and $\angle C$ is acute angle in (i), right angle in (ii) and obtuse angle in (iii).

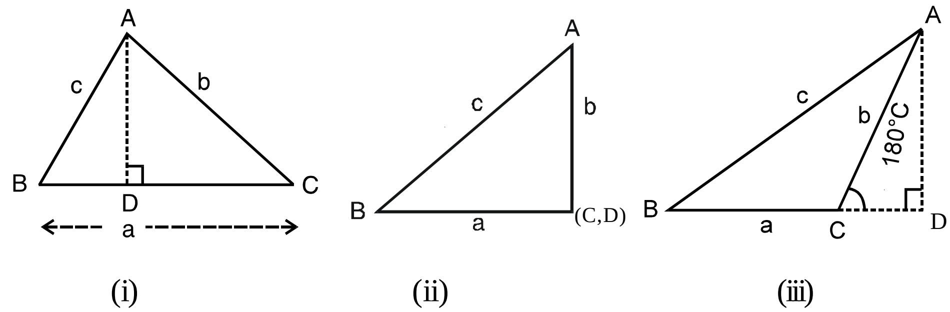


Fig. 5.1

Draw AD prependicular to BC (or BC produced, if need be)

In ΔABC, $\frac{AD}{AB} = \sin B$ or $\frac{AD}{c} = \sin B \Rightarrow AD = c \sin B$ .....(i)

In ΔADC, $\frac{AD}{AC} = \sin C$ in Fig 5.1 (i)

or, $\frac{AD}{b} = \sin C \Rightarrow AD = b \sin C$ ....(ii)

In Fig. 5.1 (ii), $\frac{AD}{AC} = 1 = \sin\frac{\pi}{2} = \sin C$ and $\frac{AD}{AB} = \sin B$

$AD = b \sin C$ and $AD = c \sin B$.

and in Fig. 5.1 (iii), $\frac{AD}{AC} = \sin(\pi - C) = \sin C$ and $\frac{AD}{AB} = \sin B$

or $\frac{AD}{b} = \sin C$ or $AD = b \sin C$ and $AD = c \sin B$

Thus, in all three figures, $AD = b \sin C$ and $AD = c \sin B$ ....(iii)

From (iii) we get

$$c \sin B = b \sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C} \quad ....(iv)$$

Similarly, by drawing prependiculars from C on AB, we can prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad .....(v)$$

From (iv) and (v), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad .....(A)$$

(A) is called the sine-formula

Note : (A) is sometimes written as

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \qquad ....(A')$$

The relations (A) and (A') help us in finding unknown angles and sides, when some others are given.

Let us take some examples :

Example 5.1 Prove that $a\cos\frac{B-C}{2}=(b+c)\sin\frac{A}{2}$, using sine-formula.

Solution : We know that, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$ (say)

$\Rightarrow$ $a=k\sin A,\ b=k\sin B,\ c=k\sin C$

$\therefore$ $$\text{R.H.S.}=k(\sin B+\sin C)\cdot\sin\frac{A}{2}$$

$$=k\cdot 2\sin\frac{B+C}{2}\cdot\cos\frac{B-C}{2}\cdot\sin\frac{A}{2}$$

Now $$\frac{B+C}{2}=90^\circ-\frac{A}{2} \qquad (\because\ A+B+C=\pi)$$

$\therefore$ $$\sin\frac{B+C}{2}=\cos\frac{A}{2}$$

$\therefore$ $$\text{R.H.S.}=2k\cos\frac{A}{2}\cdot\cos\frac{B-C}{2}\cdot\sin\frac{A}{2}$$

$$=k\cdot\sin A\cdot\cos\frac{B-C}{2}=a\cdot\cos\frac{B-C}{2}=\text{L.H.S}$$

Example 5.2 Using sine formula, prove that

$$a(\cos C-\cos B)=2(b-c)\cos^2\frac{A}{2}$$

Solution : We have $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$ (say)

$\Rightarrow$ $a=k\sin A,\ b=k\sin B,\ c=k\sin C$

$\therefore$ $$\text{R.H.S}=2k(\sin B-\sin C)\cdot\cos^2\frac{A}{2}$$

$$=2k\cdot 2\cos\frac{B+C}{2}\cdot\sin\frac{B-C}{2}\cdot\cos^2\frac{A}{2}$$

$$= 4k \sin\frac{A}{2}\cdot \sin\frac{B-C}{2}\cdot \cos^2\frac{A}{2} = 2a\sin\frac{B-C}{2}\cdot \cos\frac{A}{2}$$

$$= 2a\sin\frac{B+C}{2}\cdot \sin\frac{B-C}{2} = a(\cos C - \cos B) = \text{L.H.S.}$$

Example 5.3 In any triangle ABC, show that

$$a\sin A - b\sin B = c\sin(A-B)$$

Solution : We have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$$\text{L.H.S.} = k\sin A\cdot\sin A - k\sin B\cdot\sin B = k\left[\sin^2 A - \sin^2 B\right]$$

$$= k\sin(A+B)\cdot\sin(A-B)$$

$$A+B = \pi - C \Rightarrow \sin(A+B) = \sin C$$

$$\therefore \quad \text{L.H.S.} = k\sin C\cdot\sin(A-B) = c\sin(A-B) = \text{R.H.S.}$$

Example 5.4 In any triangle, show that

$$a(b\cos C - c\cos B) = b^2 - c^2$$

Solution : We have, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$$\text{L.H.S.} = k\sin A(k\sin B\cos C - k\sin C\cos B) = k^2\cdot\sin A[\sin(B-C)]$$

$$= k^2\cdot\sin(B+C)\cdot\sin(B-C) \qquad [\because \sin A = \sin(B+C)]$$

$$= k^2(\sin^2 B - \sin^2 C) = k^2\sin^2 B - k^2\sin^2 C = b^2 - c^2 = \text{R.H.S}$$

CHECK YOUR PROGRESS 5.1

1. Using sine-formula, show that each of the following hold :

(i) $\dfrac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \dfrac{a-b}{a+b}$ (ii) $b\cos B + c\cos C = a\cos(B-C)$

(iii) $a\sin\frac{B-C}{2} = (b-c)\cos\frac{A}{2}$ (iv) $\frac{b+c}{b-c} = \tan\frac{B+C}{2}\cdot\cot\frac{B-C}{2}$

(v) $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

2. In any triangle if $\frac{a}{\cos A} = \frac{b}{\cos B}$, prove that the tiangle is isosceles.

5.2 COSINE FORMULA

In any triangle, prove that

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ (ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$ (iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Proof :

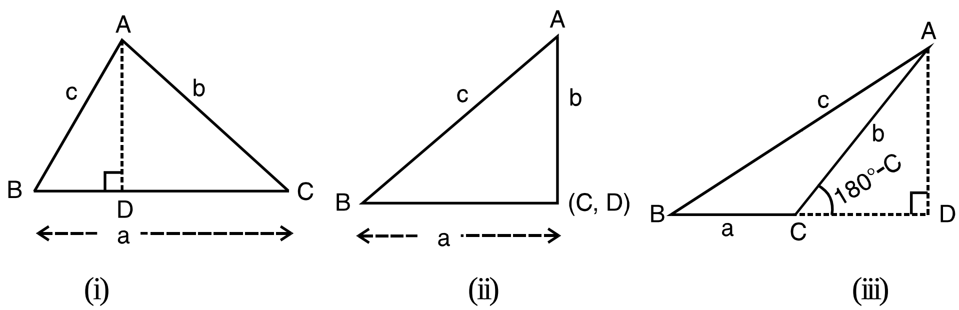


Fig. 5.2

Three cases arise :

(i) When $\angle C$ is acute (ii) When $\angle C$ is a right angle

(iii) When $\angle C$ is obtuse

Let us consider these one by one :

Case (i) When $\angle C$ is acute, $\frac{AD}{AC} = \sin C \quad \Rightarrow \quad AD = b \sin C$

Also $BD = BC - DC = a - b\cos C \quad \left[\because \frac{DC}{b} = \cos C\right]$

From Fig. 5.2 (i) $c^2 = (b\sin C)^2 + (a - b\cos C)^2$

$$= b^2 \sin^2 C + a^2 + b^2 \cos^2 C - 2ab\cos C = a^2 + b^2 - 2\,ab\,\cos C$$

$$\Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}$$

Case (ii) When $\angle C = 90°$, $c^2 = AD^2 + BD^2 = b^2 + a^2$

As $C = 90° \Rightarrow \cos C = 0 \;\therefore\; c^2 = b^2 + a^2 - 2ab \cdot \cos C$

$$\Rightarrow \quad \cos C = \frac{b^2 + a^2 - c^2}{2ab}$$

Case (iii) When $\angle C$ is obtuse

$$\frac{AD}{AC} = \sin(180° - C) = \sin C$$

$\therefore$ $AD = b \sin C$

Also, $BD = BC + CD = a + b\cos(180° - C)$

$= a - b\cos C \quad \therefore c^2 = (b\sin C)^2 + (a - b\cos C)^2$

$= a^2 + b^2 - 2ab\cos C \Rightarrow \qquad \cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\therefore$ In all the three cases, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Similarly, it can be proved that $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$ and $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Let us take some examples to show its application.

Example 5.5 In any triangle ABC, show that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

Solution : We know that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \ \cos B = \frac{c^2 + a^2 - b^2}{2ac}, \ \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \quad \text{L.H.S.} = \frac{b^2 + c^2 - a^2}{2\,abc} + \frac{c^2 + a^2 - b^2}{2\,abc} + \frac{a^2 + b^2 - c^2}{2\,abc}$$

$$= \frac{1}{2\,abc}\left[b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2\right]$$

$$= \frac{a^2 + b^2 + c^2}{2\,abc} = \text{R.H.S.}$$

Example 5.6 If $\angle A = 60°$, show that in ΔABC

$$(a + b + c)(b + c - a) = 3bc$$

Solution : $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ .....(i) Here $A = 60° \Rightarrow \cos A = \cos 60° = \dfrac{1}{2}$

$\therefore$ (i) becomes $\frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc}$ $\Rightarrow$ $b^2 + c^2 - a^2 = bc$

or $b^2 + c^2 + 2bc - a^2 = 3bc$ or $(b + c)^2 - a^2 = 3bc$

or $(b + c + a)(b + c - a) = 3bc$

Example 5.7 If the sides of a triangle are 3 cm, 5 cm and 7 cm find the greatest angle of the triangle.

Solution : Here a = 3 cm, b = 5 cm, c = 7 cm

We know that in a triangle, the angle opposite to the largest side is greatest

$\therefore$ $\angle C$ is the greatest angle. $\therefore$ $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$$= \frac{9 + 25 - 49}{30} = \frac{-15}{30} = \frac{-1}{2}$$

$\therefore$ $\cos C = \frac{-1}{2}$ $\Rightarrow$ $C = \frac{2\pi}{3}$

$\therefore$ The greatest angle of the triangle is $\frac{2\pi}{3}$ or $120°$.

Example 5.8 In ΔABC, if $\angle A = 60°$, prove that $\frac{b}{c + a} + \frac{c}{a + b} = 1$.

Solution : $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $\cos 60° = \frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc}$

$\therefore$ $b^2 + c^2 - a^2 = bc$ or $b^2 + c^2 = a^2 + bc$ .....(i)

$$\text{L.H.S.} = \frac{b}{c + a} + \frac{c}{a + b} = \frac{ab + b^2 + c^2 + ac}{(c + a)(a + b)}$$

$$= \frac{ab + ac + a^2 + bc}{(c + a)(a + b)} \quad \text{[Using (i)]}$$

$$= \frac{ab + a^2 + ac + bc}{(c + a)(a + b)} = \frac{a(a + b) + c(a + b)}{(a + c)(a + b)} = \frac{(a + c)(a + b)}{(a + c)(a + b)} = 1$$

CHECK YOUR PROGRESS 5.2

1. In any triangle ABC, show that

(i) $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$

(ii) $(a^2-b^2+c^2)\tan B=(b^2-c^2+a^2)\tan C=(c^2-a^2+b^2)\tan A$

(iii) $\frac{k}{2}(\sin 2A+\sin 2B+\sin 2C)=\frac{a^2+b^2+c^2}{2abc}$

where $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

(iv) $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$

2. The sides of a triangle are $a = 9$ cm, $b = 8$ cm, $c = 4$ cm. Show that

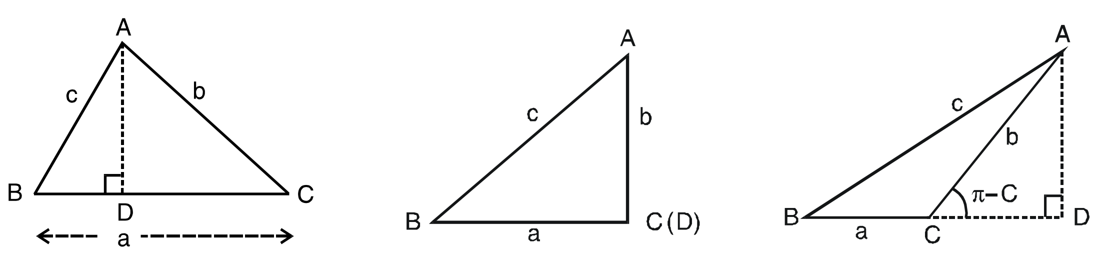
$$6\cos C = 4 + 3\cos B.$$

5.3 PROJECTION FORMULA

In ΔABC, if $BC = a$, $CA = b$ and $AB = c$, then prove that

(i) $a = b\cos C + c\cos B$ (ii) $b = c\cos A + a\cos C$ (iii) $c = a\cos B + b\cos A$

Proof :



(i) (ii) (iii)

Fig. 5.3

As in previous result, three cases arise. We will discuss them one by one.

(i) When $\angle C$ is acute :

In Δ ADB, $\frac{BD}{c}=\cos B$ $\Rightarrow$ $BD = c\cos B$

In Δ ADC, $\frac{DC}{b}=\cos C$ $\Rightarrow$ $DC = b\cos C$

$a = BD + DC = c\cos B + b\cos C$, $\therefore$ $a = c\cos B + b\cos c$

(ii) When $\angle C = 90°$

$$a = BC = \frac{BC}{AB} \cdot AB = \cos B \cdot c \; = c\cos B + 0$$

$$= c\cos B + b\cos 90° \quad (\because \cos 90° = 0) \; = c\cos B + b\cos C$$

(iii) When $\angle C$ is obtuse

In $\Delta\, ADB$, $\dfrac{BD}{c} = \cos B \;\Rightarrow\; BD = c\cos B$

In $\Delta\, ADC$, $\dfrac{CD}{b} = \cos(\pi - C) = -\cos C \quad\Rightarrow\quad CD = -b\cos C$

In Fig. 5.3 (iii),

$$BC = BD - CD \Rightarrow \; a = c\cos B - (-b\cos C) \; = c\cos B + b\cos C$$

Thus in all cases, $a = b\cos C + c\cos B$

Similarly, we can prove that

$$b = c\cos A + a\cos C \quad \text{and} \quad c = a\cos B + b\cos A$$

Let us take some examples, to show the application of these results.

Example 5.9 In any triangle ABC, show that

$$(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a + b + c$$

Solution : L.H.S. $= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$

$$= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (c\cos B + b\cos C)$$

$$= c + b + a \; = a + b + c = \text{R.H.S.}$$

Example 5.10 In any $\Delta\, ABC$, prove that

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Solution : L.H.S. $= \dfrac{1 - 2\sin^2 A}{a^2} - \dfrac{1 - 2\sin^2 B}{b^2}$

$$= \frac{1}{a^2} - \frac{2\sin^2 A}{a^2} - \frac{1}{b^2} + \frac{2\sin^2 B}{b^2}$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2k^2 + 2k^2 = \frac{1}{a^2} - \frac{1}{b^2} \qquad \left(\therefore \; \frac{\sin A}{a} = \frac{\sin B}{b} = k\right)$$

= R. H. S.

Example 5.11 In $\Delta\, ABC$, if $a\cos A = b\cos B$, where $a \neq b$ prove that $\Delta\, ABC$ is a right angled triangle.

Solution : $a \cos A = b \cos B$, $\therefore$ $$a\left[\frac{b^2+c^2-a^2}{2bc}\right]=b\left[\frac{a^2+c^2-b^2}{2ac}\right]$$

or $\quad a^2\left(b^2+c^2-a^2\right)=b^2\left(a^2+c^2-b^2\right)$

or $\quad a^2b^2+a^2c^2-a^4=a^2b^2+b^2c^2-b^4$

or $\quad c^2\left(a^2-b^2\right)=\left(a^2-b^2\right)\left(a^2+b^2\right)$

$\Rightarrow \quad c^2=a^2+b^2 \quad \therefore \Delta\, ABC$ is a right triangle.

Example 5.12 If a = 2, b = 3, c = 4, find cos A, cos B and cos C.

Solution :
$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{9+16-4}{2\times3\times4}=\frac{21}{24}=\frac{7}{8}$$

$$\cos B=\frac{c^2+a^2-b^2}{2ac}=\frac{16+4-9}{2\times4\times2}=\frac{11}{16}$$

and
$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{4+9-16}{2\times2\times3}=\frac{-3}{12}=\frac{-1}{4}$$

CHECK YOUR PROGRESS 5.3

1. If a = 3, b = 4 and c = 5, find cos A, cos B and cos C.
2. The sides of a triangle are 7 cm, $4\sqrt{3}$ cm and $\sqrt{13}$ cm. Find the smallest angle of the triangle.
3. If a : b : c = 7 : 8 : 9, prove that
   cos A : cos B : cos C = 14 : 11 : 6.
4. If the sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Show that the greatest angle of the triangle is 120°.
5. In a triangle, b cos A = a cos B, prove that the triangle is isosceles.
6. Deduce sine formula from the projection formula.

LET US SUM UP

It is possible to find out the unknown elements of a triangle, if the relevent elements are given by using

Sine-formula :

(i) $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Cosine foumulae :

(ii) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Projection formulae :

$a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$, $c = a\cos B + b\cos A$

SUPPORTIVE WEB SITES

www.mathopenref.com/trianglesideangle.html
http://en.wikipedia.org/wiki/Solution_of_triangles
www.themathpage.com/abookI/propI-18-19.htm

TERMINAL EXERCISE

In a triangle ABC, prove the following (1-10) :

1. $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0$

2. $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

3. $\dfrac{b^2 - c^2}{a^2} \cdot \sin 2A + \dfrac{c^2 - a^2}{b^2} \cdot \sin 2B + \dfrac{a^2 - b^2}{c^2} \cdot \sin 2C = 0$

4. $\dfrac{c^2 + a^2}{b^2 + c^2} = \dfrac{1 + \cos B\cos(C - A)}{1 + \cos A\cos(B - C)}$

5. $\dfrac{c - b\cos A}{b - c\cos A} = \dfrac{\cos B}{\cos C}$

6. $\dfrac{a - b\cos C}{c - b\cos A} = \dfrac{\sin C}{\sin A}$

7. $(a + b + c)\left[\tan\dfrac{A}{2} + \tan\dfrac{B}{2}\right] = 2c\cot\dfrac{C}{2}$

8. $\sin\dfrac{A - B}{2} = \dfrac{a - b}{c}\cos\dfrac{C}{2}$

9. (i) $b\cos B + c\cos C = a\cos(B - C)$ (ii) $a\cos A + b\cos B = c\cos(A - B)$

10. $b^2 = (c - a)^2\cos^2\dfrac{B}{2} + (c + a)^2\sin^2\dfrac{B}{2}$

11. In a triangle, if $b = 5$, $c = 6$, $\tan\dfrac{A}{2} = \dfrac{1}{\sqrt{2}}$, then show that $a = \sqrt{41}$.

12. In any ΔABC, show that $\dfrac{\cos A}{\cos B} = \dfrac{b - a\cos C}{a - b\cos C}$

ANSWERS

CHECK YOUR PROGRESS 5.3

1. $\cos A = \frac{4}{5}$

   $\cos B = \frac{3}{5}$

   $\cos C = \text{zero}$

2. The smallest angle of the triangle is 30°.

6

SEQUENCES AND SERIES

Succession of numbers of which one number is designated as the first, other as the second, another as the third and so on gives rise to what is called a sequence. Sequences have wide applications. In this lesson we shall discuss particular types of sequences called arithmetic sequence, geometric sequence and also find arithmetic mean (A.M), geometric mean (G.M) between two given numbers. We will also establish the relation between A.M and G.M.

Let us consider the following problems :

(a) A man places a pair of newly born rabbits into a warren and wants to know how many rabbits he would have over a certain period of time. A pair of rabbits will start producing offsprings two months after they were born and every following month one new pair of rabbits will appear. At the beginning the man will have in his warren only one pair of rabbits, during the second month he will have the same pair of rabbits, during the third month the number of pairs of rabbits in the warren will grow to two; during the fourth month there will be three pairs of rabbits in the warren. Thus, the number of pairs of rabbits in the consecutive months are :

1, 1, 2, 3, 5, 8, 13, ...

(b) The recurring decimal $0.\bar{3}$ can be written as a sum

$$0.\bar{3} = 0.3 + 0.03 + 0.003 + 0.0003 \ldots$$

(c) A man earns Rs.10 on the first day, Rs. 30 on the second day, Rs. 50 on the third day and so on. The day to day earning of the man may be written as $10, 30, 50, 70, 90, \cdots$

We may ask what his earnings will be on the 10^{th} day in a specific month.

Again let us consider the following sequences:

(1) $2, 4, 8, 16, \cdots$ (2) $\frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}, \ldots$

(3) $0.01, 0.0001, 0.000001, \cdots$

In these three sequences, each term except the first, progressess in a definite order but different from the order of other three problems. In this lesson we will discuss those sequences whose term progressess in a definite order.

OBJECTIVES

After studying this lesson, you will be able to :

- describe the concept of a sequence (progression);
- define an A.P. and cite examples;
- find common difference and general term of a A.P;
- find the fourth quantity of an A.P. given any three of the quantities a, d, n and t_n;
- calculate the common difference or any other term of the A.P. given any two terms of the A.P;
- derive the formula for the sum of 'n' terms of an A.P;
- calculate the fourth quantity of an A.P. given three of S, n, a and d;
- insert A.M. between two numbers;
- solve problems of daily life using concept of an A.P;
- state that a geometric progression is a sequence increasing or decreasing by a definite multiple of a non-zero number other than one;
- identify G.P.'s from a given set of progessions;
- find the common ratio and general term of a G.P;
- calculate the fourth quantity of a G.P when any three of the quantities t_n, a, r and n are given;
- calculate the common ratio and any term when two of the terms of the G.P. are given;
- write progression when the general term is given;
- derive the formula for sum of n terms of a G.P;
- calculate the fourth quantity of a G.P. if any three of a, r, n and S are given;
- derive the formula for sum (S_∞) of infinite number of terms of a G.P. when $|r| < 1$;
- find the third quantity when any two of S_∞, a and r are given;
- convert recurring decimals to fractions using G.P;
- insert G.M. between two numbers; and
- establish relationship between A.M. and G.M.

EXPECTED BACKGROUND KNOWLEDGE

- Laws of indices
- Simultaneous equations with two unknowns.
- Quadratic Equations.

6.1 SEQUENCE

A sequence is a collection of numbers specified in a definite order by some assigned law, whereby a definite number a_n of the set can be associated with the corresponding positive integer n. The different notations used for a sequence are.

1. $a_1, a_2, a_3, ..., a_n, ...$ 2. a_n , $n = 1, 2, 3, ...$ 3. $\{a_n\}$

Let us consider the following sequences :

1. 1, 2, 4, 8, 16, 32, ... 2. 1, 4, 9, 16, 25, ...

3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$ 4. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$

In the above examples, the expression for n^{th} term of the sequences are as given below :

(1) $a_n = 2^{n-1}$ (2) $a_n = n^2$ (3) $a_n = \frac{n}{n+1}$ (4) $a_n = \frac{1}{n}$

for all positive integer n.

Also for the first problem in the introduction, the terms can be obtained from the relation

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-2} + a_{n-1},\ n \geq 3$$

A finite sequence has a finite number of terms. An infinite sequence contains an infinite number of terms.

6.2 ARITHMETIC PROGRESSION

Let us consider the following examples of sequence, of numbers :

(1) $2, 4, 6, 8, \cdots$ (2) $1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$

(3) $10, 8, 6, 4, \cdots$ (4) $-\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{2}, \cdots$

Note that in the above four sequences of numbers, the first terms are respectively 2, 1, 10, and $-\frac{1}{2}$. The first term has an important role in this lesson. Also every following term of the sequence has certain relation with the first term. What is the relation of the terms with the first term in Example (1) ? First term = 2, Second term = 4 = 2 + 1 × 2

Third term = 6 = 2 + 2 × 2

Fourth term = 8 = 2 + 3 × 2 and so on.

The consecutive terms in the above sequence are obtained by adding 2 to its preceding term. i.e., the difference between any two consecutive terms is the same.

A finite sequence of numbers with this property is called an arithmetic progression.
A sequence of numbers with finite terms in which the difference between two consecutive terms is the same non-zero number is called the Arithmetic Progression or simply A. P.

The difference between two consecutive terms is called the common defference of the A. P. and is denoted by '*d*'.

In general, an A. P. whose first term is a and common difference is d is written as $a, a + d, a + 2d, a + 3d, \cdots$

Also we use t_n to denote the nth term of the progression.

6.2.1 GENERAL TERM OF AN A. P.

Let us consider A. P. $a, a + d, a + 2d, a + 3d, \cdots$

Here, first term $(t_1) = a$

second term $(t_2) = a + d = a + (2 - 1)\, d,$

third term $(t_3) = a + 2d = a + (3 - 1)\, d$

By observing the above pattern, n[th] term can be written as: $t_n = a + (n - 1)\, d$

Hence, if the first term and the common difference of an A. P. are known then any term of A. P. can be determined by the above formula.

Note.:

(i) If the same non-zero number is added to each term of an A. P. the resulting sequence is again an A. P.

(ii) If each term of an A. P. is multiplied by the same non-zero number, the resulting sequence is again an A. P.

Example 6.1 Find the 10[th] term of the A. P.: 2, 4, 6, ...

Solution : Here the first term $(a) = 2$ and common difference $d = 4 - 2 = 2$

Using the formula $t_n = a + (n - 1)\, d$, we have

$$t_{10} = 2 + (10 - 1)\, 2 = 2 + 18 = 20$$

Hence, the 10th term of the given A. P. is 20.

Example 6.2 The 10[th] term of an A. P. is – 15 and 31[st] term is –57, find the 15[th] term.

Solution : Let a be the first term and d be the common difference of the A. P. Then from the formula: $t_n = a + (n - 1)\, d$, we have

$$t_{10} = a + (10 - 1)\, d = a + 9d \text{ and } t_{31} = a + (31 - 1)\, d = a + 30\, d$$

We have, $a + 9d = -15$...(1), $a + 30d = -57$ ...(2)

Solve equations (1) and (2) to get the values of a and d.

Subtracting (1) from (2), we have

$$21d = -57 + 15 = -42 \quad \therefore\ d = \frac{-42}{21} = -2$$

Again from (1), $a = -15 - 9d = -15 - 9(-2) = -15 + 18 = 3$

Now $t_{15} = a + (15 - 1)d = 3 + 14(-2) = -25$

Example 6.3 Which term of the A. P.: 5, 11, 17, ... is 119 ?

Solution : Here $a = 5$, $d = 11 - 5 = 6$

$$t_n = 119$$

We know that $t_n = a + (n - 1)\, d$

$$\Rightarrow \quad 119 = 5 + (n-1) \times 6 \quad \Rightarrow \quad (n-1) = \frac{119 - 5}{6} = 19$$

$$\therefore \quad n = 20$$

Therefore, 119 is the 20th term of the given A. P.

Example 6.4 Is 600 a term of the A. P.: 2, 9, 16, ...?

Solution : Here, $a = 2$, and $d = 9 - 2 = 7$.

Let 600 be the n^{th} term of the A. P. We have $t_n = 2 + (n - 1)\, 7$

According to the question,

$$2 + (n-1)\,7 = 600 \qquad \therefore\ (n-1)\,7 = 598$$

or $$n = \frac{598}{7} + 1 \qquad \therefore\ n = 86\frac{3}{7}$$

Since n is a fraction, it cannot be a term of the given A. P. Hence, 600 is not a term of the given A. P.

Example 6.5 If $a + b + c = 0$ and $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are. in A. P., then prove that

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are also in A. P.}$$

Solution. : Since $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A. P., therefore

$$\frac{b}{c+a}-\frac{a}{b+c}=\frac{c}{a+b}-\frac{b}{c+a}$$

or, $$\left(\frac{b}{c+a}+1\right)-\left(\frac{a}{b+c}+1\right)=\left(\frac{c}{a+b}+1\right)-\left(\frac{b}{c+a}+1\right)$$

or, $$\frac{a+b+c}{c+a}-\frac{a+b+c}{b+c}=\frac{a+b+c}{a+b}-\frac{a+b+c}{c+a}$$

or, $$\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a} \quad (\text{Since } a+b+c \neq 0)$$

or, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A. P.

CHECK YOUR PROGRESS 6.1

1. Find the n^{th} term of each of the following A. P's. :

   (a) 1, 3, 5, 7, $\cdots$ (b) 3, 5, 7, 9, $\cdots$

2. If $t_n = 2n + 1$, then find the A. P.

3. Which term of the A. P. $2\frac{1}{2}, 4, 5\frac{1}{2},$ ..... is 31? Find also the 10^{th} term?

4. Is – 292 a term of the A. P. 7, 4, 1, –2, ...?

5. The m^{th} term of an A. P. is n and the n^{th} term is m. Show that its $(m + n)^{\text{th}}$ term is zero.

6. Three numbers are in A. P. The difference between the first and the last is 8 and the product of these two is 20. Find the numbers.

7. The n^{th} term of a sequence is $na + b$. Prove that the sequence is an A. P. with common difference a.

6.3 TO FIND THE SUM OF FIRST *n* TERMS IN AN A. P.

Let a be the first term and d be the common difference of an A. P. Let l denote the last term, i.e., the n^{th} term of the A. P. Then, $l = t_n = a + (n-1)d$ ... (i)

Let S_n denote the sum of the first n terms of the A. P. Then

$$S_n = a + (a + d) + (a + 2d) + ... + (l - 2d) + (l - d) + l \qquad ... \text{(ii)}$$

Reversing the order of terms in the R. H. S. of the above equation, we have

$$S_n = l + (l - d) + (l - 2d) + ... + (a + 2d) + (a + d) + a \qquad ... \text{(iii)}$$

Adding (ii) and (iii) vertically, we get

$$2S_n = (a + l) + (a + l) + (a + l) + \ldots \text{ containing } n \text{ terms} = n\,(a + l)$$

i.e., $S_n = \dfrac{n}{2}(a + l)$

Also $S_n = \dfrac{n}{2}[2a + (n-1)\,d]$ [From (i)]

It is obvious that $t_n = S_n - S_{n-1}$

Example 6.6 Find the sum of $2 + 4 + 6 + \ldots\, n$ terms.

Solution.: Here $a = 2$, $d = 4 - 2 = 2$

Using the formula $S_n = \dfrac{n}{2}[2a + (n-1)\,d]$, we get

$$S_n = \frac{n}{2}[2 \times 2 + (n-1)\,2] = \frac{n}{2}[2 + 2n] = \frac{2n\,(n+1)}{2} = n\,(n + 1)$$

Example 6.7 The 35[th] term of an A. P. is 69. Find the sum of its 69 terms.

Solution. Let a be the first term and d be the common difference of the A. P.

We have $t_{35} = a + (35 - 1)\,d = a + 34\,d$.

$\therefore \quad a + 34\,d = 69$ ... (i)

Now by the formula, $S_n = \dfrac{n}{2}[2a + (n-1)\,d]$

We have $S_{69} = \dfrac{69}{2}[2a + (69-1)\,d]$

$= 69\,(a + 34d)$ [using (i)]

$= 69 \times 69 = 4761$

Example 6.8 The first term of an A. P. is 10, the last term is 50. If the sum of all the terms is 480, find the common difference and the number of terms.

Solution : We have: $a = 10$, $l = t_n = 50$, $S_n = 480$.

By substituting the values of a, t_n and S_n in the formulae

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \text{ and } t_n = a + (n-1)\,d, \text{ we get}$$

$$480 = \frac{n}{2}\left[20 + (n-1)\,d\right] \qquad \cdots \text{(i)}$$

$$50 = 10 + (n-1)\,d \qquad \cdots \text{(ii)}$$

From (ii), $(n-1)\,d = 50 - 10 = 40$ $\qquad \cdots$ (iii)

From (i), we have $480 = \frac{n}{2}(20+40)$ $\qquad$ using (i)

or, $\quad 60n = 2 \times 480 \quad \therefore \quad n = \frac{2 \times 480}{60} = 16$

From (iii),

$$\therefore \quad d = \frac{40}{15} = \frac{8}{3} \qquad (\text{as } n-1 = 16-1 = 15)$$

Example 6. 9 Let the n^{th} term and the sum of n terms of an A. P. be p and q respectively. Prove that its first term is $\left(\frac{2q - pn}{n}\right)$.

Solution: In this case, $t_n = p$ and $S_n = q$

Let a be the first term of the A. P.

Now, $S_n = \frac{n}{2}(a + t_n)$ $\quad$ or, $\quad \frac{n}{2}(a+p) = q$

or, $\quad a + p = \frac{2q}{n}$ $\quad$ or, $\quad a = \frac{2q}{n} - p$ $\quad \therefore \quad a = \frac{2q - pn}{n}$

CHECK YOUR PROGRESS 6.2

1. Find the sum of the following A. P's.

   (a) 8, 11, 14, 17, $\cdots$ up to 15 terms $\qquad$ (b) 8, 3, –2, –7, –12, $\cdots$ up to n terms.

2. How many terms of the A. P.: 27, 23, 19, 15, ... have a sum 95?

3. A man takes an interest-free loan of Rs. 1740 from his friend agreeing to repay in monthly instalments. He gives Rs. 200 in the first month and diminishes his monthly instalments by Rs. 10 each month. How many months will it take to repay the loan?

4. How many terms of the progression 3, 6, 9, 12, $\cdots$

   must be taken at the least to have a sum not less than 2000?

5. In a children potato race, n potatoes are placed 1 metre apart in a straight line. A competitor starts from a point in the line which is 5 metre from the nearest potato. Find an expression for the total distance run in collecting the potatoes, one at a time and bringing them back one at a time to the starting point. Calculate the value of n if the total distance run is 162 metres.

6. If the sum of first n terms of a sequence be $an^2 + bn$, prove that the sequence is an A. P. and find its common difference ?

6.4 ARITHMETIC MEAN (A. M.)

When three numbers a, A and b are in A. P., then A is called the arithmetic mean of numbers a and b. We have, $A - a = b - A$

or, $$A = \frac{a+b}{2}$$

Thus, the required A. M. of two numbers a and b is $\frac{a+b}{2}$. Consider the following A. P :

3, 8, 13, 18, 23, 28, 33.

There are five terms between the first term 3 and the last term 33. These terms are called *arithmetic means* between 3 and 33. Consider another A. P. : 3, 13, 23, 33. In this case there are two arithmetic means 13, and 23 between 3 and 33.

Generally any number of arithmetic means can be inserted between any two numbers a and b. Let $A_1, A_2, A_3, \ldots, A_n$ be n arithmetic means between a and b, then.

$a, A_1, A_2, A_3 \ldots, A_n, b$ is an A. P.

Let d be the common difference of this A. P. Clearly it contains $(n + 2)$ terms

$\therefore \quad b = (n + 2)^{th}$ term

$= a + (n + 1)\, d$

$$\therefore \quad d = \frac{b-a}{n+1}$$

Now, $A_1 = a + d \Rightarrow A_1 = \left(a + \frac{b-a}{n+1}\right)$ ...(i)

$$A_2 = a + 2d \Rightarrow A_2 = \left[a + \frac{2(b-a)}{n+1}\right] \qquad \text{...(ii)}$$

$$\vdots$$

$$A_n = a + nd \Rightarrow An\left[a + \frac{n(b-a)}{n+1}\right] \qquad \text{... }(n)$$

These are required n arithmetic means between a and b.

Adding (i), (ii), ..., (n), we get

$$A_1 + A_2 + \dots + A_n = na + \dots + \frac{b-a}{n+1}[1+2+\dots n]$$

$$= na + \left(\frac{b-a}{n+1}\right)\left(\frac{n(n+1)}{2}\right) = na + \frac{n(b-a)}{2} = \frac{n(a+b)}{2}$$

$= n$ [Single A. M. between a and b]

Example 6.10 Insert five arithmetic means between 8 and 26.

Solution : Let A_1, A_2, A_3, A_4 and A_5 be five arithmetic means between 8 and 26.

Therefore, 8, A_1, A_2, A_3, A_4, A_5, 26 are in A. P. with $a = 8$, $b = 26$, $n = 7$

We have $26 = 8 + (7 - 1)\, d \quad \therefore \quad d = 3$

$\therefore \quad A_1 = a + d = 8 + 3 = 11$, $A_2 = a + 2d = 8 + 2 \times 3 = 14$

$A_3 = a + 3d = 17$, $A_4 = a + 4d = 20$, $A_5 = a + 5d = 23$

Hence, the five arithmetic means between 8 and 26 are 11, 14, 17, 20 and 23.

Example 6.11 The 'n', A. M's between 20 and 80 are such that the ratio of the first mean and the last mean is 1 : 3. Find the value of n.

Solution : Here, 80 is the $(n+2)^{th}$ term of the A. P., whose first term is 20. Let d be the common difference.

$\therefore \quad 80 = 20 + (n+2-1)\, d \quad$ or, $\quad 80 - 20 = (n+1)\, d \quad$ or, $\quad d = \dfrac{60}{n+1}$

The first A. M. $= 20 + \dfrac{60}{n+1} = \dfrac{20n + 20 + 60}{n+1} = \dfrac{20n + 80}{n+1}$

The last A. M. $= 20 + n \times \dfrac{60}{n+1} = \dfrac{80n + 20}{n+1}$

We have $\frac{20n+80}{n+1} : \frac{80n+20}{n+1} = 1:3$ or, $\frac{n+4}{4n+1} = \frac{1}{3}$

or, $4n + 1 = 3n + 12$ or, $n = 11$

$\therefore$ The number of A. M's between 20 and 80 is 11.

CHECK YOUR PROGRESS 6.3

1. Prove that if the number of terms of an A. P. is odd then the middle term is the A. M. between the first and last terms.

2. Between 7 and 85, m number of arithmetic means are inserted so that the ratio of $(m-3)^{th}$ and m^{th} means is 11 : 24. Find the value of m.

3. Prove that the sum of n arithmetic means between two numbers is n times the single A. M. between them.

4. If the A. M. between p^{th} and q^{th} terms of an A. P., be equal and to the A. M. between r^{th} and s^{th} terms of the A. P., then show that $p + q = r + s$.

6.5 GEOMETRIC PROGRESSION

Let us consider the following sequence of numbers :

(1) $1, 2, 4, 8, 16, \cdots$ (2) $3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \cdots$

(3) $1, -3, 9, -27, \cdots$ (4) $x, x^2, x^3, x^4, \cdots$

If we see the patterns of the terms of every sequence in the above examples each term is related to the leading term by a definite rule.

For Example (1), the first term is 1, the second term is twice the first term, the third term is 2^2 times of the leading term.

Again for Example (2), the first term is 3, the second term is $\frac{1}{3}$ times of the first term, third term is $\frac{1}{3^2}$ times of the first term.

A sequence with this property is called a *gemetric progression.*

A sequence of numbers in which the ratio of any term to the term which immediately precedes is the same non zero number (other than1), is called a geometric progression or simply G. P. This ratio is called the common ratio.

Thus, $\frac{Second\ term}{First\ term} = \frac{Third\ term}{Second\ term} =$ ..... is called the common ratio of the geometric progression.

Examples (1) to (4) are geometric progressions with the first term 1, 3, 1, x and with common ratio $2, \frac{1}{3}, -3$, and x respectively.

The most general form of a G. P. with the first term a and common ratio r is $a, ar, ar^2, ar^3, \ldots$

6.5.1 GENERAL TERM

Let us consider a geometric progression with the first term a and common ratio r. Then its terms are given by $a, ar, ar^2, ar^3, \ldots$

In this case, $t_1 = a = ar^{1-1}$ $\quad t_2 = ar = ar^{2-1}$

$t_3 = ar^2 = ar^{3-1}$ $\quad t_4 = ar^3 = ar^{4-1}$

... ...

On generalisation, we get the expression for the n^{th} term as $t_n = ar^{n-1}$ ... (A)

6.5.2 SOME PROPERTIES OF G. P.

(i) If all the terms of a G. P. are multiplied by the same non-zero quantity, the resulting series is also in G. P. The resulting G. P. has the same common ratio as the original one.

If $a, b, c, d, \ldots$ are in G. P.

then $ak, bk, ck, dk \ldots$ are also in G. P. $\quad (k \neq 0)$

(ii) If all the terms of a G. P. are raised to the same power, the resulting series is also in G. P.

Let $a, b, c, d \ldots$ are in G. P.

the $a^k, b^k, c^k, d^k, \ldots$ are also in G. P. $\quad (k \neq 0)$

The common ratio of the resulting G. P. will be obtained by raising the same power to the original common ratio.

Example 6.12 Find the 6^{th} term of the G. P.: 4, 8, 16, ...

Solution : In this case the first term (a) = 4 Common ratio (r) = 8 ÷ 4 = 2

Now using the formula $t_n = ar^{n-1}$, we get $t_6 = 4 \times 2^{6-1} = 4 \times 32 = 128$

Hence, the 6^{th} term of the G. P. is 128.

Example 6.13 The 4^{th} and the 9^{th} term of a G. P. are 8 and 256 respectively. Find the G. P.

Solution : Let a be the first term and r be the common ratio of the G. P., then

$t_4 = ar^{4-1} = ar^3, \; t_9 = ar^{9-1} = ar^8$

According to the question, $ar^8 = 256$ $\quad \cdots (1)$

and $\quad ar^3 = 8$ $\quad \cdots (2)$

$$\therefore \qquad \frac{ar^8}{ar^3} = \frac{256}{8} \text{ or } \qquad r^5 = 32 = 2^5 \qquad \therefore\ r = 2$$

Again from (2), $a \times 2^3 = 8 \qquad \therefore \qquad a = \frac{8}{8} = 1$

Therefore, the G. P. is 1, 2, 4, 8, 16, ...

Example 6.14 Which term of the G. P.: 5, –10, 20, – 40, ... is 320?

Solution : In this case, $a = 5$; $r = \frac{-10}{5} = -2$.

Suppose that 320 is the n^{th} term of the G. P.

By the formula, $t_n = ar^{n-1}$, we get $t_n = 5.\ (-2)^{n-1}$

$\therefore \quad 5.\ (-2)^{n-1} = 320 \qquad$ (Given)

$\therefore \quad (-2)^{n-1} = 64 = (-2)^6$

$\therefore \quad n - 1 = 6\ \therefore \qquad n = 7$ Hence, 320 is the 7^{th} term of the G. P.

Example 6.15 If $a, b, c,$ and d are in G. P., then show that $(a + b)^2$, $(b + c)^2$, and $(c + d)^2$ are also in G. P.

Solution. Since $a, b, c,$ and d are in G. P., $\therefore \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

$\therefore \quad b^2 = ac,\ c^2 = bd,\ ad = bc \qquad$ ...(1)

Now, $(a + b)^2 (c + d)^2 = \left[(a+b)(c+d)\right]^2 = (ac + bc + ad + bd)^2$

$= (b^2 + c^2 + 2bc)^2 \qquad$ ...[Using (1)]

$= \left[(b+c)^2\right]^2$

$\therefore \quad \frac{(c+d)^2}{(b+c)^2} = \frac{(b+c)^2}{(a+b)^2}$ Thus, $(a + b)^2$, $(b + c)^2$, $(c + d)^2$ are in G. P.

CHECK YOUR PROGRESS 6.4

1. The first term and the common ratio of a G. P. are respectively 3 and $-\frac{1}{2}$. Write down the first five terms.

2. Which term of the G. P. 1, 2, 4, 8, 16, ... is 1024? Is 520 a term of the G. P.?
3. Three numbers are in G. P. Their sum is 43 and their product is 216. Find the numbers in proper order.
4. The n^{th} term of a G. P. is 2×3^n for all n. Find (*a*) the first term (*b*) the common ratio of the G. P.

6.6 SUM OF *n* TERMS OF A G. P.

Let a denote the first term and r the common ratio of a G. P. Let S_n represent the sum of first n terms of the G. P. Thus, $S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1}$ ... (1)

Multiplying (1) by r, we get $r\,S_n = ar + ar^2 + ... + ar^{n-2} + ar^{n-1} + ar^n$ ... (2)

$(1) - (2) \Rightarrow S_n - rS_n = a - ar^n$ or $S_n(1-r) = a(1-r^n)$

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r} \qquad ....(A)$$

$$= \frac{a(r^n-1)}{r-1} \qquad ....(B)$$

Either (A) or (B) gives the sum up to the n^{th} term when $r \neq 1$. It is convenient to use formula (A) when $|r| < 1$ and (B) when $|r| > 1$.

Example 6.16 Find the sum of the G. P.: 1, 3, 9, 27, ... up to the 10^{th} term.

Solution : Here the first term (a) = 1 and the common ratio $(r) = \frac{3}{1} = 3$

Now using the formula, $S_n = \frac{a(r^n-1)}{r-1}$, $(\because r > 1)$ we get $S_{10} = \frac{1.(3^{10}-1)}{3-1} = \frac{3^{10}-1}{2}$

Example 6.17 Find the sum of the G. P.: $\frac{1}{\sqrt{3}}, 1, \sqrt{3},\ , \ldots, 81$

Solution : Here, $a = \frac{1}{\sqrt{3}}; r = \sqrt{3}$ and $t_n = l = 81$

Now $t_n = 81 = \frac{1}{\sqrt{3}}(\sqrt{3})^{n-1} = (\sqrt{3})^{n-2}$

$$\therefore \quad (\sqrt{3})^{n-2} = 3^4 = (\sqrt{3})^8 \quad \therefore n - 2 = 8 \text{ or } n = 10$$

$$\therefore \quad S_n = \frac{\frac{1}{\sqrt{3}}\left[\sqrt{3}^{10} - 1\right]}{\sqrt{3}-1} = \frac{\left(\sqrt{3}\right)^{10} - 1}{3-\sqrt{3}}$$

Example 6.18 Find the sum of the G. P.: 0.6, 0.06, 0.006, 0.0006, $\cdots$ to n terms.

Solution. Here, $a = 0.6 = \frac{6}{10}$ and $r = \frac{0.06}{0.6} = \frac{1}{10}$

Using the formula $S_n = \frac{a(1-r^n)}{1-r}$, we have $\qquad [\because\ r < 1]$

$$S_n = \frac{\frac{6}{10}\left\{1-\left(\frac{1}{10}\right)^n\right\}}{1-\frac{1}{10}} = \frac{6}{9}\left(1-\frac{1}{10^n}\right) = \frac{2}{3}\left(1-\frac{1}{10^n}\right)$$

Hence, the required sum is $\frac{2}{3}\left(1-\frac{1}{10^n}\right)$.

Example 6.19 How many terms of the G. P.: 64, 32, 16, $\cdots$ has the sum $127\frac{1}{2}$?

Solution : Here, $a = 64$, $r = \frac{32}{64} = \frac{1}{2}$ (<1) and $S_n = 127\frac{1}{2} = \frac{255}{2}$.

Using the formula $S_n = \frac{a(1-r^n)}{1-r}$, we get

$$S_n = \frac{64\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} \quad \Rightarrow \quad \frac{64\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = \frac{255}{2} \quad \text{... (given)}$$

or $\quad 128\left[1-\left(\frac{1}{2}\right)^n\right] = \frac{255}{2}$ or $1-\left(\frac{1}{2}\right)^n = \frac{255}{256}$

or $\quad \left(\frac{1}{2}\right)^n = 1-\frac{255}{256} = \frac{1}{256} = \left(\frac{1}{2}\right)^8 \qquad \therefore \qquad n = 8$

Thus, the required number of terms is 8.

Example 6.20 Find the sum of the following sequence :

2, 22, 222, ... to n terms.

Solution : Let S denote the sum. Then

$$S = 2 + 22 + 222 + \cdots \text{ to } n \text{ terms} = 2\,(1 + 11 + 111 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{2}{9}\,(9 + 99 + 999 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{2}{9}\left\{(10-1)+(10^2-1)+(10^3-1)+ \quad ...to\ n\ terms)\right\}$$

$$= \frac{2}{9}\left\{(10-10^2+10^3+ ...to\ n\ terms)-(1+1+1+ \quad ...to\ n\ terms)\right\}$$

$$= \frac{2}{9}\left\{\frac{(10^n-1)}{10-1}-n\right\} \qquad [\because 10-10^2+10^3+\cdots \text{ is a G P with } r = -10<1]$$

$$= \frac{2}{9}\left\{\frac{10^n-1-9n}{9}\right\} = \frac{2}{81}\left(10^n-1-9n\right)$$

Example 6. 21 Find the sum up to n terms of the sequence:

$$0.7,\ 0.77,\ 0.777,\ \cdots$$

Solution : Let S denote the sum, then

$$S = 0.7 + 0.77 + 0.777 + \cdots \text{ to } n \text{ terms}$$

$$= 7(0.1 + 0.11 + 0.111 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\,(0.9 + 0.99 + 0.999 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\,\{(1–0.1) + (1–0.01) + (1 – 0.001) + \cdots \text{ to } n \text{ terms}\}$$

$$= \frac{7}{9}\,\{(1 + 1 + 1 + ... n \text{ terms}) – (0.1 + 0.01 + 0.001 + \cdots \text{ to } n \text{ terms})\}$$

$$= \frac{7}{9}\left\{n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots to\ n\ terms\right)\right\}$$

$$= \frac{7}{9}\left\{ n - \frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}} \right\} \qquad \text{(Since } r < 1\text{)}$$

$$= \frac{7}{9}\left\{ n - \frac{1}{9}\left[1-\frac{1}{10^n}\right] \right\} = \frac{7}{9}\left\{ \frac{9n-1+10^{-n}}{9} \right\} = \frac{7}{81}\left[9n-1+10^{-n}\right]$$

CHECK YOUR PROGRESS 6.5

1. Find the sum of each of the following G. P's :

   (a) 6, 12, 24, ... to 10 terms (b) $1, -\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} -$ ... to 20 terms.

2. How many terms of the G. P. 8, 16, 32, 64, $\cdots$ have their sum 8184 ?

3. Show that the sum of the G. P. $a + b + ... + l$ is $\frac{bl - a^2}{b - a}$

4. Find the sum of each of the following sequences up to n terms.

   (a) 8, 88, 888, ... (b) 0.2, 0.22, 0.222, ...

6.7 INFINITE GEOMETRIC PROGRESSION

So far, we have found the sum of a finite number of terms of a G. P. We will now learn to find out the sum of infinitely many terms of a G P such as. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$

We will proceed as follows: Here $a = 1,\ r = \frac{1}{2}$.

The n^{th} term of the G. P. is $t_n = \frac{1}{2^{n-1}}$ and sum to n terms

i.e., $$S_n = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}} < 2.$$

So, no matter, how large n may be, the sum of n terms is never more than 2.

So, if we take the sum of all the infinitely many terms, we shall not get more than 2 as answer.

Also note that the recurring decimal 0.3 is really 0.3 + 0.03 + 0.003 + 0.0003 + ...

i.e., 0.3 is actually the sum of the above infinite sequence.

On the other hand it is at once obvious that if we sum infinitely many terms of the G. P. 1, 2, 4, 8, 16, ... we shall get a infiinite sum.

So, sometimes we may be able to add the infinitely many terms of G. P. and sometimes we may not. We shall discuss this question now.

6.7.1 SUM OF INFINITE TERMS OF A G. P.

Let us consider a G. P. with infinite number of terms and common ratio *r*.

Case 1 : We assume that $|r| > 1$

The expression for the sum of *n* terms of the G. P. is then given by

$$S_n = \frac{a(r^n - 1)}{r-1} = \frac{a\,r^n}{r-1} - \frac{a}{r-1} \qquad \text{... (A)}$$

Now as *n* becomes larger and larger r^n also becomes larger and larger. Thus, when *n* is infinitely large and $|r| > 1$ then the sum is also infinitely large which has no importance in Mathematics. We now consider the other possibility.

Case 2 : Let $|r| < 1$

Formula (A) can be written as $S = \dfrac{a\,(1-r^n)}{1-r} = \dfrac{a}{1-r} - \dfrac{ar^n}{1-r}$

Now as *n* becomes infinitely large, r^n becomes infinitely small, i.e., as $n \to \infty$, $r^n \to 0$, then the above expression for sum takes the form $S = \dfrac{a}{1-r}$

Hence, the sum of an infinite G. P. with the first term *a* and common ratio *r* is given by

$$S = \frac{a}{1-r}, \text{ when } |r| < 1 \qquad \text{...(i)}$$

Example 6.22 Find the sum of the infinite G. P. $\dfrac{1}{3}, -\dfrac{2}{9}, \dfrac{4}{27}, -\dfrac{8}{81}, \cdots$

Solution : Here, the first term of the infinite G. P. is $a = \dfrac{1}{3}$, and $r = \dfrac{-\frac{2}{9}}{\frac{1}{3}} = -\dfrac{2}{3}$.

Here, $|r| = \left|-\dfrac{2}{3}\right| = \dfrac{2}{3} < 1$

$\therefore$ Using the formula for sum $S = \dfrac{a}{1-r}$ we have $S = \dfrac{\frac{1}{3}}{1-\left(\frac{-2}{3}\right)} = \dfrac{\frac{1}{3}}{1+\frac{2}{3}} = \dfrac{1}{5}$

Hence, the sum of the given G. P. is $\dfrac{1}{5}$.

Example 6.23 Express the recurring decimal $0.\bar{3}$ as an infinite G. P. and find its value in rational form.

Solution. $0.\bar{3}$ = 0.3333333 ...

= 0.3 + 0.03 + 0.003 + 0.0003 + ...

$= \dfrac{3}{10} + \dfrac{3}{10^2} + \dfrac{3}{10^3} + \dfrac{3}{10^4} + \ \ldots$

The above is an infinite G. P. with the first term $a = \dfrac{3}{10}$ and $r = \dfrac{\frac{3}{10^2}}{\frac{3}{10}} = \dfrac{1}{10} < 1$

Hence, by using the formula $S = \dfrac{a}{1-r}$, we get $0.\bar{3} = \dfrac{\frac{3}{10}}{1-\frac{1}{10}} = \dfrac{\frac{3}{10}}{\frac{9}{10}} = \dfrac{3}{9} = \dfrac{1}{3}$

Hence, the recurring decimal $0.\bar{3} = \dfrac{1}{3}$.

Example 6.24 The distance travelled (in cm) by a simple pendulum in consecutive seconds are 16, 12, 9, ... How much distance will it travel before coming to rest ?

Solution : The distance travelled by the pendulum in consecutive seconds are, 16, 12, 9, ... is an infinite geometric progression with the first term $a = 16$ and r $= \dfrac{12}{16} = \dfrac{3}{4} < 1$.

Hence, using the formula $S = \dfrac{a}{1-r}$ we have

$S = \dfrac{16}{1-\frac{3}{4}} = \dfrac{16}{\frac{1}{4}} = 64$ $\therefore$ Distance travelled by the pendulum is 64 cm.

Example 6.25 The sum of an infinite G. P. is 3 and sum of its first two terms is $\frac{8}{3}$. Find the first term.

Solution: In this problem $S = 3$. Let a be the first term and r be the common ratio of the given infinite G. P.

Then according to the question. $a + ar = \frac{8}{3}$

or, $3a(1+r) = 8$ ... (1)

Also from $S = \frac{a}{1-r}$, we have $3 = \frac{a}{1-r}$

or, $a = 3(1-r)$ ... (2)

From (1) and (2), we get.

$3.3\,(1 - r)\,(1 + r) = 8$

or, $1 - r^2 = \frac{8}{9}$ or, $r^2 = \frac{1}{9}$

or, $r = \pm\frac{1}{3}$

From (2), $a = 3\left(1 \mp \frac{1}{3}\right) = 2$ or 4 according as $r = \pm\frac{1}{3}$.

CHECK YOUR PROGRESS 6.6

(1) Find the sum of each of the following inifinite G. P's :

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \infty$ (b) $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \cdots \infty$

2. Express the following recurring decimals as an infinite G. P. and then find out their values as a rational number. (a) $0.\overline{7}$ (b) $0.3\overline{15}$

3. The sum of an infinite G. P. is 15 and the sum of the squares of the terms is 45. Find the G.P.

4. The sum of an infinite G. P. is $\frac{1}{3}$ and the first term is $\frac{1}{4}$. Find the G.P.

6.8 GEOMETRIC MEAN (G. M.)

If a, G, b are in G. P., then G is called the geometric mean between a and b.

If three numbers are in G. P., the middle one is called the geometric mean between the other two.

If $a, G_1, G_2, ..., G_n, b$ are in G. P.,

then $G_1, G_2, ... G_n$ are called n G. M.'s between a and b.

The geometric mean of n numbers is defined as the n^{th} root of their product.

Thus if $a_1, a_2, ..., a_n$ are n numbers, then their

$$\text{G. M.} = (a_1, a_2, ... a_n)^{\frac{1}{n}}$$

Let G be the G. M. between a and b, then a, G, b are in G. P $\therefore \frac{G}{a} = \frac{b}{G}$

or, $G^2 = ab$ or, $G = \sqrt{ab}$

$\therefore$ Geometric mean $= \sqrt{\text{Product of extremes}}$

Given any two positive numbers a and b, any number of geometric means can be inserted between them Let $a_1, a_2, a_3 ..., a_n$ be n geometric means between a and b.

Then $a_1, a_1, a_3, ... a_n, b$ is a G. P.

Thus, b being the $(n + 2)^{th}$ term, we have

$$b = a\,r^{n+1}$$

or, $r^{n+1} = \frac{b}{a}$ or, $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Hence, $a_1 = ar = a \times \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$, $a_2 = ar^2 = a \times \left(\frac{b}{a}\right)^{\frac{2}{n+1}}$

$\dots \qquad \dots$

$\dots \qquad \dots$

$$a_n = ar^n = a \times \left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Further we can show that the product of these n G. M.'s is equal to n^{th} power of the single geometric mean between a and b.

Multiplying $a_1 . a_2, ... a_n$, we have

$$a_1, a_2 \cdots a_n = a^n \left(\frac{b}{a}\right)^{\frac{1}{n+1}+\frac{2}{n+1}+\cdots+\frac{n}{n+1}} = a^n \left(\frac{b}{a}\right)^{\frac{1+2+\cdots+n}{n+1}} = a^n \left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n}{2}} = (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n = G^n = (\text{single G. M. between } a \text{ and } b)^n$$

Example 6.26 Find the G. M. between $\frac{3}{2}$ and $\frac{27}{2}$

Solution : We know that if a is the G. M. between a and b, then $G = \sqrt{ab}$

$\therefore$ G. M. between $\frac{3}{2}$ and $\frac{27}{2} = \sqrt{\frac{3}{2} \times \frac{27}{2}} = \frac{9}{2}$

Example 6.27 Insert three geometric means between 1 and 256.

Solution : Let G_1, G_2, G_3, be the three geometric means between 1 and 256.

Then 1, G_1, G_2, G_3, 256 are in G. P.

If r be the common ratio, then $t_5 = 256$ i.e, $ar^4 = 256 \Rightarrow 1.\ r^4 = 256$

or, $r^2 = 16$ or, $r = \pm 4$

When $r = 4$, $G_1 = 1.\ 4 = 4$, $G_2 = 1.\ (4)^2 = 16$ and $G_3 = 1.\ (4)^3 = 64$

When $r = -4$, $G_1 = -4$, $G_2 = (1)\ (-4)^2 = 16$ and $G_3 = (1)\ (-4)^3 = -64$

$\therefore$ G.M. between 1 and 256 are 4, 16, 64, or, – 4, 16, –64.

Example 6.28 If 4, 36, 324 are in G. P. insert two more numbers in this progression so that it again forms a G. P.

Solution : G. M. between 4 and 36 = $\sqrt{4 \times 36} = \sqrt{144} = 12$

G. M. between 36 and $324 = \sqrt{36 \times 324} = 6 \times 18 = 108$

If we introduce 12 between 4 and 36 and 108 betwen 36 and 324, the numbers 4, 12, 36, 108, 324 form *a* G. P.

$\therefore$ The two new numbers inserted are 12 and 108.

Example 6.29 Find the value of n such that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Solution : If x be G. M. between a and b, then $x = a^{\frac{1}{2}} \times b^{\frac{1}{2}}$

$$\therefore \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = a^{\frac{1}{2}}\, b^{\frac{1}{2}} \text{ or, } \quad a^{n+1}+b^{n+1} = \left(a^{\frac{1}{2}}\, b^{\frac{1}{2}}\right)\left(a^n+b^n\right)$$

$$\text{or, } \quad a^{n+1}+b^{n+1} = a^{n+\frac{1}{2}}\, b^{\frac{1}{2}} + a^{\frac{1}{2}}\, b^{n+\frac{1}{2}} \text{ or, } a^{n+1} - a^{n+\frac{1}{2}}.\, b^{\frac{1}{2}} = a^{\frac{1}{2}}\, b^{n+\frac{1}{2}} - b^{n+1}$$

$$\text{or, } \quad a^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = b^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) \text{ or, } \quad a^{n+\frac{1}{2}} = b^{n+\frac{1}{2}}$$

$$\text{or, } \quad \frac{a^{n+\frac{1}{2}}}{b^{n+\frac{1}{2}}} = 1 \text{ or, } \quad \left(\frac{a}{b}\right)^{n+\frac{1}{2}} = \left(\frac{a}{b}\right)^{0}$$

$$\therefore \quad n+\frac{1}{2} = 0 \text{ or, } n = \frac{-1}{2}$$

6.8.1 RELATIONSHIP BETWEEN A. M. AND G.M.

Let a and b be the two numbers.

Let A and G be the A. M. and G. M. respectively between a and b

$$\therefore \quad A = \frac{a+b}{2}, G = \sqrt{ab}$$

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{ab}}{2} = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$$

$$\therefore \quad A > G$$

Example 6.30 The arithemetic mean between two numbers is 34 and their geometric mean is 16. Find the numbers.

Solution : Let the numbers be a and b. Since A. M. between a and b is 34,

$$\therefore \quad \frac{a+b}{2} = 34, \text{ or, } a+b = 68 \qquad \text{... (1)}$$

Since G. M. between a and b is 16,

$$\therefore \quad \sqrt{ab} = 16 \text{ or, } ab = 256 \text{ we know that } (a-b)^2 = (a+b)^2 - 4\,ab \qquad \cdots(2)$$

$$= (68)^2 - 4 \times 256 = 4624 - 1024 = 3600$$

MODULE - II
Sequences And Series

Notes

$\therefore \quad a - b = \sqrt{3600} = 60 \qquad \cdots(3)$

Adding (1) and (3), we get, $2a = 128 \quad \therefore \quad a = 64$

Subtracting (3) from (1), we get

$2b = 8 \quad$ or, $\quad b = 4$

$\therefore$ Required numbers are 64 and 4.

Example 6.31 The arithmetic mean between two quantities b and c is a and the two geometric means between them are g_1 and g_2. Prove that $g_1^3 + g_2^3 = 2\,abc$

Solution : The A. M. between b and c is a $\therefore \quad \frac{b+c}{2} = a$, or, $b + c = 2a$

Again g_1 and g_2 are two G. M.'s between b and c $\therefore \quad b, g_1, g_2, c$ are in G. P.

If r be the common ratio, then $c = br^3$ or, $r = \left(\frac{c}{b}\right)^{\frac{1}{3}}$

$$g_1 = br = b\left(\frac{c}{b}\right)^{\frac{1}{3}} \quad \text{and} \quad g_2 = br^2 = b\left(\frac{c}{b}\right)^{\frac{2}{3}}$$

$$\therefore \quad g_1^3 + g_2^3 = b^3\left[\left(\frac{c}{b}\right) + \left(\frac{c}{b}\right)^2\right] = b^3 \times \frac{c}{b}\left(1 + \frac{c}{b}\right) = b^2 c \times \left(\frac{b+c}{b}\right)$$

$= bc\,(2a) \qquad$ [since $b + c = 2a$]

$= 2abc$

Example 6.32 The product of first three terms of a G. P. is 1000. If we add 6 to its second term and 7 to its 3rd term, the three terms form an A. P. Find the terms of the G. P.

Solution : Let $t_1 = \frac{a}{r}, t_2 = a$ and $t_3 = ar$ be the first three terms of G. P.

Then, their product $= \frac{a}{r}.\, a.\, ar = 1000$ or, $a^3 = 1000$, or, $a = 10$

By the question, $t_1, t_2 + 6, t_3 + 7$ are in A. P. $\qquad$ ...(1)

i.e. $\frac{a}{r}$, $a + 6$, $ar + 7$ are in A. P.

$\therefore \quad (a + 6) - \frac{a}{r} = (ar + 7) - (a + 6)$ or, $\quad 2(a+6) = \frac{a}{r} + (ar+7)$

or, $\quad 2(10+6) = \frac{10}{r} + (10r+7)$ [using (1)]

or, $\quad 32r = 10 + 10r^2 + 7r$ or, $\quad 10r^2 - 25r + 10 = 0$

$$\therefore \quad r = \frac{25 \pm \sqrt{625 - 400}}{20} = \frac{25 \pm 15}{20} = 2, \frac{1}{2}$$

When $a = 10$, $r = 2$. then the terms are $\frac{10}{2}$, $10(2)$ i.e., 5, 10, 20

When $a = 10$, $r = \frac{1}{2}$ then the terms are $10(2)$, 10, $10\left(\frac{1}{2}\right)$ i.e., 20, 10, 5

CHECK YOUR PROGRESS 6.7

1. Insert 8 G. M.'s between 8 and $\frac{1}{64}$.

2. If a_1 is the first of n geometric means between a and b, show that $a_1^{n+1} = a^n b$

3. If G is the G. M. between a and b, prove that $\frac{1}{G^2 - a^2} + \frac{1}{G^2 - b^2} = \frac{1}{G^2}$

4. If the A. M. and G. M. between two numbers are in the ratio $m : n$, then prove that the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$

5. If A and G are respectvely arithmetic and geometric means between two numbers a and b, then show that A > G.

6. The sum of first three terms of a G. P. is $\frac{13}{12}$ and their product is –1. Find the G. P.

7. The product of three terms of a G. P. is 512. If 8 is added to first and 6 is added to second term, the numbers form an A. P., Find the numbers.

LET US SUM UP

- A sequence in which the difference of two cousecutive terms is always constant ($\neq 0$) is called an Arithmetic Progression (A. P.)

- The general term of an A. P.

  $a, a + d, a + 2d, \ldots$ is given by $t_n = a + (n - 1)d$

- $S_{n,}$ the sum of the first n terms of the A.P a, a+d, a+2d,... is given by

  $$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

  $$= \frac{n}{2}(a + l), \text{ where } l = a + (n-1)d.$$

- $t_n = S_n - S_{n-1}$

- An arithmetic mean between a and b is $\frac{a+b}{2}$.

- A sequence in which the ratio of two consecutive terms is always constant ($\neq 0$) is called a Geometric Progression (G. P.)

- The n^{th} term of a G. P.: $a, ar, ar^2, \ldots$ is ar^{n-1}

- Sum of the first n terms of a G. P.: $a, ar, ar^2, \ldots$ is

  $$S_n = \frac{a(r^n - 1)}{r - 1} \text{ for } |r| > |$$

  $$= \frac{a(1 - r^n)}{1 - r} \text{ for } |r| < 1$$

- The sums of an infintite G. P. $a, ar, ar^2, \ldots$ is given by

  $$S = \frac{a}{1 - r} \text{ for } |r| < 1$$

- Geometric mean G between two numbers a and b is $\sqrt{ab}$

- The arithmetic mean A between two numbers a and b is always greater than the corresponding Geometric mean G i.e., A > G.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=_cooC3yG_p0

http://www.youtube.com/watch?v=pXo0bG4iAyg

http://www.youtube.com/watch?v=dIGLhLMsy2U

http://www.youtube.com/watch?v=cYw4MFWsB6c

http://www.youtube.com/watch?v=Uy_L8tnihDM

http://www.bbc.co.uk/education/asguru/maths/13pure/03sequences/index.shtml

TERMINAL EXERCISE

1. Find the sum of all the natural numbers between 100 and 200 which are divisible by 7.

2. The sum of the first n terms of two A. P.'s are in the ratio $(2n - 1) : (2n + 1)$. Find the ratio of their 10^{th} terms.

3. If a, b, c are in A. P. then show that $b + c, c + a, a + b$ are also in A. P.

4. If $a_1, a_2, ..., a_n$ are in A. P., then prove that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \ldots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}$$

5. If $(b - c)^2, (c - a)^2, (a - b)^2$ are in A. P., then prove that

$\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$, are also in A. P.

6. If the p^{th}, q^{th} and r^{th} terms are P, Q, R respectively. Prove that $P(Q - R) + Q(R - P) + r(P - Q) = 0$.

7. If a, b, c are in G. P. then prove that $a^2 b^2 c^2 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) = a^3 + b^3 + c^3$

8. If a, b, c, d are in G. P., show that each of the following form a G. P. :

(a) $(a^2 - b^2), (b^2 - c^2), (c^2 - d^2)$ (b) $\frac{1}{a^2 + b^2}, \frac{1}{b^2 + c^2}, \frac{1}{c^2 - d^2}$

9. If x, y, z are the p^{th}, q^{th} and r^{th} terms of a G. P., prove that $x^{q-r} y^{r-p} z^{p-q} = 1$

10. If a, b, c are in A. P. and x, y, z are in G. P. then prove that $x^{b-c} y^{c-a} z^{a-b} = 1$

11. If the sum of the first n terms of a G. P. is represented by S_n, then prove that

$S_n (S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$

12. If p, q, r are in A. P. then prove that the p^{th}, q^{th} and r^{th} terms of a G. P. are also in G. P.

13. If $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} \ldots + \frac{1}{2^{n-1}}$, find the least value of n such that

$2 - S_n < \frac{1}{100}$

14. If the sum of the first n terms of a G. P. is S and the product of these terms is p and the sum of their reciprocals is R, then prove that $p^2 = \left(\frac{S}{R}\right)^n$

ANSWERS

CHECK YOUR PROGRESS 6.1

1. (a) $2n-1$ (b) $2n+1$ 2. 3, 5, 7, 9, ... 3. 20, 16

4. no 5. $m+n$ 6. 10, 6, 2,

CHECK YOUR PROGRESS 6.2

1. (a) 435 (b) $\frac{n}{2}\left[21-5n^2\right]$ 2. 5 3. 12

4. 37 5. $n^2+9n, 9$ 6. $2a$

CHECK YOUR PROGRESS 6. 3

2. 5

CHECK YOUR PROGRESS 6. 4

1. $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \frac{3}{16}$ 2. 11th, no 3. 36, 6, 1 or 1, 6, 36

4. (a) 6 (b) 3

CHECK YOUR PROGRESS 6.5

1. (a) 6138 (b) $\frac{2}{3}\left(1-\frac{1}{2^{20}}\right)$ 2. 10.

4. (a) $\frac{80}{81}\left(10^n-1\right)-\frac{8n}{9}$ (b) $\frac{2n}{9}-\frac{2}{81}\left(1-\frac{1}{10^n}\right)$

CHECK YOUR PROGRESS 6. 6

1. (a) $\frac{3}{2}$ (b) $\frac{13}{24}$ 2. (a) $\frac{7}{9}$ (b) $\frac{52}{165}$

3. $5, \frac{10}{3}, \frac{20}{9}, \frac{40}{27}, \ldots \infty$

4. $\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{4^4}, \ldots \infty$

CHECK YOUR PROGRESS 6.7

1. $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32},$ 6. $\frac{4}{3}, -1, \frac{3}{4}$ ... or $\frac{3}{4}, -1, \frac{4}{3}$ ... 7. 4, 8, 16

TERMINAL EXERCISE

1. 2107 2. 37 : 39

7

SOME SPECIAL SEQUENCES

Suppose you are asked to collect pebbles every day in such a way that on the first day if you collect one pebble, second day you collect double of the pebbles that you have collected on the first day, third day you collect double of the pebbles that you have collected on the second day, and so on. Then you write the number of pebbles collected daywise, you will have a sequence, $1, 2, 2^2, 2^3, ...$

From a sequence we derive a series. The series corresponding to the above sequence is

$1 + 2 + 2^2 + 2^3 + ...$

One well known series is Fibonacci series 1 + 1 + 2 + 3 + 5+ 8 + 13 + ...

In this lesson we shall study some special types of series in detail.

OBJECTIVES

After studying this lesson, you will be able to :

- define a series;
- calculate the terms of a series for given values of n from t_n;
- evaluate $\sum n, \sum n^2, \sum n^3$ using method of differences and mathematical induction; and
- evaluate simple series like 1.3 + 3.5 + 5.7 +..... n terms.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a sequence
- Concept of A. P. and G. P., sum of n terms.
- Knowldge of converting recurring decimals to fractions by using G. P.

7.1 SERIES

An expression of the form $u_1 + u_2 + u_3 + ... + u_n + ...$ is called a series, where $u_1, u_2, u_3 ..., u_n$ ... is a sequence of numbers. The above series is denoted by $\sum_{r=1}^{n} u_r$. If n is finite then the series is a finite series, otherwise the series is infinite. Thus we find that a series is associated to a sequence. Thus a series is a sum of terms arranged in order, according to some definite law.

Consider the following sets of numbers :

(a) 1, 6, 11, ..., (b) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12} \cdots$

(c) 48, 24, 12, ..., (d) $1^2, 2^2, 3^2, \ldots.$

(a), (b), (c), (d) form sequences, since they are connected by a definite law. The series associated with them are :

$$1 + 6 + 11 + \ldots, \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \ldots, 48 + 24 + 12 + \ldots, 1^2 + 2^2 + 3^2 + \ldots$$

Example 7.1 Write the first 6 terms of each of the following sequences, whose n[th] term is given by

(a) $T_n = 2n + 1$, (b) $a_n = n^2 - n + 1$ (c) $f_n = (-1)^n . 5^n$

Hence find the series associated to each of the above sequences.

Solution : (a) $T_n = 2n + 1$, For $n = 1$, $T_1 = 2.1 + 1 = 3$, For $n = 2$, $T_2 = 2.2 + 1 = 5$

For $n = 3$, $T_3 = 2.3 + 1 = 7$, For $n = 4$, $T_4 = 2.4 + 1 = 9$

For $n = 5$, $T_5 = 2.5 + 1 = 11$, For $n = 6$, $T_6 = 2.6 + 1 = 13$

Hence the series associated to the above sequence is $3 + 5 + 7 + 9 + 11 + 13 + \ldots$

(b) $a_n = n^2 - n + 1$, For $n = 1$, $a_1 = 1^2 - 1 + 1 = 1$

For $n = 2$, $a_2 = 2^2 - 2 + 1 = 3$, For $n = 3$, $a_3 = 3^2 - 3 + 1 = 7$

For $n = 4$, $a_4 = 4^2 - 4 + 1 = 13$,For $n = 5$, $a_5 = 5^2 - 5 + 1 = 21$

For $n = 6$, $a_6 = 6^2 - 6 + 1 = 31$

Hence the series associated to the above sequence is $1 + 3 + 7 + 13 + \ldots$

(c) Here $f_n = (-1)^n 5^n$, For $n = 1$, $f_1 = (-1)^1 5^1 = -5$

For $n = 2$, $f_2 = (-1)^2 5^2 = 25$, For $n = 3$, $f_3 = (-1)^3 5^3 = -125$

For $n = 4$, $f_4 = (-1)^4 5^4 = 625$, For $n = 5$, $f_5 = (-1)^5 5^5 = -3125$

For $n = 6$, $f_6 = (-1)^6 5^6 = 15625$

The corresponding series relative to the sequence

$f_n = (-1)^n 5^n$ is $-5 + 25 - 125 + 625 - 3125 + 15625 -$

Example 7.2 Write the n^{th} term of each of the following series :

(a) $-2 + 4 - 6 + 8 - ....$ (b) $1 - 1 + 1 - 1 + ...$

(c) $4 + 16 + 64 + 256 + ...$ (d) $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + ...$

Solution : (a) The series is $-2 + 4 - 6 + 8$ ....

Here the odd terms are negative and the even terms are positive. The above series is obtained by multiplying the series. $-1 + 2 - 3 + 4 - ....$ by 2

$\therefore$ $T_n = 2(-1)^n n = (-1)^n 2n$

(b) The series is $1 - 1 + 1 - 1 + 1 -$ ......

$\therefore$ $T_n = (-1)^{n+1}$

(c) The series is $4 + 16 + 64 + 256 + ...$

The above series can be writen as $4 + 4^2 + 4^3 + 4^4 + ...$

i.e., n^{th} term, $T_n = 4^n$.

(d) The series is $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + ...$ *i.e.,* $\sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + ...$

$\therefore$ n^{th} term is $T_n = \sqrt{n+1}$.

CHECK YOUR PROGRESS 7.1

1. Write the first 6 terms of each of the following series, whose n^{th} term is given by

(a) $T_n = \dfrac{n(n+1)(n+2)}{6}$ (b) $a_n = \dfrac{n^2 - 1}{2n - 3}$

2. If $A_1 = 1$ and $A_2 = 2$, find A_6 if $A_n = \dfrac{A_{n-1}}{A_{n-2}}, (n > 2)$

3. Write the n^{th} term of each of the following series:

(a) $-1 + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \cdots$ (b) $3 - 6 + 9 - 12 + ...$

7.2 SUM OF THE POWERS OF THE FIRST *n* NATURAL NUMBERS

(a) The series of first n natural numbers is

$1 + 2 + 3 + 4 + ... + n.$

Let $S_n = 1 + 2 + 3 + ... + n$

This is an arithmetic series whose the first term is 1, the common difference is 1 and the number of terms is n. $\therefore \quad S_n = \frac{n}{2}\left[2.1+(n-1)1\right] = \frac{n}{2}\left[2n-1\right]$

i.e., $\quad S_n = \frac{n(n+1)}{2}$

$\therefore$ We can write $\sum n = \frac{n(n+1)}{2}$

(b) Determine the sum of the squares of the first n natural numbers.

Let $S_n = 1^2 + 2^2 + 3^2 + ... + n^2$

Consider the identity : $n^3 - (n-1)^3 = 3n^2 - 3n + 1$

By giving the values for $n = 1, 2, 3, ..., n-1, n$ in the above identity, we have.

$1^3 - 0^3 = 3.1^2 - 3.1 + 1$

$2^3 - 1^3 = 3.2^2 - 3.2 + 1$

$3^3 - 2^3 = 3.3^2 - 3.3 + 1$

....

....

$n^3 - (n-1)^3 = 3n^2 - 3n + 1$

Adding these we get

$n^3 - 0^3 = 3\,(1^2 + 2^2 + 3^2 + ... + n^2) - 3\,(1 + 2 + 3 + ... + n) +$

$(1 + 1 + 1 + ... \; n$ times)

or, $\quad n^3 = 3\,S_n - 3\left[\frac{n\,(n+1)}{2}\right] + n \;....\left[\because \sum n = \frac{n\,(n+1)}{2}\right]$

or, $\quad 3\,S_n = n^3 + \frac{3n\,(n+1)}{2} - n \;=\; n\,(n^2 - 1) + \frac{3n}{2}\,(n+1)$

$= n\,(n+1)\left[n - 1 + \frac{3}{2}\right] = \frac{n\,(n+1)\,(2n+1)}{2}$

$\therefore \quad S_n = \frac{n\,(n+1)\,(2n+1)}{6}$ i.e., $\quad \sum n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$

(c) Determine the sum of the cubes of the first n natural numbers.

Here $S_n = 1^3 + 2^3 + 3^3 + ... + n^3$

Consider the identity : $n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$

Putting successively 1, 2, 3, ... for n we have

$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1$$

$$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1$$

$$3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1$$

$$\ldots \qquad \ldots$$

$$n^4 - (n-1)^4 = 4.n^3 - 6.n^2 + 4.n - 1$$

Adding these, we get

$$n^4 - 0^4 = 4(1^3 + 2^3 + \ldots + n^3) - 6(1^2 + 2^2 + \ldots + n^2) + 4(1 + 2 + 3 + \ldots + n) - (1 + 1 + \ldots\ n \text{ times})$$

$$\Rightarrow \quad n^4 = 4.S_n - 6\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 4n\frac{n+1}{2} - n$$

$$\Rightarrow \quad 4S_n = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$

$$= n^4 + n(2n^2 + 3n + 1) - 2n^2 - 2n + n$$

$$= n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n = n^4 + 2n^3 + n^2 = n^2(n^2 + 2n + 1)$$

i.e., $4S_n = n^2(n+1)^2$

$$\therefore \quad S_n = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2$$

$$\therefore \quad \sum n^3 = \left[\frac{n(n+1)}{2}\right]^2 \quad \text{or,} \quad \sum n^3 = \left(\sum n\right)^2$$

Note : *In problems on finding sum of the series, we shall find the nth term of the series (t_n) and then use $S_n = \sum t_n$.*

Example 7.3 Find the sum of first n terms of the series 1.3 + 3.5 + 5.7 + ...

Solution :

Let $S_n = 1.3 + 3.5 + 5.7 + \ldots$

The n^{th} term of the series

$$t_n = \{n^{th} \text{ term of } 1, 3, 5, \ldots\} \times \{n^{th} \text{ term of } 3, 5, 7, \ldots\}$$

$$= (2n-1)(2n+1) = 4n^2 - 1$$

$$S_n = \sum t_n = \sum \left[4n^2 - 1\right]$$

$$= 4\sum n^2 - \sum(1) = 4\,\frac{n\,(n+1)\,(2n+1)}{6} - n$$

$$= \frac{2n\,(n+1)\,(2n+1) - 3n}{3} = \frac{n}{3}\left[2\,(2n^2 + 3n + 1) - 3\right]$$

$$= \frac{n}{3}\left[4n^2 + 6n - 1\right]$$

Example 7.4 Find the sum of first n terms of the series

$$1.2^2 + 2.3^2 + 3.4^2 + \ldots\ldots$$

Solution : Here $t_n = n\,\{2 + (n-1)\}^2 = n\,(n+1)^2 = n.\,(n^2 + 2n + 1)$

i.e., $t_n = n^3 + 2n^2 + n$

Let $S_n = 1.2^2 + 2.3^2 + 2.3^2 + 3.4^2 + \ldots. + n.\,(n+1)^2$.

$\therefore$ $S_n = \sum t_n = \sum\,(n^3 + 2n^2 + n) = \sum n^3 + 2\,\sum n^2 + \sum n.$

$$= \left\{\frac{n\,(n+1)}{2}\right\}^2 + 2\left\{\frac{n\,(n+1)\,(2n+1)}{6}\right\} + \frac{n\,(n+1)}{2}$$

$$= n\,(n+1)\left\{\frac{n\,(n+1)}{4} + \frac{2n+1}{3} + \frac{1}{2}\right\}$$

$$= \frac{n\,(n+1)}{12}\,(3n^2 + 11n + 10) = \frac{1}{12}\,n\,(n+1)\,(n+2)\,(3n+5)$$

Example 7.5 Find the sum of first n terms of the series

$$2.\ 3.\ 5 + 3.\ 5.\ 7 + 4.\ 7.\ 9 + \ldots.$$

Solution : Let $S_n = 2.\ 3.\ 5. + 3.\ 5.\ 7 + 4.\ 7.\ 9 + \ldots.$

n^{th} term of the series

$t_n =$ $\{n^{th}$ term of $2, 3, 4, \ldots\} \times \{n^{th}$ term of $3, 5, 7, \ldots\} \times \{n^{th}$ term of $5, 7, 9, \ldots.\}$

$= (n+1) \times (2n+1) \times (2n+3)$

$= (n+1)\,[4n^2 + 8n + 3] = 4n^3 + 12n^2 + 11n + 3$

$\therefore$ $S_n = \sum t_n = \sum\,[4n^3 + 12n^2 + 11n + 3]$

$= 4\,\sum n^3 + 12\,\sum n^2 + 11\,\sum n + \sum(3)$

$$= 4\,\frac{n^2\,(n+1)^2}{4} + \frac{12\,n\,(n+1)\,(2n+1)}{6} + \frac{11\,n\,(n+1)}{2} + 3n$$

$$= n^2\,(n+1)^2 + 2n\,(n+1)\,(2n+1) + \frac{11n\,(n+1)}{2} + 3n$$

$$= \frac{n}{2}\left[2n\,(n+1)^2 + 4\,(n+1)\,(2n+1) + 11\,(n+1) + 6\right]$$

$$= \frac{n}{2}\left[2n\,(n^2 + 2n + 1) + 4\,(2n^2 + 3n + 1) + 11n + 17\right]$$

$$= \frac{n}{2}\left[2n^3 + 12n^2 + 25n + 21\right]$$

Example 7.6 Find the sum of first n terms of the following series :

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots$$

Solution : $t_n = \dfrac{1}{(2n-1)\,(2n+1)}$

$$= \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

Now putting successively for $n = 1, 2, 3, \ldots$

$$t_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$t_2 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$t_3 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right)$$

$$\ldots \qquad \ldots$$

$$t_n = \frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right)$$

Adding, $t_1 + t_2 + \cdots + t_n = \dfrac{1}{2}\left[1 - \dfrac{1}{2n+1}\right] = \dfrac{n}{(2n+1)}$

CHECK YOUR PROGRESS 7.2

1. Find the sum of first n terms of each of the following series :

   (a) $1 + (1 + 3) + (1 + 3 + 5) + ...$

   (b) $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots$

   (c) $(1) + (1 + 3) + (1 + 3 + 3^2) + (1 + 3 + 3^2 + 3^3) + ...$

2. Find the sum of n terms of the series. whose n^{th} term is $n(n + 1)(n + 4)$

3. Find the sum of the series $1.\ 2.\ 3 + 2.\ 3.\ 4. + 3.\ 4.\ 5 + \cdots$ upto n terms

LET US SUM UP

- An expression of the form $u_1 + u_2 + u_3 + ... + u_n + ...$ is called a series, where $u_1, u_2, u_3 .... u_n, ...$ is a sequence of numbers
- $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$
- $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$
- $S_n = \sum t_n$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Sequence_and_series
http://mathworld.wolfram.com/Series.html

TERMINAL EXERCISE

1. Find the sum of each of the following series :

   (a) $2 + 4 + 6 + ...$ up to 40 terms.

(b) $2 + 6 + 18 + \ldots$ up to 6 terms.

2. Sum each of the following series to n terms :

(a) $1 + 3 + 7 + 15 + 31 + \ldots..$

(b) $$\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \ldots..$$

(c) $$\frac{3}{1.4} + \frac{5}{4.9} + \frac{7}{9.16} + \frac{9}{16.25} + \ldots.$$

3. Find the sum of first n terms of the series $1^2 + 3^2 + 5^2 + \ldots.$

4. Find the sum to n terms of the series $5 + 7 + 13 + 31 + \ldots.$

5. Find the sum to n terms of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$

6. Find the sum of $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$

7. Show that

$$\frac{1\times 2^2 + 2\times 3^2 + \ldots\ldots + n\times (n+1)^2}{1^2\times 2 + 2^2\times 3 + \ldots\ldots + n^2\times (n+1)} = \frac{3n+5}{3n+1}$$

ANSWERS

CHECK YOUR PROGRESS 7.1

1. (a) 1, 4, 10, 20, 35, 56 (b) $0, 3, \frac{8}{3}, 3, \frac{24}{7}, \frac{35}{9}$

2. $\frac{1}{2}$

3. (a) $(-1)^n \frac{1}{n}$ (b) $(-1)^{n+1}\, 3n$

CHECK YOUR PROGRESS 7.2

1. (a) $\frac{1}{6} n\,(n+1)\,(2n+1)$ (b) $\frac{n}{3n+1}$ (c) $\frac{1}{4}(3^{n+1} - 2n - 3)$

2. $\frac{n(n+1)}{12}\left[3n^2 + 23n + 34\right]$

3. $\frac{1}{4} n(n+1)(n+2)(n+3)$

TERMINAL EXERCISE

1. (a) 1640 (b) 728

2. (a) $2^{n+1} - n - 2$ (b) $\frac{1}{12} - \frac{1}{4\,(2n+1)\,(2n+3)}$ (c) $1 - \frac{1}{(n+1)^2}$

3. $\frac{n}{3}(4n^2 - 1)$

4. $\frac{1}{2}(3^n + 8n - 1)$

5. $\frac{5}{4} + \frac{15}{16}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{3n-2}{4.(5^{n-1})}$

6. $\frac{2n\,(n+1)\,(2n+1)}{3}$

8

COMPLEX NUMBERS

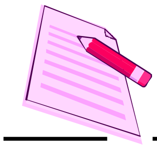
We started our study of number systems with the set of natural numbers, then the number zero was included to form the system of whole numbers; negative of natural numbers were defined. Thus, we extended our number system to whole numbers and integers.

To solve the problems of the type $p \div q$ we included rational numbers in the system of integers. The system of rational numbers has been extended further to irrational numbers as all lengths cannot be measured in terms of lengths expressed in rational numbers. Rational and irrational numbers taken together are termed as real numbers. But the system of real numbers is not sufficient to solve all algebraic equations. There are no real numbers which satisfy the equation $x^2+1 = 0$ or $x^2 = -1$. In order to solve such equations, i.e., to find square roots of negative numbers, we extend the system of real numbers to a new system of numbers known as complex numbers. In this lesson the learner will be acquinted with complex numbers, its representation and algebraic operations on complex numbers.

OBJECTIVES

After studying this lesson, you will be able to:

- describe the need for extending the set of real numbers to the set of complex numbers;
- define a complex number and cite examples;
- identify the real and imaginary parts of a complex number;
- state the condition for equality of two complex numbers;
- recognise that there is a unique complex number $x + iy$ associated with the point $P(x, y)$ in the Argand Plane and vice-versa;
- define and find the conjugate of a complex number;
- define and find the modulus and argument of a complex number;
- represent a complex number in the polar form;
- perform algebraic operations (addition, subtraction, multiplication and division) on complex numbers;
- state and use the properties of algebraic operations (closure, commutativity, associativity, identity, inverse and distributivity) of complex numbers; and



Notes

- state and use the following properties of complex numbers in solving problems:

  (i) $|z| = 0 \Leftrightarrow z = 0$ and $z_1 = z_2 \Rightarrow |z_1| = |z_2|$

  (ii) $|z| = |-z| = |\bar{z}|$ (iii) $|z_1 + z_2| \le |z_1| + |z_2|$

  (iv) $|z_1 . z_2| = |z_1| . |z_2|$ (v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$

- to find the square root of a complex number.

EXPECTED BACKGROUND KNOWLEDGE

- Properties of real numbers.
- Solution of linear and quadratic equations
- Representation of a real number on the number line
- Representation of point in a plane.

8.1 COMPLEX NUMBERS

Consider the equation $x^2 + 1 = 0$. ...(A)

This can be written as $x^2 = -1$ or $x = \pm\sqrt{-1}$

But there is no real number which satisfy $x^2 = -1$.In other words, we can say that there is no real number whose square is -1.In order to solve such equations, let us imagine that there exist a number *'i'* which equal to $\sqrt{-1}$.

In 1748, a great mathematician, L. Euler named a number 'i' as *Iota* whose square is -1. This *Iota or 'i'* is defined as imaginary unit. With the introduction of the new symbol *'i'*, we can interpret the square root of a negative number as a product of a real number with i.

Therefore, we can denote the solution of (A) as $x = \pm i$

Thus, $-4 = 4(-1)$

$$\therefore \quad \sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{i^2 . 2^2} = i2$$

Conventionally written as 2i.

So, we have $\sqrt{-4} = 2i$, $\sqrt{-7} = \sqrt{7}i$

$\sqrt{-4}$, $\sqrt{-7}$ are all examples of complex numbers.

Consider another quadratic equation: $x^2 - 6x + 13 = 0$

This can be solved as under:

$(x - 3)^2 + 4 = 0$ or, $(x - 3)^2 = -4$

or, $x - 3 = \pm 2i$ or, $x = 3 \pm 2i$

We get numbers of the form x + yi where x and y are real numbers and $i = \sqrt{-1}$.

Any number which can be expressed in the form a + bi where a,b are real numbers and $i = \sqrt{-1}$, is called a complex number.

A complex number is, generally, denoted by the letter z.

i.e. z = a + bi, 'a' is called the real part of z and is written as Re (a+bi) and 'b' is called the imaginary part of z and is written as Imag (a + bi).

If a = 0 and b ≠ 0, then the complex number becomes b*i* which is a purely imaginary complex number.

$-7i, \frac{1}{2}i, \sqrt{3}i$ and πi are all examples of purely imaginary numbers.

If $a \neq 0$ and b = 0 then the complex number becomes 'a' which is a real number.

5, 2.5 and $\sqrt{7}$ are all examples of real numbers.

If a = 0 and b = 0, then the complex number becomes 0 (zero). Hence the real numbers are particular cases of complex numbers.

Example 8.1 Simplify each of the following using 'i'.

(i) $\sqrt{-36}$ (ii) $\sqrt{25}.\sqrt{-4}$

Solution: (i) $\sqrt{-36} = \sqrt{36(-1)} = 6i$

(ii) $\sqrt{25}.\sqrt{-4} = 5 \times 2i = 10i$

8.2 POSITIVE INTEGRAL POWERS OF i

We know that

$i^2 = -1,\ i^3 = i^2.i = -1.i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1,\ i^5 = (i^2)^2.i = 1.i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1,\ i^7 = (i^2)^3(i) = -i,\ i^8 = (i^2)^4 = 1$

Thus, we find that any higher powers of 'i' can be expressed in terms of one of four values i, –1, –i, 1

If n is a positive integer such that n>4, then to find i^n, we first divide n by 4.

Let m be the quotient and r be the remainder.

Then n = 4m + r. where $0 \leq r < 4$.

Thus, $i^n = i^{(4m+r)} = i^{4m}.i^r = (i^4)^m.i^r = i^r \;(\because i^4=1)$

***Note :** For any two real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when atleast one of a and b is either 0 or positive.*

If fact $\quad \sqrt{-a} \times \sqrt{-b}$

$= i\sqrt{a} \times i\sqrt{b} = i^2 \sqrt{ab}$

$= -\sqrt{ab}$ where a and b are positive real numbers.

Example 8.2 Find the value of $1 + i^{10} + i^{20} + i^{30}$

Solution: $1 + i^{10} + i^{20} + i^{30}$

$= 1 + (i^2)^5 + (i^2)^{10} + (i^2)^{15} = 1 + (-1)^5 + (-1)^{10} + (-1)^{15}$

$= 1 + (-1) + 1 + (-1) = 1 - 1 + 1 - 1 = 0$

Thus, $1 + i^{10} + i^{20} + i^{30} = 0$.

Example 8.3 Express $8i^3 + 6i^{16} - 12i^{11}$ in the form of a + bi

Solution: $8i^3 + 6i^{16} - 12i^{11}$ can be written as $8(i^2).i + 6(i^2)^8 - 12(i^2)^5.i$

$= 8(-1).i + 6(-1)^8 - 12(-1)^5.i = -8i + 6 - 12(-1).i$

$= -8i + 6 + 12i = 6 + 4i$

which is of the form a + bi where 'a' is 6 and 'b' is 4.

CHECK YOUR PROGRESS 8.1

1. Simplify each of the following using 'i'.

   (a) $\sqrt{-27}$ (b) $-\sqrt{-9}$ (c) $\sqrt{-13}$

2. Express each of the following in the form of a + bi

   (a) 5 (b) –3i (c) 0

3. Simplify $10i^3 + 6i^{13} - 12i^{10}$

4. Show that $i^m + i^{m+1} + i^{m+2} + i^{m+3} = 0$ for all $m \in N$.

8.3 CONJUGATE OF A COMPLEX NUMBER

The complex conjugate (or simply conjugate) of a complex number z = a + bi is defined as the complex number a – bi and is denoted by $\bar{z}$.

Thus, if $z = a + bi$ then $\bar{z} = a - bi$.

Note : The conjugate of a complex number is obtained by changing the sing of the imaginary part.

Following are some examples of complex conjugates:

(i) If $z = 2 + 3i$, then $\bar{z} = 2 - 3i$

(ii) If $z = 1 - i$, then $\bar{z} = 1 + i$

(iii) If $z = -2 + 10i$, then $\bar{z} = -2 - 10i$

8.3.1 PROPERTIES OF COMPLEX CONJUGATES

(i) If z is a real number then $z = \bar{z}$ i.e., the conjugate of a real number is the number itself.

For example, let $z = 5$

This can be written as $z = 5 + 0i$

$\therefore \quad \bar{z} = 5 - 0i = 5, \qquad \therefore \quad z = 5 = \bar{z}$.

(ii) If z is a purely imaginary number then $\bar{z} = -z$

For example, if $z = 3i$. This can be written as $z = 0 + 3i$

$\therefore \quad \bar{z} = 0 - 3i = -3i = -z$

$\therefore \quad \bar{z} = -z$.

(iii) Conjugate of the conjugate of a complex number is the number itself.

i.e., $\overline{(\bar{z})} = z$

For example, if $z = a + bi$ then $\bar{z} = a - bi$

Again $\overline{(\bar{z})} = \overline{(a - bi)} = a + bi = z$

$\therefore \quad \overline{(\bar{z})} = z$

Example 8.4 Find the conjugate of each of the following complex numbers:

(i) $3 - 4i$ (ii) $(2 + i)^2$

Solution : (i) Let $z = 3 - 4i$ then $\bar{z} = \overline{(3 - 4i)} = 3 + 4i$

Hence, $3 + 4i$ is the conjugate of $3 - 4i$.

(iii) Let $z = (2 + i)^2$

i.e. $z = (2)^2 + (i)^2 + 2(2)(i) = 4 - 1 + 4i = 3 + 4i$

Then $\bar{z} = \left(\overline{3 + 4i}\right) = 3 - 4i$

Hence, $3 - 4i$ is the conjugate of $(2 + i)^2$

8.4 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

Let z = a + bi be a complex number. Let two mutually perpendicular lines xox' and yoy' be taken as x-axis and y-axis respectively, O being the origin.

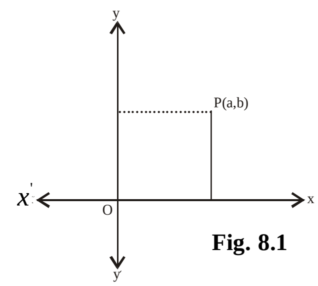


Fig. 8.1

Let P be any point whose coordinates are (a,b). We say that the complex number z = a + bi is represented by the point P (a , b) as shown in Fig. 8.1

If b = 0, then z is real and the point representing complex number z = a + 0i is denoted by (a, 0). This point (a, 0) lies on the x-axis.

So, xox' is called the real axis. In the Fig. 8.2 the point Q (a, 0) represent the complex number z = a + 0i.

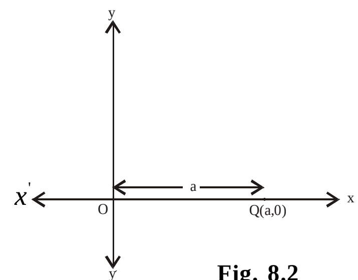


Fig. 8.2

If a = 0, then z is purely imaginary and the point representing complex number z = 0 + bi is denoted by (0, b). The point (0, b) lies on the y-axis.

So, yoy' is called the imaginary axis. In Fig.8.3, the point R (0, b) represents the complex number z = 0 + bi.

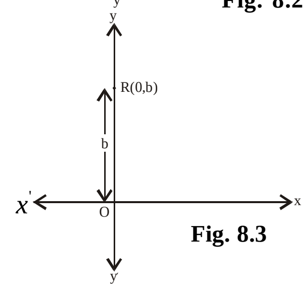


Fig. 8.3

The plane of two axes representing complex numbers as points is called the complex plane or Argand Plane.

The diagram which represents complex number in the Argand Plane is called Argand Diagram.

Example 8.5

Represent complex numbers 2 + 3i, –2–3i, 2–3i in the same Argand Plane

Solution: (a) 2+3i is represented by the point P (2, 3)

(b) –2–3i is represented by the point Q (–2,–3)

(c) 2–3i is represented by the point R (2, –3)

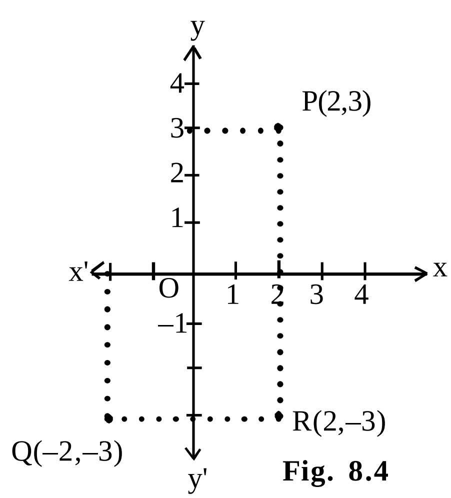


Fig. 8.4

8.5 MODULUS OF A COMPLEX NUMBER

We have learnt that any complex number z = a + bi can be represented by a point in the Argand Plane. How can we find the distance of the point from the origin? Let P(a, b) be a point in the plane representing a + bi. Draw perpendiculars PM and PL on x-axis and y-axis respectively. Let OM = a and MP = b. We have to find the distance of P from the origin.

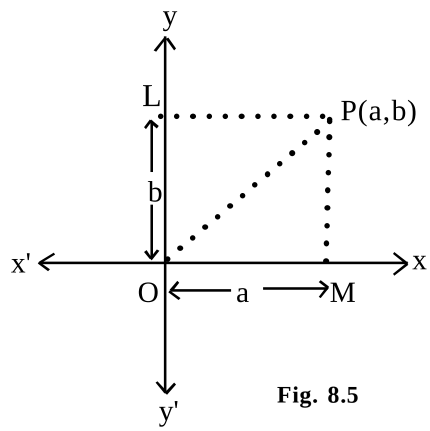


Fig. 8.5

$$\therefore \quad OP = \sqrt{OM^2 + MP^2}$$

$$= \sqrt{a^2 + b^2}$$

OP is called the modulus or absolute value of the complex number a + bi.

$\therefore$ Modulus of any complex number z such that $z = a + bi$, $a \in R$, $b \in R$ is denoted by $|z|$ and is given by $\sqrt{a^2 + b^2}$

$$\therefore \quad |z| = |a + ib| = \sqrt{a^2 + b^2}$$

8.5.1 Properties of Modulus

(a) $|z| = 0 \Leftrightarrow z = 0.$

Proof : Let $z = a + bi$, $a \in R$, $b \in R$

then $|z| = \sqrt{a^2 + b^2}$, $|z| = 0 \Leftrightarrow a^2 + b^2 = 0$

$\Leftrightarrow$ $a = 0$ and $b = 0$ (since a^2 and b^2 both are positive), $\Leftrightarrow z = 0$

(b) $|z| = |\bar{z}|$

Proof : Let z = a + bi then $|z| = \sqrt{a^2 + b^2}$

Now, $\bar{z} = a - bi \quad \therefore \quad |\bar{z}| = \sqrt{a^2 + (-b^2)} = \sqrt{a^2 + b^2}$

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Notes

Thus, $|z| = \sqrt{a^2 + b^2} = |\bar{z}|$ ...(i)

(c) $|z| = |-z|$

Proof : Let $z = a + bi$ then $|z| = \sqrt{a^2 + b^2}$

$-z = -a - bi$ then $|-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Thus, $|z| = \sqrt{a^2 + b^2} = |-z|$ ...(ii)

By (i) and (ii) it can be proved that $|z| = |-z| = |\bar{z}|$ ...(iii)

Now, we consider the following examples:

Example 8.6 Find the modulus of z, –z and $\bar{z}$ where z = 1 + 2i

Solution : z = 1 + 2i then –z = –1–2i and $\bar{z}$ = 1 –2i

$$|z| = \sqrt{1^2 + 2^2} = \sqrt{5},\ |-z| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

and $$|\bar{z}| = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}$$

Thus, $$|z| = |-z| = \sqrt{5} = |\bar{z}|$$

Example 8.7 Find the modulus of the complex numbers shown in an Argand Plane (Fig. 8.6)

Solution: (i) P(4, 3) represents the complex number z = 4 + 3i

$\therefore \quad |z| = \sqrt{4^2 + 3^2} = \sqrt{25}$

or $\quad |z| = 5$

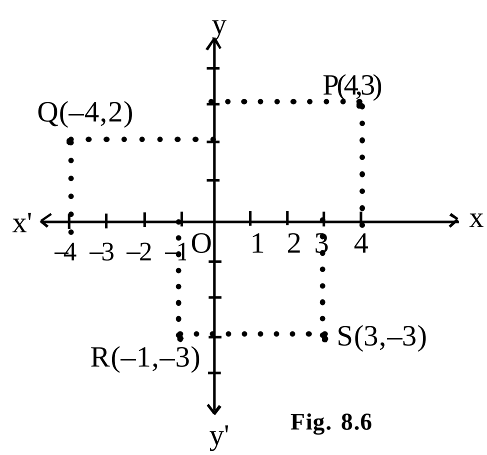


Fig. 8.6

(ii) Q(–4, 2) represents the complex number z = –4 + 2i

$\therefore \quad |z| = \sqrt{(-4)^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20}$

or $\quad |z| = 2\sqrt{5}$

(iii) R(–1, –3) represents the complex number $z = -1 - 3i$

$\therefore \quad |z| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{1+9}$

or $\quad |z| = \sqrt{10}$

(iv) S(3, –3) represents the complex number $z = 3 - 3i$

$\therefore \quad |z| = \sqrt{(3)^2 + (-3)^2} = \sqrt{9+9}$

or $\quad |z| = \sqrt{18} = 3\sqrt{2}$

CHECK YOUR PROGRESS 8.2

1. Find the conjugate of each of the following:

   (a) $-2i$ (b) $-5 - 3i$ (c) $-\sqrt{2}$ (d) $(-2 + i)^2$

2. Represent the following complex numbers on Argand Plane :

   (a) (i) $2 + 0i$ (ii) $-3 + 0i$ (iii) $0 - 0i$ (iv) $3 - 0i$

   (b) (i) $0 + 2i$ (ii) $0 - 3i$ (iii) $4i$ (iv) $-5i$

   (c) (i) $2 + 5i$ and $5 + 2i$ (ii) $3 - 4i$ and $-4 + 3i$

   (iii) $-7 + 2i$ and $2 - 7i$ (iv) $-2 - 9i$ and $-9 - 2i$

   (d) (i) $1 + i$ and $-1 - i$ (ii) $6 + 5i$ and $-6 - 5i$

   (iii) $-3 + 4i$ and $3 - 4i$ (iv) $4 - i$ and $-4 + i$

   (e) (i) $1 + i$ and $1 - i$ (ii) $-3 + 4i$ and $-3 - 4i$

   (iii) $6 - 7i$ and $6 + 7i$ (iv) $-5 - i$ and $-5 + i$

3. (a) Find the modulus of following complex numbers :

   (i) 3 (ii) $(i + 1)(2 - i)$ (iii) $2 - 3i$ (iv) $4 + \sqrt{5}i$

   (b) For the following complex numbers, verify that $|z| = |\bar{z}|$

   (i) $-6 + 8i$ (ii) $-3 - 7i$

   (c) For the following complex numbers, verify that $|z| = |-z|$

(i) $14 + i$ (ii) $11 - 2i$

(d) For the following complex numbers, verify that $|z| = |-z| = |\bar{z}|$

(i) $2 - 3i$ (ii) $-6 - i$ (iii) $7 - 2i$

8.6 EQUALITY OF TWO COMPLEX NUMBERS

Two complex numbers are equal if and only if their real parts and imaginary parts are respectively equal.

In general $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Example 8.8 For what value of x and y, $5x + 6yi$ and $10 + 18i$ are equal?

Solution : It is given that $5x + 6yi = 10 + 18i$

Comparing real and imaginary parts, we have

$5x = 10$ or $x = 2$

and $6y = 18$ or $y = 3$

For $x = 2$ and $y = 3$, the given complex numbers are equal.

8.7 ADDITION OF COMPLEX NUMBERS

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then their sum $z_1 + z_2$ is defined by

$$z_1 + z_2 = (a + c) + (b + d)i$$

For example, if $z_1 = 2 + 3i$ and $z_2 = -4 + 5i$,

then $z_1 + z_2 = [2 + (-4)] + [3 + 5]i = -2 + 8i$.

Example 8.9 Simplify

(i) $(3 + 2i) + (4 - 3i)$ (ii) $(2 + 5i) + (-3 - 7i) + (1 - i)$

Solution : (i) $(3 + 2i) + (4 - 3i) = (3 + 4) + (2 - 3)i = 7 - i$

(ii) $(2 + 5i) + (-3 - 7i) + (1 - i) = (2 - 3 + 1) + (5 - 7 - 1)i = 0 - 3i$

or $(2 + 5i) + (-3 - 7i) + (1 - i) = -3i$

8.7.1 Geometrical Represention of Addition of Two Complex Numbers

Let two complex numbers z_1 and z_2 be represented by the points P(a, b) and Q(c, d).

Their sum, $z_1 + z_2$ is represented by the point R (a + c, b + d) in the same Argand Plane.

Join OP, OQ, OR, PR and QR.

Draw perpendiculars PM, QN, RL from P, Q, R respectvely on X-axis.

Draw perpendicular PK to RL

In ΔQON

$ON = c$

and $QN = d.$

In ΔROL In ΔPOM

$RL = b + d$ $PM = b$

and $OL = a + c$ $OM = a$

Also $PK = ML = OL - OM$

$= a + c - a = c = ON$

$RK = RL - KL = RL - PM$

$= b + d - b = d = QN.$

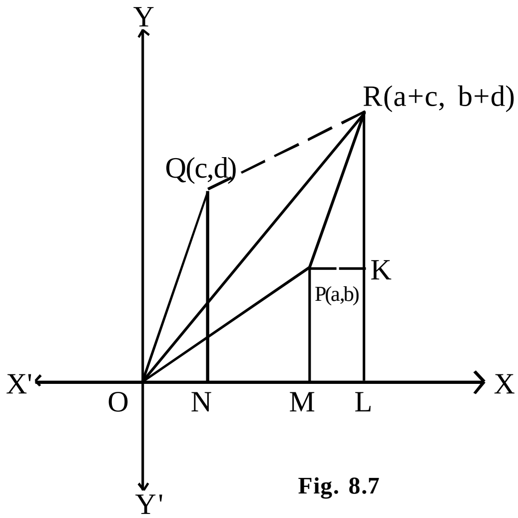


Fig. 8.7

In ΔQON and ΔRPK,

$ON = PK$, $QN = RK$ and $\angle QNO = \angle RKP = 90^o$

$\therefore$ $\Delta QON \cong \Delta RPK$

$\therefore$ $OQ = PR$ and $OQ \parallel PR$

$\Rightarrow$ OPRQ is a parallelogram and OR its diagonal.

Therefore, we can say that the sum of two complex numbers is represented by the diagonal of a parallelogram.

Example 8.10 Prove that $|z_1 + z_2| \le |z_1| + |z_2|$

Solution: We have proved that the sum of two complex numbers z_1 and z_2 represented by the diagonal of a parallelogram OPRQ (see fig. 8.8).

In ΔOPR, $OR \le OP + PR$

or $OR \le OP + OQ$ (since $OQ = PR$)

or $|z_1 + z_2| \le |z_1| + |z_2|$

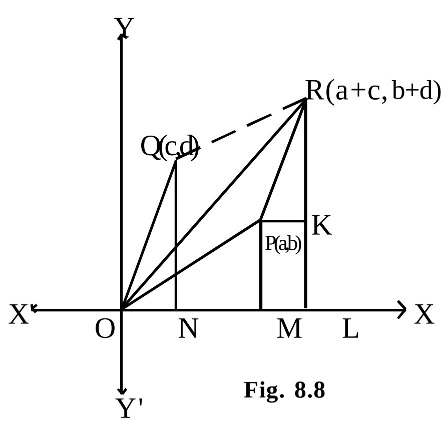


Fig. 8.8

Example 8.11 If $z_1 = 2 + 3i$ and $z_2 = 1 + i$,

verify that $| z_1 + z_2 | \le | z_1 | + | z_1 |$

Solution: $z_1 = 2 + 3i$ and $z_2 = 1 + i$ represented by the points (2, 3) and (1, 1) respectively. Their sum $(z_1 + z_2)$ will be represented by the point (2+1, 3+1) i.e. (3, 4)

Verification

$$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.6 \text{ approx.}$$

$$|z_2| = \sqrt{1^2 + 1^2} = \sqrt{2} = 1.41 \text{ approx.}$$

$$|z_1 + z_2| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$| z_1 | + | z_2 | = 3.6 + 1.41 = 5.01$$

$\therefore \quad | z_1 + z_2 | \le | z_1 | + | z_2 |$

8.7.2 Subtraction of Complex Numbers

Let two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ be represented by the points (a, b) and (c, d) respectively.

$\therefore \quad (z_1) - (z_2) = (a + bi) - (c + di) = (a - c) + (b - d)i$

which represents a point $(a - c, b - d)$

$\therefore$ The difference i.e. $z_1 - z_2$ is represented by the point $(a - c, b - d)$.

Thus, to subtract a complex number from another, we subtract corresponding real and imaginary parts separately.

Example 8.12 Find $z_1 - z_2$ if:

$z_1 = 3 - 4i, \qquad z_2 = -3 + 7i$

Solution: $z_1 - z_2 = (3 - 4i) - (-3 + 7i) = (3 - 4i) + (3 - 7i)$

$= (3 + 3) + (-4 - 7)i = 6 + (-11i) = 6 - 11i$

Examle 8.13 What should be added to i to obtain 5 + 4i?

Solution: Let $z = a + bi$ be added to i to obtain $5 + 4i$

$\therefore \quad i + (a + bi) = 5 + 4i$

or, $\quad a + (b + 1)i = 5 + 4i$

Equating real and imaginary parts, we have

$a = 5$ and $b + 1 = 4$ or $b = 3$, $\therefore$ $z = 5 + 3i$ is to be added to i to obtain $5 + 4i$

8.8 PROPERTIES: WITH RESPECT TO ADDITION OF COMPLEX NUMBERS.

1. Closure : The sum of two complex numbers will always be a complex number.

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, $a_1, b_1, a_2, b_2 \in R$.

Now, $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ which is again a complex nu mber.

This proves the closure property of complex numbers.

2. Commutative : If z_1 and z_2 are two complex numbers then

$$z_1 + z_2 = z_2 + z_1$$

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

Now $z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$

$= (a_2 + a_1) + (b_2 + b_1)i$ [commutative property of real numbers]

$= (a_2 + b_2 i) + (a_2 + b_1 i) = z_2 + z_1$

i.e. $z_1 + z_2 = z_2 + z_2$ Hence, addition of complex numbers is commutative.

3. Associative

If $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$ are three complex numbers, then

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

Now $z_1 + (z_2 + z_3) = (a_1 + b_1 i) + \{(a_2 + b_2 i) + (a_3 + b_3 i)\}$

$= (a_1 + b_1 i) + \{(a_2 + a_3) + (b_2 + b_3)i\} = \{a_1 + (a_2 + a_3)\} + \{b_1 + (b_2 + b_3)\}i$

$= \{(a_1 + a_2) + (b_1 + b_2)i\} + (a_3 + b_3 i) = \{(a_1 + b_1 i) + (a_2 + b_2 i)\} + (a_3 + b_3 i)$

$= (z_1 + z_2) + z_3$

Hence, the associativity property holds good in the case of addition of complex numbers.

4. Existence of Additive Identitiy

if $z = a + bi$ is any complex number, then $(a + bi) + (0 + 0i) = a + bi$

i.e. $(0 + 0i)$ is called the additive identity for $a + ib$.

5. Existence of Additive Inverse

For every complex number $a + bi$ there exists a unique complex number $-a - bi$ such that $(a + bi) + (-a - bi) = 0 + 0i$. $-a - ib$ is called the additive inverse of $a + ib$.

In general, additive inverse of a complex number is obtained by changing the signs of real and imaginary parts.

CHECK YOUR PROGRESS 8.3

1. Simplify:

(a) $\left(\sqrt{2}+\sqrt{5}i\right)+\left(\sqrt{5}-\sqrt{2}i\right)$ (b) $\dfrac{2+i}{3}+\dfrac{2-i}{6}$

(c) $(1+i)-(1-6i)$ (d) $\left(\sqrt{2}-\sqrt{3}i\right)-\left(-2-7i\right)$

2. If $z_1 = (5+i)$ and $z_2 = (6+2i)$, then:

(a) find $z_1 + z_2$ (b) find $z_2 + z_1$ (c) Is $z_1 + z_2 = z_2 + z_1$?

(d) find $z_1 - z_2$ (e) find $z_2 - z_1$ (f) Is $z_1 - z_2 = z_2 - z_1$?

3. If $z_1 = (1+i)$, $z_2 = (1-i)$ and $z_3 = (2+3i)$, then:

(a) find $z_1 + (z_2 + z_3)$ (b) find $(z_1 + z_2) + z_3$

(c) Is $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$? (d) find $z_1 - (z_2 - z_3)$

(e) find $(z_1 - z_2) - z_3$ (f) Is $z_1 - (z_2 - z_3) = (z_1 - z_2) - z_3$?

4. Find the additive inverse of the following:

(a) $12 - 7i$ (b) $4 - 3i$

5. What shoud be added to $(-15 + 4i)$ to obtain $(3 - 2i)$?

6. Show that $\left\{\overline{(3+7i)-(5+2i)}\right\} = \overline{(3+7i)} - \overline{(5+2i)}$

8.9 ARGUMENT OF A COMPLEX NUMBER

Let P(a, b) represent the complex number $z = a + bi$, $a \in R$, $b \in R$, and OP makes an angle θ with the positive direction of x-axis. Draw $PM \perp OX$, Let $OP = r$

In right Δ OMP, OM = a, MP = b

$\therefore$ $r\cos\theta = a$, $r\sin\theta = b$

Then $z = a + bi$ can be written as

$z = r(\cos\theta + i\sin\theta)$ ...(i)

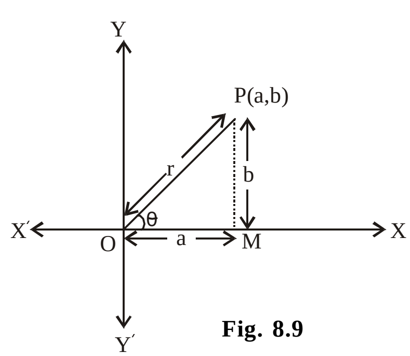


Fig. 8.9

where $r = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$ or $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

(i) is known as the polar form of the complex number z, and r and θ are respectively called the modulus and argument of the complex number.

8.10 MULTIPLICATION OF TWO COMPLEX NUMBERS

Two complex numbers can be multiplied by the usual laws of addition and multiplication as is done in the case of numbers.

Let $z_1 = (a + bi)$ and $z_2 = (c + di)$ then, $z_1.z_2 = (a + bi).(c + di)$

$= a(c + di) + bi(c + di)$

or $= ac + adi + bci + bdi^2$

or $= (ac - bd) + (ad + bc)i.$ [since $i^2 = -1$]

If (a + bi) and (c + di) are two complex numbers, their product is defined as the complex number $(ac - bd) + (ad + bc)i$

Example 8.14 Evaluate: $(1 + 2i)(1 - 3i)$,

Solution:

$$(1 + 2i)(1 - 3i) = \{1 - (-6)\} + (-3 + 2)i = 7 - i$$

8.10.1 Prove that

$$|z_1.z_2| = |z_1|.|z_2|$$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\therefore \quad |z_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$$

Similarly, $|z_2| = r_2$.

Now, $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\, r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)i]$

$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

[Since $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_1 - \sin\theta_1\sin\theta_2$ and
$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$]

$$|z_1.z_2| = r_1r_2\sqrt{\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)} = r_1r_2$$

$\therefore \quad |z_1.z_2| = r_1r_2 = |z_1|.|z_2|$

and argument of $z_1z_2 = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

Example 8.15 Find the modulus of the complex number (1 + i) (4 – 3i)

Solution: Let z = (1 + i) (4 – 3i)

then $\quad |z| = |(1+i)(4-3i)|$

$= |(1+i)|\,.\,|(4-3i)| \qquad$ (since $|z_1z_2| = |z_1|.|z_2|$)

But $\quad |1+i| = \sqrt{1^2+1^2} = \sqrt{2}, \; |4-3i| = \sqrt{4^2+(-3)^2} = 5$

$\therefore \quad |z| = \sqrt{2}.5 = 5\sqrt{2}$

8.11 DIVISION OF TWO COMPLEX NUMBERS

Division of complex numbers involves multiplying both numerator and denominator with the conjugate of the denominator. We will explain it through an example.

Let $\quad z_1 = a + bi$ **and** $z_2 = c+di$**, then.**

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}\left(c+di \neq 0\right)$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

(multiplying numerator and denominator with the conjugate of the denominator)

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Thus, $$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Example 8.16 Divide 3+i by 4–2i

Solution: $$\frac{3+i}{4-2i} = \frac{(3+i)(4+2i)}{(4-2i)(4+2i)}$$

Multiplying numerator and denominator by the conjugate of (4 – 2i) we get

$$= \frac{10+10i}{20} = \frac{1}{2} + \frac{1}{2}i$$

Thus, $\dfrac{3+i}{4-2i} = \dfrac{1}{2} + \dfrac{1}{2}i$

8.11.1 Prove that $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

Proof: $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$|z_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$$

Similarly, $|z_2| = r_2$

and $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Then, $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right]$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

Thus, $= \left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2}\sqrt{\cos^2(\theta_1-\theta_2) + \sin^2(\theta_1-\theta_2)} = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$

$\therefore$ Argument of $\left|\dfrac{z_1}{z_2}\right| = \theta_1 - \theta_2 = arg(z_1) - arg(z_2)$

Example 8.17 Find the modulus of the complex number $\frac{2+i}{3-i}$

Solution : Let $z = \frac{2+i}{3-i}$

$$\therefore \quad |z| = \left|\frac{2+i}{3-i}\right| = \frac{|2+i|}{|3-i|} \quad \left(\text{since} \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right)$$

$$= \frac{\sqrt{2^2+1^2}}{\sqrt{3^2+(-1)^2}} = \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}} \qquad \therefore \quad |z| = \frac{1}{\sqrt{2}}$$

8.12 PROPERTIES OF MULTIPLICATION OF TWO COMPLEX NUMBERS

1. Closure If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then their product z_1z_2 is also a complex number.

2. Cummutative If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then $z_1z_2 = z_2z_1$.

3. Associativity If $z_1 = (a + bi)$, $z_2 = c + di$ and $z_3 = (e + fi)$ then

$$z_1(z_2.z_3) = (z_1.z_3).z_3$$

4. Existence of Multiplicative Identity: For every non-zero complex number $z_1 = a + bi$ there exists a unique complex number $(1 + 0i)$ such that

$$(a + bi).(1 + 0i) = (1 + 0i)\,(a + bi) = a + bi$$

Let $z_1 = x + yi$ be the multipicative identity of $z = a + bi$ Then $z.z_1 = z$.

i.e. $(a + bi)\,(x + yi) = a + bi$

or $(ax - by) + (ay + bx)i = a + bi$

or $ax - by = a$ and $ay + bx = b$

pr $x = 1$ and $y = 0$, i.e. $z_1 = x + yi = 1 + 0i$ is the multiplicative identity.

The complex number $1 + 0i$ is the identity for multiplication.

5. Existence of Multiplicative inverse: Multiplicative inverse is a complex number that when multiplied to a given non-zero complex munber yields one. In other words, for every non-zero complex number $z = a + bi$, there exists a unique complex number $(x + yi)$ such that their product is $(1 + 0i)$. i.e. $(a + bi)\,(x + yi) = 1 + 0i$ or $(ax - by) + (bx + ay)i = 1 + 0i$

Equating real and imaging parts, we have

$ax - by = 1$ and $bx + ay = 0$

By cross multiplication

$$\frac{x}{a} = \frac{y}{-b} = \frac{1}{a^2 + b^2} \Rightarrow \quad x = \frac{a}{a^2+b^2} = \frac{\text{Re}(z)}{|z|^2} \text{ and } y = \frac{-b}{a^2+b^2} = -\frac{\text{Im}(z)}{|z|^2}$$

Thus, the multiplicative inverse of a non-zero compelx number $z = (a + bi)$ is

$$x + yi = \left(\frac{\text{Re}(z)}{|z|^2} - \frac{\text{Im}(z)}{|z|^2} i\right) = \frac{\bar{z}}{|z|^2}$$

Example 8. 18 Find the multiplication inverse of $2 - 4i$.

Solution: Let $z = 2 - 4i$ **We have,** $\bar{z} = 2 + 4i$ and $|z|^2 = \left|2^2 + (-4)^2\right| = 20$

$\therefore$ Required multiplicative inverse is $\dfrac{\bar{z}}{|z|^2} = \dfrac{2+4i}{20} = \dfrac{1}{10} + \dfrac{1}{5}i$

6. Distributive Property of Multiplication over Addition

Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$

Then $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

CHECK YOUR PROGRESS 8.4

1. Simplify each of the following:

   (a) $(1+2i)(\sqrt{2} - i)$ (b) $(\sqrt{2} + i)^2$ (c) $(3 + i)(1 - i)(-1 + i)$

   (d) $(2 + 3i) \div (1 - 2i)$ (e) $(1 + 2i) \div (1 + i)$ (f) $(1 + 0i) \div (3 + 7i)$

2. Compute multiplicative inverse of each of the following complex numbers:

   (a) $3 - 4i$ (b) $\sqrt{3} + 7i$ (c) $\dfrac{3+5i}{2-3i}$

3. If $z_1 = 4 + 3i$, $z_2 = 3 - 2i$ and $z_3 = i + 5$, verify that $z_1(z_2 + z_3) = z_1 z_2 + z_2 z_3$.

4. If $z_1 = 2 + i$, $z_2 = -2 + i$ and $z_3 = 2 - i$ then verify that $(z_1.z_2)z_3 = z_1(z_2.z_3)$

8.13 SQUARE ROOT OF A COMPLEX NUMBER

Let $a + ib$ be a complex number and $x + iy$ be its square root

i.e., $\sqrt{a+ib} = x + iy$

$\Rightarrow \quad a + ib = x^2 - y^2 + 2ixy$

Equating real and imaginary parts we have

$x^2 - y^2 = a$ ....(1)

and $2xy = b$ ...(2)

Using the algebraic identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$ we get

$(x^2 + y^2)^2 = a^2 + b^2 \Rightarrow x^2 + y^2 = \sqrt{a^2+b^2}$ ...(3)

From equations (1) and (3), we get

$$\left.\begin{aligned} 2x^2 &= \sqrt{a^2+b^2} + a \Rightarrow x = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)} \\ \text{and } 2y^2 &= \sqrt{a^2+b^2} - a \Rightarrow y = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)} \end{aligned}\right\} \quad \text{...(4)}$$

Out of these four pairs of values of x and y (given by equation (4) we have to choose the values which satisfy (1) and (2) both.

From (2) if b is + ve then both x and y should be of same sign and in that case

$$\sqrt{a+ib} = \sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)} + i\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}-a)}$$

and $$-\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)} - i\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}-a)}$$

and if b is –ve then x and y should be of opposite sign. Therefore in that case

$$\sqrt{a+ib} = -\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)} + i\sqrt{\frac{1}{2}(\sqrt{b^2+b^2}-a)}$$

and $$\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)} - i\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}$$

Hence $a + ib$ has two square roots in each case and the two square roots just differ in sign.

Example 8.19 Find the square root of $7 + 24i$

Solution : Let $\sqrt{7+24i} = a + ib$ ...(1)

Squaring both sides, we get $7 + 24i = a^2 - b^2 + 2\,iab$

Comparing real and imaginary parts, we have $a^2 - b^2 = 7$ ...(2)

and $2\ ab = 24 \Rightarrow ab = 12$ ....(3)

Now $(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$

$\Rightarrow \quad (a^2+b^2)^2 = 49 + 4\times 144$

$\Rightarrow \quad (a^2+b^2)^2 = 625$

$\Rightarrow \quad a^2 + b^2 = 25$ ...(4)

Solving (2) and (4), we get $2a^2 = 32 \Rightarrow a^2 = 16 \Rightarrow a = \pm 4$

and $2b^2 = 18 \Rightarrow b^2 = 9 \Rightarrow b = \pm 3$

From (3), $ab = 12$ which is +ve $\Rightarrow$ a and b should be of same sign

$\therefore$ Either $a = 4,\ b = 3$ or $= -4,\ b = -3$

Hence, the two square roots of $7 + 24i$ are $4 + 3i$ and $-4 - 3i$

Example 8.20 Find the square root of $-i$

Solution : Let $\sqrt{-i} = a + ib$

$\Rightarrow \quad -i = a^2 - b^2 + 2\ iab$ ...(1)

Equating real and imaginary parts of (1), we get $a^2 - b^2 = 0$ ...(2)

and $2\ ab = -1 \Rightarrow ab = -\dfrac{1}{2}$ ...(3)

Now, $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2 = 0 + 4\left(\dfrac{1}{4}\right) = 1$

$\Rightarrow \quad a^2 + b^2 = 1$ ...(4)

From (2) and (4), $2b^2 = 1 \Rightarrow b^2 = \dfrac{1}{2} \Rightarrow b = \pm\dfrac{1}{\sqrt{2}}$

and $2a^2 = 1 \Rightarrow a^2 = \dfrac{1}{2} \Rightarrow a = \pm\dfrac{1}{\sqrt{2}}$.

Equation (3) suggests; that a and b should be of opposite sign therefore two square roots of $-i$ are $\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$ and $-\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$

CHECK YOUR PROGRESS 8.5

Find the square root of the following complex numbers :

(*i*) $-21 - 20i$ (ii) $-4 - 3i$ (iii) $-48 - 14i$

LET US SUM UP

- $z = a + bi$ is a complex number in the standard form where $a, b \in R$ and $i = \sqrt{-1}$.
- Any higher powers of 'i' can be expressed in terms of one of the four values i, –1, –i, 1.
- Conjugate of a complex number $z = a + bi$ is $a - bi$ and is denoted by $\bar{z}$.
- Modulus of a complex number $z = a + bi$ is $\sqrt{a^2 + b^2}$ i.e. $|z| = |a + bi| = \sqrt{a^2 + b^2}$

  (a) $|z| = 0 \Leftrightarrow z = 0$ (b) $|z| = |\bar{z}|$ (c) $|z_1 + z_2| \le |z_1| + |z_2|$
- $z = r(\cos\theta + i\sin\theta)$ represents the polar form of a complex number $z = a + bi$ where

  $r = \sqrt{a^2 + b^2}$ is modulus and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ is its argument.
- Multiplicative inverse of a complex number $z = a + bi$ is $\frac{\bar{z}}{|z|^2}$
- Square root of a complex number is also a complex number.
- Two square roots of a complex number only differ in sign.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=MEuPzvh0roM
http://www.youtube.com/watch?v=kpywdu1afas
http://www.youtube.com/watch?v=bPqB9a1uk_8
http://www.youtube.com/watch?v=SfbjqVyQljk
http://www.youtube.com/watch?v=tvXRaZbIjO8
http://www.youtube.com/watch?v=cWn6g8Qqvs4
http://www.youtube.com/watch?v=Z8j5RDOibV4
http://www.youtube.com/watch?v=dbxJ6LD0344

TERMINAL EXERCISE

1. Find real and imaginary parts of each of the following:

   (a) $2 + 7i$ (b) $3 + 0i$ (c) $-\frac{1}{2}$ (d) $5i$ (e) $\frac{1}{2 + 3i}$

2. Simplify each of the following:

(a) $\sqrt{-3}.\sqrt{-27}$ (b) $\sqrt{-3}\sqrt{-4}\sqrt{-72}$ (c) $3i^{15} - 5i^8 + 1$

3. Form the complex numbers whose real and imaginary parts are given in the form of ordered pairs.

(a) $z(3, -5)$ (b) $z(0, -4)$ (c) $z(8, \pi)$

4. Find the conjugate of each of the following:

(a) $1 - 2i$ (b) $-1 - 2i$ (c) $6 - \sqrt{2}i$ (d) $4i$ (e) $-4i$

5. Find the modulus of each of the following:

(a) $1 - i$ (b) $3 + \pi i$ (c) $-\frac{3}{2}i$ (d) $-2 + \sqrt{3}i$

6. Express $7i^{17} - 6i^6 + 3i^3 - 2i^2 + 1$ in the form of a + bi.

7. Find the values of x and y if:

(a) $(x - yi) + 7 - 2i = 9 - i$ (b) $2x + 3yi = 4 - 9i$ (c) $x - 3yi = 7 + 9i$

8. Simplify each of the following:

(a) $(3 + i) - (1 - i) + (-1 + i)$ (b) $\left(\frac{1}{7}+i\right)-\left(\frac{2}{7}-i\right)+\left(\frac{3}{7}-2i\right)$

9. Write additive inverse and multiplicative inverse of each of the following:

(a) $3 - 7i$ (b) $11 - 2i$ (c) $\sqrt{3} + 2i$ (d) $1 - \sqrt{2}i$ (e) $\frac{1+5i}{1-i}$

10. Find the modulus of each of the following complex numbers:

(a) $\frac{1+i}{3-i}$ (b) $\frac{5+2i}{\sqrt{2}+\sqrt{3}i}$ (c) $(3 + 2i)(1 - i)$ (d) $(1 - 3i)(-2i^3 + i^2 + 3)$

11. For the following pairs of complex numbers verify that $|z_1 z_2| = |z_2||z_1|$

(a) $z_1 = 3 - 2i$, $z_2 = 1 - 5i$ (b) $z_1 = 3 - \sqrt{7}i$, $z_2 = \sqrt{3} - i$

12. For the following pairs of complex numbers verify that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(a) $z_1 = 1 + 3i$, $z_2 = 2 + 5i$ (b) $z_1 = -2 + 5i$, $z_2 = 3 - 4i$

13. Find the square root of $2 + 3i$

14. Find the square root of $-2+2\sqrt{-3}$.

15. Find the square root of i.

ANSWERS

CHECK YOUR PROGRESS 8.1

1 (a) $3\sqrt{3}i$ (b) $-3i$ (c) $\sqrt{13}i$

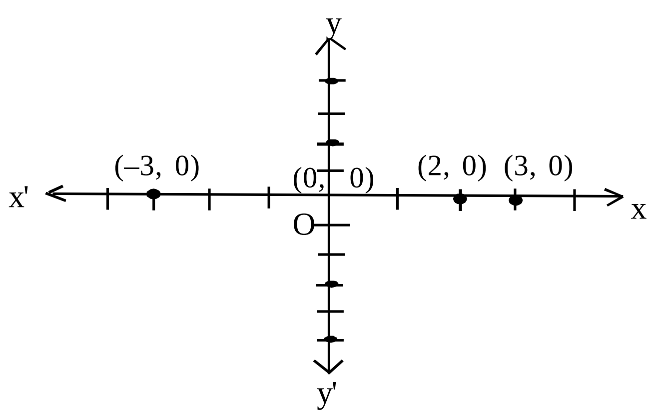
2. (a) $5 + 0i$ (b) $0 - 3i$ (c) $0 + 0i$

3. $12 - 4i$

CHECK YOUR PROGRESS 8.2

1 (a) $2i$ (b) $-5 + 3i$ (c) $-\sqrt{2}$ (d) $3 + 4i$

2. **(a)**



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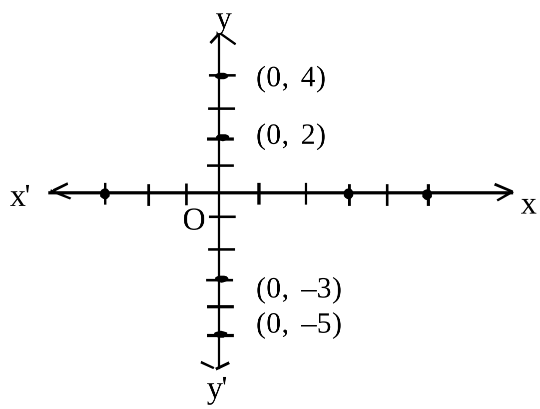

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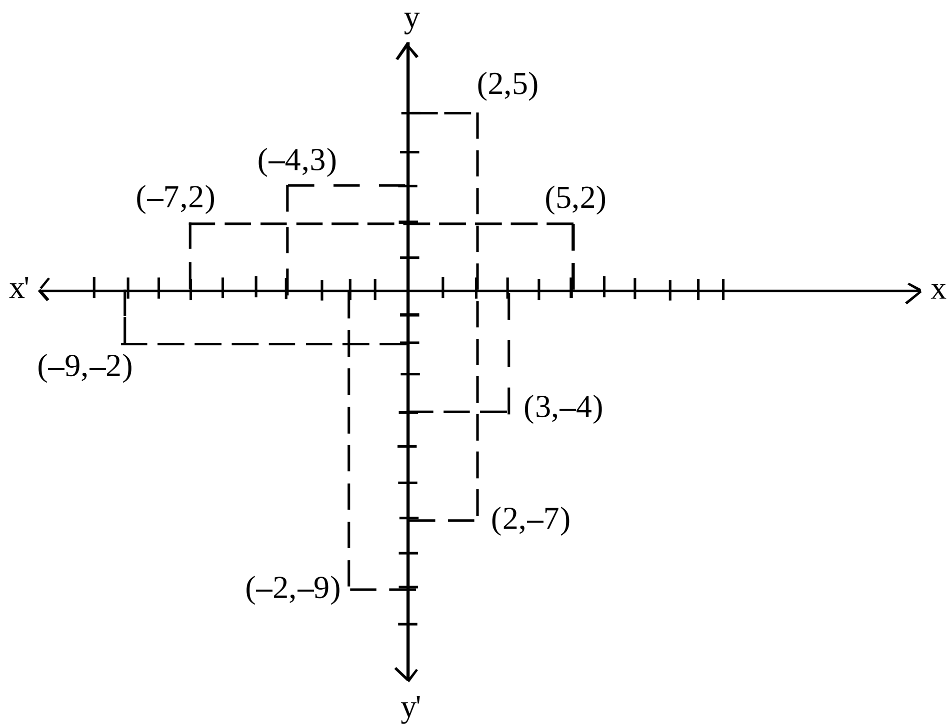

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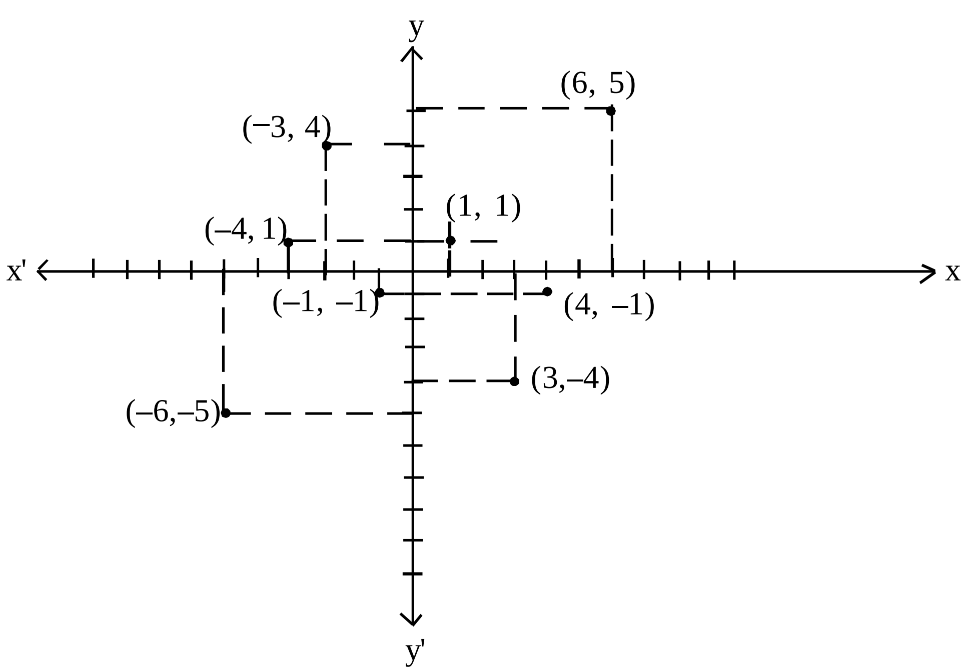

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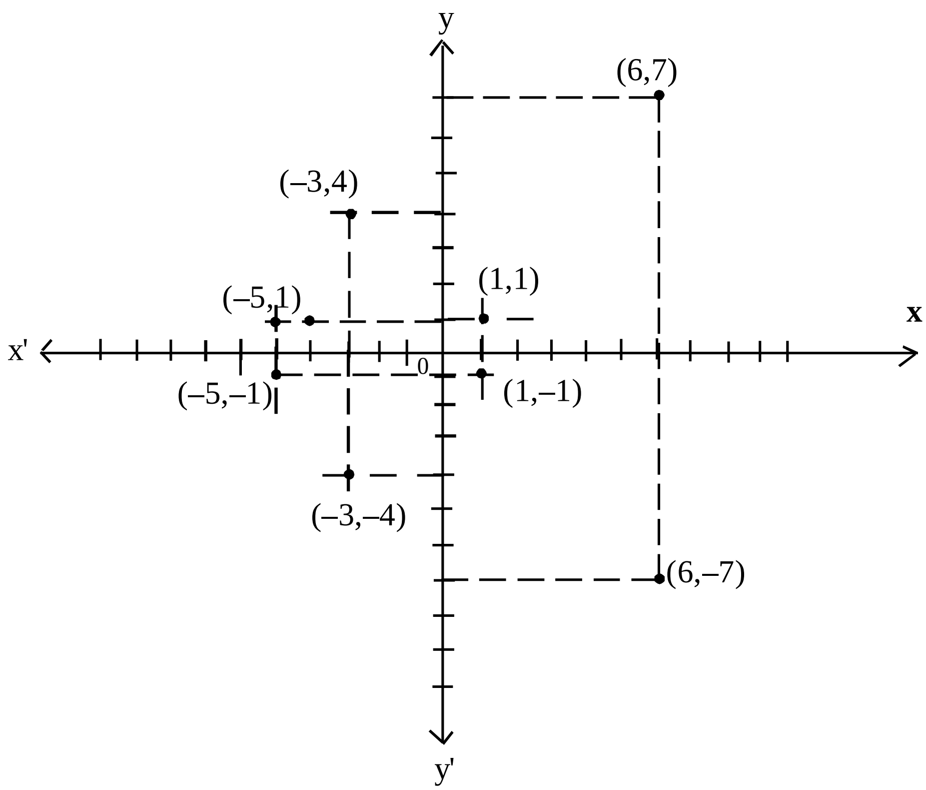


3. (a) (i) 3 (ii) $\sqrt{10}$ (iii) $\sqrt{13}$ (iv) $\sqrt{21}$

CHECK YOUR PROGRESS 8.3

1. (a) $\left(\sqrt{2}+\sqrt{5}\right)+\left(\sqrt{5}-\sqrt{2}\right)i$ (b) $\frac{1}{6}(6+i)$

   (c) 7i (d) $\sqrt{2}\left(\sqrt{2}+1\right)+(7-\sqrt{3})$

2. (a) 11 + 3i (b) 11 + 3i (c) Yes

   (d) –1 – i (e) 1 + i (f) No

3. (a) 4 + 3i (b) 4 + 3i (c) Yes

   (d) 2 + 5i (e) – 2 – i (f) No.

4. (a) –12 + 7i (b) –4 + 3i

5. 18 – 6i

CHECK YOUR PROGRESS 8.4

1. (a) $\left(\sqrt{2}+2\right)+\left(2\sqrt{2}-1\right)i$ (b) $1+2\sqrt{2}i$

(c) $-2 + 6i$ (d) $\frac{1}{\sqrt{5}}(-4 + 7i)$

(e) $\frac{1}{2}(3 + i)$ (f) $\frac{1}{58}(3 - 7i)$

2. (a) $\frac{1}{25}(3 + 4i)$ (b) $\frac{1}{52}(\sqrt{3} - 7i)$

(c) $\frac{1}{34}(-9 - 19i)$

CHECK YOUR PROGRESS 8.5

(i) $2 - 5i, -2 + 5i$ (ii) $\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}}i, \frac{-1}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$

(iii) $1 - 7i, -1 + 7i$

TERMINAL EXERCISE

1. (a) 2, 7 (b) 3, 0

(c) $-\frac{1}{2}$, 0 (d) 0, 5

(e) $\frac{2}{13}, -\frac{3}{13}$

2. (a) -9 (b) $-12\sqrt{6}i$

(c) $-4 - 3i$

3. (a) $3 - 5i$ (b) $0 - 4i$

(c) $8 + \pi i$

4. (a) $1 + 2i$ (b) $-1 + 2i$

(c) $6 + \sqrt{2}i$ (d) $-4i$

(e) $4i$

5. (a) $\sqrt{2}$ (b) $\sqrt{9+\pi^2}$

(c) $\frac{3}{2}$ (d) $\sqrt{7}$

6. $9 + 4i$

7. (a) $x = 2, \quad y = -1$ (b) $x = 2, \; y = -3$

(c) $x = 7, \; y = -3$

8. (a) $1 + 3i$ (b) $\frac{2}{7} + 0i$

9. (a) $-3 + 7i, \; \frac{1}{58}(3+7i)$ (b) $-11+2i, \; \frac{1}{125}(-11+2i)$

(c) $-\sqrt{3}-2i, \; \frac{1}{7}\left(\sqrt{3}-2i\right)$ (d) $-1+\sqrt{2}i, \; \frac{1}{3}(1+\sqrt{2}i)$

(e) $2-3i, \; \frac{1}{13}\left(2+3i\right)$

10. (a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{5}\sqrt{145}$

(c) $\sqrt{26}$ (d) $4\sqrt{5}$

13. $\pm\left(\sqrt{\frac{\sqrt{13}+2}{2}}+\sqrt{\frac{\sqrt{13}-2}{2}}i\right)$

14. $1+\sqrt{3}i, -1-\sqrt{3}i$

15. $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i, \frac{-1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

9

QUADRATIC EQUATIONS AND LINEAR INEQUALITIES

Recall that an algebraic equation of the second degree is written in general form as $ax^2 + bx + c = 0,\ a \neq 0$. It is called a quadratic equation in *x*. The coefficient 'a' is the first or leading coefficient, 'b' is the second or middle coefficient and 'c' is the constant term (or third coefficient). For example, $7x^2 + 2x + 5 = 0$, $\frac{5}{2}x^2 + \frac{1}{2}x + 1 = 0$,

$3x^2 - x = 0$, $x^2 + \frac{1}{2} = 0$, $\sqrt{2}\,x^2 + 7x = 0$, are all quadratic equations.

Some times, it is not possible to translate a word problem in the form of an equation. Let us consider the following situation:

Alok goes to market with Rs. 30 to buy pencils. The cost of one pencil is Rs. 2.60. If x denotes the number of pencils which he buys, then he will spend an amount of Rs. 2.60x. This amount cannot be equal to Rs. 30 as x is a natural number. Thus.

$$2.60\,x < 30 \qquad \ldots \text{(i)}$$

Let us consider one more situation where a person wants to buy chairs and tables with Rs. 50,000 in hand. A table costs Rs. 550 while a chair costs Rs. 450. Let x be the number of chairs and y be the number of tables he buys, then his total cost = Rs.(550 x + 450 y)

Thus, in this case we can write, $550x + 450y \leq 50{,}000$

or $\quad 11x + 9y \leq 1000 \qquad \ldots \text{(ii)}$

Statement (i) involves the sign of inequality '<' and statement (ii) consists of two statements: $11x+9y < 1000$, $11x+9y = 1000$ in which the first one is not an equation: Such statements are called Inequalities. In this lesson, we will discuss linear inequalities and their solution.

We will also discuss how to solve quadratic equations with real and complex coefficients and establish relation between roots and coefficients.

OBJECTIVES

After studying this lesson, you will be able to:

- solve a quadratic equation with real coefficients by factorization and by using quadratic formula;

- find relationship between roots and coefficients;
- form a quadratic equation when roots are given;
- differentiate between a linear equation and a linear inequality;
- state that a planl region represents the solution of a linear inequality;
- represent graphically a linear inequality in two variables;
- show the solution of an inequality by shading the appropriate region;
- solve graphically a system of two or three linear inequalities in two variables;

EXPECTED BACKGROUND KNOWLEDGE

- Real numbers
- Quadratic Equations with real coefficients.
- Solution of linear equations in one or two variables.
- Graph of linear equations in one or two variables in a plane.
- Graphical solution of a system of linear equations in two variables.

9.1 ROOTS OF A QUADRATIC EQUATION

The value which when substituted for the variable in an equation, satisfies it, is called a root (or solution) of the equation.

If α be one of the roots of the quadratic equation

$$ax^2 + bx + c = 0,\ a \neq 0 \qquad \text{... (i)}$$

then $a\alpha^2 + b\alpha + c = 0$

In other words, $x - \alpha$ is a factor of the quadratic equation (i)

In particular, consider a quadratic equation $x^2 + x - 6 = 0$ ...(ii)

If we substitute x = 2 in (ii), we get L.H.S = $2^2 + 2 - 6 = 0$

$\therefore$ L.H.S = R.H.S.

Again put x = -3 in (ii), we get L.H.S. = $(-3)^2 - 3 - 6 = 0$

$\therefore$ L.H.S = R.H.S.

Again put x = -1 in (ii) ,we get L.H.S = $(-1)^2 + (-1) - 6 = -6 \neq 0$ = R.H.S.

$\therefore$ $x = 2$ and $x = -3$ are the only values of x which satisfy the quadratic equation (ii)

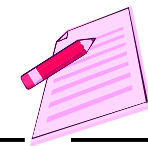
There are no other values which satisfy (ii)

$\therefore$ $x = 2$, $x = -3$ are the only two roots of the quadratic equation (ii)

Note: *If α, β be two roots of the quadratic equation*

$$ax^2 + bx + c = 0,\ a \neq 0 \qquad \text{...(A)}$$

then $(x - \alpha)$ and $(x - \beta)$ will be the factors of (A). The given quadratic equation can be written in terms of these factors as $(x - \alpha)(x - \beta) = 0$



Notes

9.2 SOLVING QUADRATIC EQUATION BY FACTORIZATION

Recall that you have learnt how to factorize quadratic polynomial of the form $p(x) = ax^2 + bx + c,\ a \neq 0,$ by splitting the middle term and taking the common factors. Same method can be applied while solving a quadratic equation by factorization.

If $x - \frac{p}{q}$ and $x - \frac{r}{s}$ are two factors of the quadratic equation

$ax^2 + bx + c = 0$, $a \neq 0$ then $\left(x - \frac{p}{q}\right)\left(x - \frac{r}{s}\right) = 0$

$\therefore$ either $x = \frac{p}{q}$ or, $x = \frac{r}{s}$

$\therefore$ The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{p}{q}, \frac{r}{s}$

Example 9.1 Using factorization method, solve the quadratic equation: $6x^2 + 5x - 6 = 0$

Solution: The given quadratic equation is $6x^2 + 5x - 6 = 0$ ... (i)

Splitting the middle term, we have $6x^2 + 9x - 4x - 6 = 0$

or, $3x(2x + 3) - 2(2x + 3) = 0$ or, $(2x + 3)(3x - 2) = 0$

$\therefore$ Either $2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$ or, $3x - 2 = 0 \Rightarrow x = \frac{2}{3}$

$\therefore$ Two roots of the given quadratic equation are $-\frac{3}{2}, \frac{2}{3}$

Example 9.2 Using factorization method, solve the quadratic equation:

$$3\sqrt{2}x^2 + 7x - 3\sqrt{2} = 0$$

Solution: Splitting the middle term, we have $3\sqrt{2}x^2 + 9x - 2x - 3\sqrt{2} = 0$

or, $3x(\sqrt{2}x + 3) - \sqrt{2}(\sqrt{2}x + 3) = 0$ or, $(\sqrt{2}x + 3)(3x - \sqrt{2}) = 0$

$\therefore$ Either $\sqrt{2}x + 3 = 0 \Rightarrow x = -\frac{3}{\sqrt{2}}$ or, $3x - \sqrt{2} = 0 \Rightarrow x = \frac{\sqrt{2}}{3}$

$\therefore$ Two roots of the given quadratic equation are $-\frac{3}{\sqrt{2}}, \frac{\sqrt{2}}{3}$

Example 9.3 Using factorization method, solve the quadratic equation:

$$(a + b)^2 x^2 + 6(a^2 - b^2)x + 9(a - b)^2 = 0$$

Solution: The given quadratic equation is $(a + b)^2 x^2 + 6(a^2 - b^2)x + 9(a - b)^2 = 0$

Splitting the middle term, we have

$(a + b)^2 x^2 + 3(a^2 - b^2)x + 3(a^2 - b^2)x + 9(a - b)^2 = 0$

or, $(a + b)x\{(a + b)x + 3(a - b)\} + 3(a - b)\{(a + b)x + 3(a - b)\} = 0$

or, $\{(a + b)x + 3(a - b)\}\{(a + b)x + 3(a - b)\} = 0$

$\therefore$ either $(a + b)x + 3(a - b) = 0 \Rightarrow x = \frac{-3(a-b)}{a+b} = \frac{3(b-a)}{a+b}$

or, $(a + b)x + 3(a - b) = 0 \Rightarrow x = \frac{-3(a-b)}{a+b} = \frac{3(b-a)}{a+b}$

The equal roots of the given quadratic equation are $\frac{3(b-a)}{a+b}, \frac{3(b-a)}{a+b}$

Alternative Method

The given quadratic equation is $(a + b)^2 x^2 + 6(a^2 - b^2)x + 9(a - b)^2 = 0$

This can be rewritten as

$\{(a + b)x\}^2 + 2.(a + b)x . 3(a - b) + \{3(a - b)\}^2 = 0$

or, $\{(a + b)x + 3(a - b)\}^2 = 0$ or, $x = -\frac{3(a-b)}{a+b} = \frac{3(b-a)}{a+b}$

$\therefore$ The quadratic equation has equal roots $\frac{3(b-a)}{a+b}, \frac{3(b-a)}{a+b}$

CHECK YOUR PROGRESS 9.1

1. Solve each of the following quadratic equations by factorization method:

(i) $\sqrt{3}x^2 + 10x + 8\sqrt{3} = 0$ (ii) $x^2 - 2ax + a^2 - b = 0$

(iii) $x^2 + \left(\frac{ab}{c} - \frac{c}{ab}\right)x - 1 = 0$ (iv) $x^2 - 4\sqrt{2}x + 6 = 0$

9.3 SOLVING QUADRATIC EQUATION BY QUADRATIC FORMULA

Recall the solution of a standard quadratic equation

$ax^2 + bx + c = 0, a \neq 0$ by the **"Method of Completing Squares"**

Roots of the above quadratic equation are given by

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{D}}{2a}, \qquad = \frac{-b-\sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is called the discriminant of the quadratic equation.

For a quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ **if**

(i) **D>0, the equation will have two real and unequal roots**

(ii) **D=0, the equation will have two real and equal roots and both roots are equal to** $-\dfrac{b}{2a}$

(iii) **D<0, the equation will have two conjugate complex (imaginary) roots.**

Example 9.4 Examine the nature of roots in each of the following quadratic equations and also verify them by formula.

(i) $x^2 + 9x + 10 = 0$ (ii) $9y^2 - 6\sqrt{2}y + 2 = 0$

(iii) $\sqrt{2}\,t^2 - 3t + 3\sqrt{2} = 0$

Solution:

(i) The given quadratic equation is $x^2 + 9x + 10 = 0$

Here, $a = 1$, $b = 9$ and $c = 10$

$\therefore$ $D = b^2 - 4ac = 81 - 4.1.10 = 41 > 0.$

$\therefore$ The equation will have two real and unequal roots

Verification: By quadratic formula, we have $x = \dfrac{-9 \pm \sqrt{41}}{2}$

$\therefore$ The two roots are $\dfrac{-9+\sqrt{41}}{2}, \dfrac{-9-\sqrt{41}}{2}$ which are real and unequal.

(ii) The given quadratic equation is $9y^2 - 6\sqrt{2}\,y + 2 = 0$

Here, $D = b^2 - 4ac = (-6\sqrt{2})^2 - 4.9.2 = 72 - 72 = 0$

$\therefore$ The equation will have two real and equal roots.

Verification: By quadratic formula, we have $y = \dfrac{6\sqrt{2} \pm \sqrt{0}}{2.9} = \dfrac{\sqrt{2}}{3}$

$\therefore$ The two equal roots are $\dfrac{\sqrt{2}}{3}, \dfrac{\sqrt{2}}{3}$.

(iii) The given quadratic equation is $\sqrt{2}\,t^2 - 3t + 3\sqrt{2} = 0$

Here, $D = (-3)^2 - 4.\sqrt{2}.3\sqrt{2} = -15 < 0$

$\therefore$ The equation will have two conjugate complex roots.

MODULE-III
Algebra -I

Notes

Verification: By quadratic formula, we have $t = \frac{3 \pm \sqrt{-15}}{2\sqrt{2}} = \frac{3 \pm \sqrt{15}i}{2\sqrt{2}}$, where $i = \sqrt{-1}$

$\therefore$ Two conjugate complex roots are $\frac{3+\sqrt{15}\,i}{2\sqrt{2}}, \frac{3-\sqrt{15}\,i}{2\sqrt{2}}$

Example 9.5 Prove that the quadratic equation $x^2 + px - 1 = 0$ has real and distinct roots for all real values of p.

Solution: Here, $D = p^2 + 4$ which is always positive for all real values of p.

$\therefore$ The quadratic equation will have real and distinct roots for all real values of p.

Example 9.6 For what values of k the quadratic equation

$(4k+1)x^2 + (k+1)x + 1 = 0$ will have equal roots ?

Solution: The given quadratic equation is $(4k+1)x^2 + (k+1)x + 1 = 0$

Here, $D = (k+1)^2 - 4.(4k+1).1$

For equal roots, $D = 0 \therefore (k+1)^2 - 4(4k+1) = 0$

$\Rightarrow k^2 - 14k - 3 = 0$

$\therefore k = \frac{14 \pm \sqrt{196+12}}{2}$ or $k = \frac{14 \pm \sqrt{208}}{2} = 7 \pm 2\sqrt{13}$ or $7 + 2\sqrt{13}, 7 - 2\sqrt{13}$

which are the required values of k.

Example 9.7 Prove that the roots of the equation

$x^2(a^2+b^2) + 2x(ac+bd) + (c^2+d^2) = 0$ are imaginary. But if $ad = bc$, roots are real and equal.

Solution: The given equation is $x^2(a^2+b^2) + 2x(ac+bd) + (c^2+d^2) = 0$

Discriminant $= 4(ac+bd)^2 - 4(a^2+b^2)(c^2+d^2)$

$= 8abcd - 4(a^2d^2 + b^2c^2) = -4(-2abcd + a^2d^2 + b^2c^2)$

$= -4(ad - bc)^2, <0$ for all a, b, c, d

$\therefore$ The roots of the given equation are imaginary.

For real and equal roots, discriminant is equal to zero.

$\Rightarrow -4(ad - bc)^2 = 0$ or, $ad = bc$

Hence, if ad=bc, the roots are real and equal.

CHECK YOUR PROGRESS 9.2

1. Solve each of the following quadratic equations by quadratic formula:

(i) $2x^2 - 3x + 3 = 0$ (ii) $-x^2 + \sqrt{2}x - 1 = 0$

(iii) $-4x^2 + \sqrt{5}x - 3 = 0$ (iv) $3x^2 + \sqrt{2}x + 5 = 0$

2. For what values of k will the equation

   $y^2 - 2(1 + 2k)y + 3 + 2k = 0$ have equal roots ?

3. Show that the roots of the equation

   $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are always real and they can not be equal unless $a = b = c$.

9.4 RELATION BETWEEN ROOTS AND COEFFICIENTS OF A QUADRATIC EQUATION

You have learnt that, the roots of a quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$

are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

Let $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ ...(i) and $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ ... (ii)

Adding (i) and (ii), we have $\alpha + \beta = \dfrac{-2b}{2a} = \dfrac{-b}{a}$

$\therefore$ **Sum of the roots** $= -\dfrac{\textit{coefficient of } x}{\textit{coefficient of } x^2} = -\dfrac{b}{a}$ ... (iii)

$$\alpha\ \beta = \frac{+b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

$\therefore$ **Product of the roots** $= \dfrac{\text{constant term}}{\text{coefficient of } x^2} = \dfrac{c}{a}$ ...(iv)

(iii) and (iv) are the required relationships between roots and coefficients of a given quadratic equation. These relationships helps to find out a quadratic equation when two roots are given.

Example 9.8 If, α, β are the roots of the equation $3x^2 - 5x + 9 = 0$ find the value of:

(a) $\alpha^2 + \beta^2$ (b) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

Solution: (a) It is given that α, β are the roots of the quadratic equation $3x^2 - 5x + 9 = 0$.

$\therefore$ $\alpha + \beta = \dfrac{5}{3}$ ... (i)

and $\alpha\beta = \dfrac{9}{3} = 3$ ... (ii)

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(\frac{5}{3}\right)^2 - 2.3$ [By (i) and (ii)]

$$= -\frac{29}{9}$$

(b) Now, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \frac{\frac{-29}{9}}{9}$ [By (i) and (ii)]

$$= -\frac{29}{81}$$

Example 9.9 If α, β are the roots of the equation $3y^2 + 4y + 1 = 0$, form a quadratic equation whose roots are α^2, β^2

Solution: It is given that α, β are two roots of the quadratic equation $3y^2 + 4y + 1 = 0$.

$\therefore$ Sum of the roots

i.e., $\alpha + \beta = -\frac{\text{coefficient of } y}{\text{coefficient of } y^2} = -\frac{4}{3}$ ... (i)

Product of the roots i.e., $\alpha\beta = \frac{\text{constant term}}{\text{coefficient of } y^2} = \frac{1}{3}$ ... (ii)

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(-\frac{4}{3}\right)^2 - 2.\frac{1}{3}$$ [By (i) and (ii)]

$$= \frac{16}{9} - \frac{2}{3} = \frac{10}{9}$$

and $\alpha^2\beta^2 = (\alpha\beta)^2 = \frac{1}{9}$ [By (i)]

$\therefore$ The required quadratic equation is $y^2 - (\alpha^2 + \beta^2)y + \alpha^2\beta^2 = 0$

or, $y^2 - \frac{10}{9}y + \frac{1}{9} = 0$ or, $9y^2 - 10y + 1 = 0$

Example 9.10 If one root of the equation $ax^2 + bx + c = 0,\ a \neq 0$ be the square of the other, prove that $b^3 + ac^2 + a^2c = 3abc$

Solution: Let α, α^2 be two roots of the equation $ax^2 + bx + c = 0$.

$\therefore \qquad \alpha + \alpha^2 = -\frac{b}{a}$ ... (i)

and $\qquad \alpha . \alpha^2 = \frac{c}{a}$

i.e., $\qquad \alpha^3 = \frac{c}{a}$ . ... (ii)

From (i) we have $\alpha(\alpha + 1) = -\frac{b}{a}$

or, $\qquad \{\alpha(\alpha+1)\}^3 = \left(-\frac{b}{a}\right)^3 = -\frac{b^3}{a^3}$ or, $\alpha^3(\alpha^3 + 3\alpha^2 + 3\alpha + 1) = -\frac{b^3}{a^3}$

or, $\qquad \frac{c}{a}\left\{\frac{c}{a} + 3\left(-\frac{b}{a}\right) + 1\right\} = -\frac{b^3}{a^3}$ ... [By (i) and (ii)]

or, $\qquad \frac{c^2}{a^2} - \frac{3bc}{a^2} + \frac{c}{a} = -\frac{b^3}{a^3}$ or, $ac^2 - 3abc + a^2c = -b^3$

or, $b^3 + ac^2 + a^2c = 3abc$, which is the required result.

Example 9.11 Find the condition that the roots of the equation $ax^2 + bx + c = 0$ are in the ratio m : n

Solution: Let $m\alpha$ and $n\alpha$ be the roots of the equation $ax^2 + bx + c = 0$

Now, $\qquad m\alpha + n\alpha = -\frac{b}{a}$ ... (i)

and $\qquad mn\alpha^2 = \frac{c}{a}$ ... (ii)

From (i) we have, $\qquad \alpha(m + n) = -\frac{b}{a}$ or, $\alpha^2 (m + n)^2 = \frac{b^2}{a^2}$

or, $\qquad \frac{c}{a}(m + n)^2 = mn\frac{b^2}{a^2}$ [By (ii)]

or, $\qquad ac(m+n)^2 = mn\, b^2$, which is the required condition

CHECK YOUR PROGRESS 9.3

1. If α, β are the roots of the equation $ay^2 + by + c = 0$ then find the value of :

(i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ (ii) $\frac{1}{\alpha^4}+\frac{1}{\beta^4}$

2. If α, β are the roots of the equation $5x^2 - 6x + 3 = 0$, form a quadratic equation whose roots are:

(i) α^2, β^2 (ii) $\alpha^3\beta, \alpha\beta^3$

3. If the roots of the equation $ay^2 + by + c = 0$ be in the ratio 3:4, prove that $12b^2 = 49\,ac$

4. Find the condition that one root of the quadratic equation $px^2 - qx + p = 0$ may be 1 more than the other.

9.5 SOLUTION OF A QUADRATIC EQUATION WHEN D < 0

Let us consider the following quadratic equation:

(a) Solve for t : $t^2 + 3t + 4 = 0$

$$\therefore \quad t = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$$

Here, $D = -7 < 0$

$$\therefore \quad \text{The roots are } \frac{-3+\sqrt{-7}}{2} \text{ and } \frac{-3-\sqrt{-7}}{2}$$

$$\text{or,} \quad \frac{-3+\sqrt{7}\,i}{2}, \frac{-3-\sqrt{7}\,i}{2}$$

Thus, the roots are complex and conjugate.

(b) Solve for y :

$$-3y^2 + \sqrt{5}\,y - 2 = 0$$

$$\therefore \quad y = \frac{-\sqrt{5} \pm \sqrt{5-4(-3).(-2)}}{2(-3)} \text{ or } y = \frac{-\sqrt{5} \pm \sqrt{-19}}{-6}$$

Here, $D = -19 < 0$

$$\therefore \quad \text{The roots are } \frac{-\sqrt{5}+\sqrt{19}i}{-6}, -\frac{\sqrt{5}-\sqrt{19}i}{-6}$$

Here, also roots are complex and conjugate. From the above examples , we can make the following conclusions:

(i) $D < 0$ in both the cases

(ii) Roots are complex and conjugate to each other.

Is it always true that complex roots occur in conjugate pairs ?

Let us form a quadratic equation whose roots are $2 + 3i$ and $4 - 5i$

The equation will be $\{x-(2+3i)\}\ \{x-(4-5i)\}=0$

or, $x^2-(2+3i)x-(4-5i)x+(2+3i)(4-5i)=0$

or, $x^2+(-6+2i)x+23+2i=0$, which is an equation with complex coefficients.

> ***Note :*** *If the quadratic equation has two complex roots, which are not conjugate of each other, the quadratic equation is an equation with complex coefficients.*

9.6 Fundamental Theorem of Algebra

You may be interested to know as to how many roots does an equation have? In this regard the following theorem known as fundamental theorem of algebra, is stated (without proof). 'A polynomial equation has at least one root'.

As a consequence of this theorem, the following result, which is of immense importance is arrived at.

'A polynomial equation of degree *n* has exactly *n* roots'

CHECK YOUR PROGRESS 9.4

Solve each of the following equations.

1. $-x^2+x+2=0$
2. $\sqrt{3}x^2-\sqrt{2}x+3\sqrt{3}=0$
3. $x^2+\dfrac{1}{\sqrt{2}}x+1=0$

4. $\sqrt{5}x^2+x+\sqrt{5}=0$
4. $x^2+3x+5=0$

9.7 INEQUALITIES (INEQUATIONS)

Now we will discuss about linear inequalities and their applications from daily life. A statement involving a sign of equality (=) is an equation.

Similarly, a statement involving a sign of inequality, $<, >, \leq$, or $\geq$ is called an inequalities.

Some examples of inequalities are:

(i) $2x+5>0$ (ii) $3x-7<0$

(iii) $ax+b\geq 0$, $a\neq 0$ (iv) $ax+b\leq c$, $a\neq 0$

(v) $3x+4y\leq 12$ (vi) $x^2-5x+6<0$

(vii) $ax+by+c\geq 0$

(v) and (vii) are inequalities in two variables and all other inequalities are in one variable. (i) to (v) and (vii) are linear inequalities and (vi) is a quadratic inequalities.

In this lesson, we shall study about linear inequalities in one or two variables only.

9.8 SOLUTIONS OF LINEAR INEQUALITIES IN ONE/TWO VARIABLES

Solving an inequalities means to find the value (or values) of the variable (s), which when substituted in the inequalities, satisfies it.

For example, for the inequalities $2.60x<30$ (statement) (i) all values of $x \leq 11$ are the solutions. (x is a whole number)

For the inequalities $2x + 16>0$, where x is a real number, all values of x which are > -8 are the solutions.

For the linear inequation in two variables, like $ax + by + c \geq 0$, we shall have to find the pairs of values of x and y which make the given inequalities true.

Let us consider the following situation :

Anil has Rs. 60 and wants to buy pens and pencils from a shop. The cost of a pen is Rs. 5 and that of a pencil is Rs. 3 If x denotes the number of pens and y, the number of pencils which Anil buys, then we have the inequality $5x + 3y \leq 60$ ... (i)

Here, $x = 6$, $y = 10$ is one of the solutions of the inequalities (i). Similarly $x = 5$, $y = 11$; $x = 4$, $y = 13$; $x = 10$, $y = 3$ are some more solutions of the inequalities.

In solving inequalities, we follow the rules which are as follows :

1. Equal numbers may be added (or subtracted) from both sides of an inequalities.

   Thus (i) if $a > b$ then $a + c > b + c$ and $a - c > b - c$

   and (ii) if $a \leq b$ then $a + d \leq b + d$ and $a - d \leq b - d$

2. Both sides of an inequalities can be multiplied (or divided) by the same positive number.

   Thus (i) if $a > b$ and $c > 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

   and (ii) if $a \leq b$ and c > 0 then $ac \leq bc$ and $\frac{a}{c} \leq \frac{b}{c}$

3. When both sides of an inequalities are multiplied by the same negative number, the sign of inequality gets reversed.

   Thus (i) if $a > b$ and $d < 0$ then $ad < bd$ and $\frac{a}{d} < \frac{b}{d}$

   and (ii) if $a \leq b$ and $c < 0$ then $ac \geq bc$ and $\frac{a}{c} \geq \frac{b}{c}$

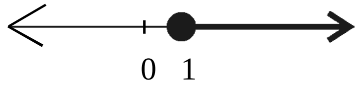
Example 9.12 Solve $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.

Solution: We have

$$\frac{3x-4}{2} \geq \frac{x+1}{4} - 1 \text{ or } \frac{3x-4}{2} \geq \frac{x+3}{4}$$

$$\text{or} \quad 2(3x-4) \geq (x-3) \text{ or } 6x - 8 \geq x - 3 \text{ or } 5x \geq 5 \text{ or } x \geq 1$$

The graphical representation of solutions is given in Fig.



Example 9.13 The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 60 marks.

Solution: Let x be the marks obtained by student in the annual examination. Then

$$\frac{62+48+x}{3} \geq 60 \text{ or } 110+x \geq 180 \text{ or } x \geq 70$$

Thus, the student must obtain a minimum of 70 marks to get an average of at least 60 marks.

Example 9.14 A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

Solution: Let x litres of 30% acid solution is required to be added. Then

Total mixture = (x + 600) litres

Therefore 30% x + 12% of 600 > 15% of (x + 600)

and 30% x + 12% of 600 < 18% of (x + 600)

or $$\frac{30x}{100}+\frac{12}{100}(600) > \frac{15}{100}(x+600)$$

and $$\frac{30x}{100}+\frac{12}{100}(600) < \frac{18}{100}(x+600)$$

or $30x + 7200 > 15x + 9000$

and $30x + 7200 < 18x + 10800$

or $15x > 1800$ and $12x < 3600$

or $x > 120$ and $x < 300$,

i.e. $120 < x < 300$

Thus, the number of litres of the 30% solution of acid will have to be more than 120 litres but less than 300 litres.

6.3 GRAPHICAL REPRESENTATION OF LINEAR INEQUALITIES IN ONE OR TWO VARIABLES.

In Section 6.2, while translating word problem of purchasing pens and pencils, we obtained the following linear inequalities in two variables x and y :

$5x + 3y \leq 60$ .... (i)

Let us now find all solutions of this inequation, keeping in mind that x and y here can be only whole numbers.

To start with, let $x = 0$.

Thus, we have $3y \le 60$ or $y \le 20$, i,e the values of y corresponding to $x = 0$ can be 0,1,2,3..........., 20 only Thus, the solutions with $x = 0$ are

(0,0), (0,1), (0,2) ....(0, 20)

Similarly the other solutions of the inequalities, when $x = 1,2, ...12$ are

(1,0) (1,1) (1,2) .... (1,18)

(2,0) (2,1) (2,2) .... (2,16)

....

....

(10,0) (10,1) (10,2), (10,3)

(11,0) (11,1)

(12,0)

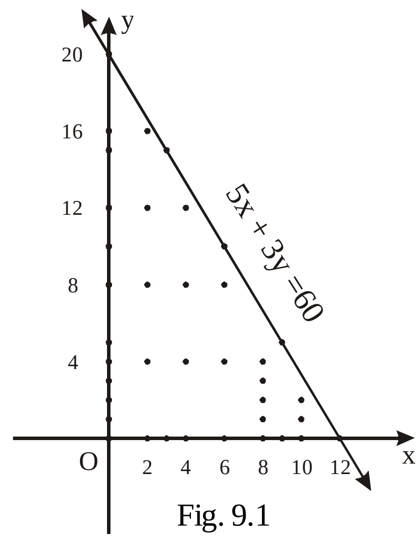


Fig. 9.1

You may note that out of the above ordered pairs, some pairs such as (0,20), (3, 15), (6, 10), (9, 5), (12,0) satisfy the equation $5x + 3y = 60$ which is a part of the given inequation and all other possible solutions lie on **<u>one of the two half planes</u>** in which the line $5x + 3y = 60$, divides the xy - plane.

If we now extend the domain of x and y from whole numbers to real numbers, the inequation $5x + 3y \le 60$ will represent one of the two half planes in which the line $5x + 3y = 60$, divides the xy-plane.

Thus we can generalize as follows :

If $a,b,c,$ are real numbers, then $ax + by + c = 0$ is called a linear equalities in two variables x and y, where as $ax + by + c \le 0$ or $ax + by + c \ge 0$, $ax + by + c > 0$ and $ax + by + c < 0$ are called linear inequations in two variables x and y.

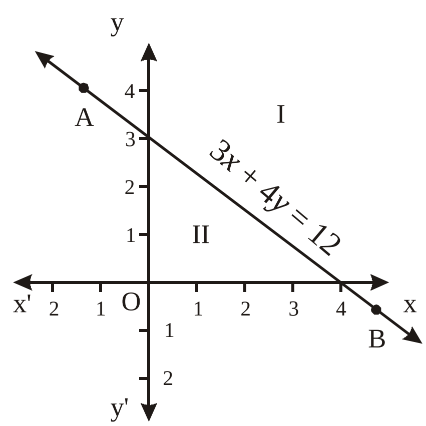


Fig. 9.2

The equation $ax + by + c = 0$ is a straight line which divides the xy plane into two half planes which are represented by $ax + by + c \ge 0$ and $ax + by + c \le 0$.

For example $3x + 4y - 12 = 0$ can be represented by line AB, in the xy - plane as shown in Fig. 9.2

The line AB divides the cordinate plane into two half -plane regions :

(i) half plane region I above the line AB

(ii) half plane region II below the line AB. One of the above region represents the inequality $3x + 4y - 12 \leq 0$ ...(i) and the other region will be represented by $3x + 4y - 12 \geq 0$. ... (ii)

To identify the half plane represented by inequation (i), we take any arbitrary point, preferably origin, if it does not lie on AB. If the point satisfies the inequation (i), then the half plane in which the arbitrary point lies, is the desired half plane. In this case, taking origin as the arbitrary point we have

$0+0-12 \leq 0$ i.e $-12 \leq 0$. Thus origin satisfies the inequalities $3x + 4y - 12 \leq 0$. Now, origin lies in half plane region II. Hence the inequaty $3x + 4y - 12 \leq 0$ represents half plane II and the inequality $3x + 4y - 12 \geq 0$ will represent the half plane I

Example 9.15 Show on graph the region represented by the inequalities $x + 2y \geq 5$.

Solution : The given inequalities is $x + 2y \geq 5$

Let us first take the corresponding linear equation $x + 2y = 5$ and draw its graph with the help of the following table :

x	1	3	5
y	2	1	0

Since (0,0) does not lie on the line AB, so we can select (0,0) as the arbitrary point. Since $0 + 0 \geq 5$ is not true

$\therefore$ The desired half plane is one, in which origin does not lie

$\therefore$ The desired half plane is the shaded one (See Fig. 9.3)

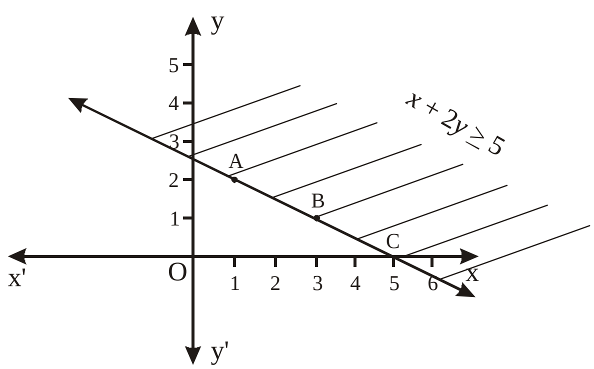


Fig. 9.3

Before taking more examples, it is important to define the following :

(i) Closed Half Plane: A half plane is said to be closed half plane if all points on the line separating the two half planes are also included in the solution of the inequation. The Half plane in Example 6.1 is a closed half plane.

(ii) An Open Half Plane : A half plane in the *xy* plane is said to be an open half plane if the points on the line separting the planes are not included in the half plane.

Example 9.16 Draw the graph of inequation $x - 5y > 0$

Solution : The given inequation is $x - 5y > 0$

The corresponding linear equation is $x - 5y = 0$ we have the following table.

x	0	5	–5
y	0	1	–1

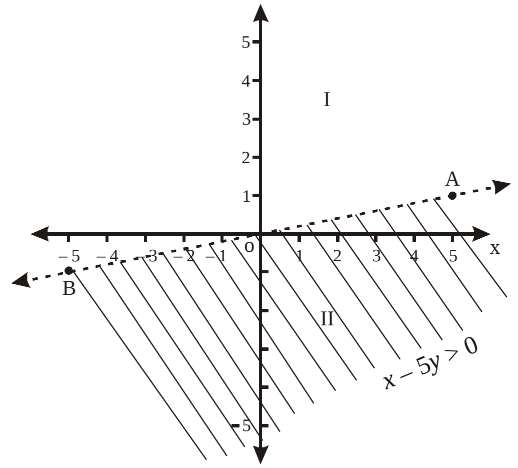


Fig. 9.4

The line AOB divides xy - plane into two half planes I and II. As the line AOB passes through origin, we consider any other arbitrary point (say) P (3,4) which is in half plane I. Let us see whether it satisfies the given inequation $x - 5y > 0$

∴ Then $3 - 5(4) > 0$ or $3 - 20 > 0$, or $-17 > 0$ which is not true

∴ The desired half plane is II

Again the inequation is a **strict** inequation $x - 5y > 0$

∴ Line AOB is not a part of the graph and hence has been shown as a dotted line.

Hence, the graph of the given inequation is the shaded region half plane II excluding the line AOB.

Example 9.17 Represent graphically the inequlities $3x - 12 \geq 0$

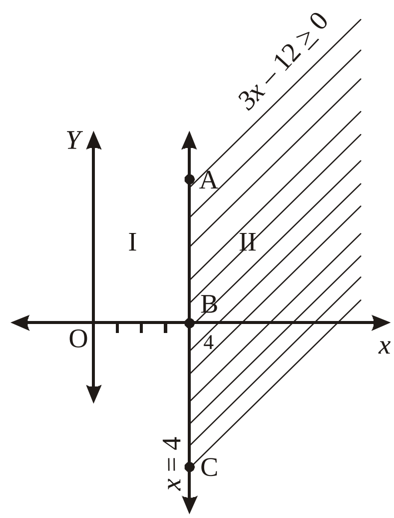


Fig. 9.5

Solution : Given inequation is $3x - 12 \geq 0$ and the corresponding linear equation is $3x - 12 = 0$ or $x - 4 = 0$ or $x = 4$ which is represented by the line ABC on the xy - plane (See Fig. 9.5). Taking (0,0) as the arbitrary point, we can say that $0 \neq 4$ and so, half plane II represents the inequation $3x - 12 \geq 0$

Example 9.18 Solve graphically the inequation $2y + 4 \geq 0$

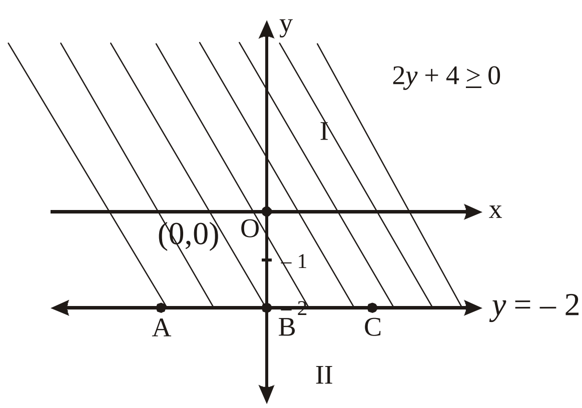


Fig. 9.6

Solution : Here the inequation is

$2y + 4 \geq 0$ and the corresponding equation is $2y + 4 = 0$ or $y = -2$

The line ABC represents the line
$y = -2$ which divides the xy - plane into two half planes and the inequation
$2y + 4 \geq 0$ is represented by the half plane I.

CHECK YOUR PROGRESS 9.5

Represent the solution of each of the following inequations graphically in two dimensional plane:

1. $2x + y \geq 8$	2. $x - 2y \leq 0$
3. $3x + 6 \geq 0$	4. $8 - 2y \geq 2$
5. $3y \geq 6 - 2x$	6. $3x \geq 0$
7. $y \leq 4$	8. $y > 2x - 8$
9. $-y < x - 5$	10. $2y \leq 8 - 4x$

6.4 GRAPHICAL SOLUTION OF A SYSTEM OF LINEAR INEQUATIONS IN TWO VARIABLES.

You already know how to solve a system of linear equations in two variables.

Now, you have also learnt how to solve **linear inequations** in two variables graphically. We will now discuss the technique of finding the solutions of a system of simultaneous linear inequations. By the term solution of a system of simultaneons linear inequations we mean, finding all ordered pairs (x,y) for which each linear inequation of the system is satisfied.

A system of simultaneous inequations may have no solution or an infinite number of solutions represented by the region bounded or unbounded by straight lines corresponding to linear inequations.

We take the following example to explain the technique.

Example 9.19 Solve the following system of inequations graphically:

$$x + y \geq 6 ; \qquad 2x - y \geq 0.$$

Solution : Given inequations are

$$x + y \geq 6 \text{ ..... (i)}$$

and $\quad 2x - y \geq 0$ ..... (ii)

We draw the graphs of the lines $x + y = 6$ and $2x - y = 0$ (Fig. 9.7)

The inequation (i) represent the shaded region above the line $x + y = 6$ and inequations (ii) represents the region on the right of the line $2x - y = 0$

The common region represented by the double shade in Fig. 9.7 represents the solution of the given system of linear inequations.

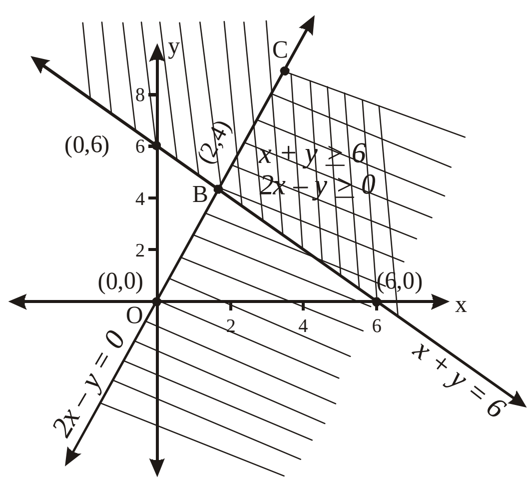


Fig. 9.7

Example 9.20 Find graphically the solution of the following system of linear inequations :

$x + y \leq 5, \quad 4x + y \geq 4,$
$x + 5y \geq 5, \quad x \leq 4, \ y \leq 3.$

Solution : Given inequations are

$x + y \leq 5$ ..... (i)
$4x + y \geq 4$ ..... (ii)
$x + 5y \geq 5$ ..... (iii)
$x \leq 4$ ..... (iv)
and $y \leq 3$ ..... (v)

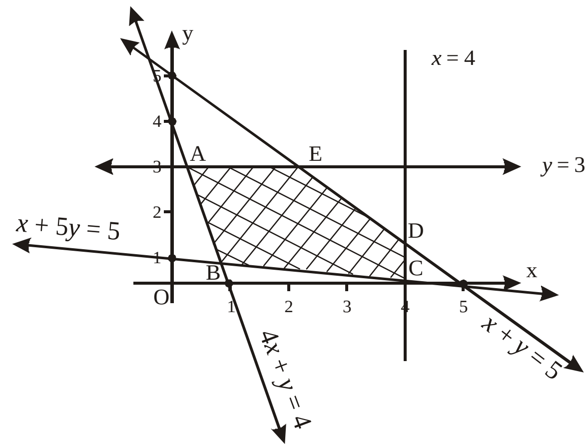


Fig. 9.8

We draw the graphs of the lines
$x + y = 5, \ 4x + y = 4, \ x + 5y = 5,$
$x = 4$ and $y = 3$ (Fig. 9.8)

The inequlities (i) represents the region below the line $x + y = 5$. The inequations (ii) represents the region on the right of equation 4x + y = 4 and the region above the line $x + 5y = 5$ represents the inequation (iii). Similarly after shading the regions for inequations (iv) and (v) we get the common region as the bounded region ABCDE as shown in (Fig. 9.8) The co-ordinates of the points of the shaded region satisfy the given system of inequations and therefore all these points represent solution of the given system.

Example 9.21 Solve graphically the following system of inequations :

$x + 2y \leq 3, 3x + 4y \geq 12, \ x \geq 0, y \geq 0.$

Solution : We represent the inequations $x + 2y \leq 3$, $3x + 4y \geq 12$, $x \geq 0$, $y \geq 0$ by shading the corresponding regions on the graph as shown in Fig. 9.9

Here we find that there is no common region represented by these inequations.

We thus conclude that there is no solution of the given system of linear inequations.

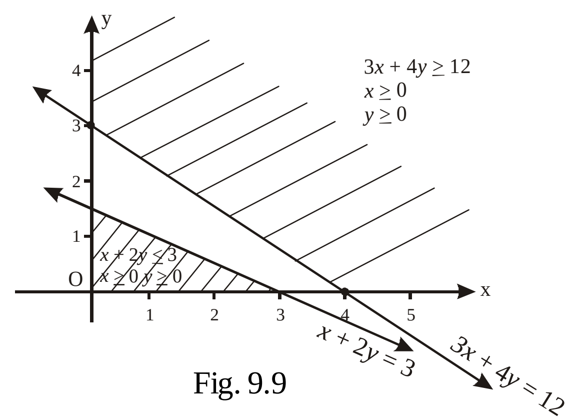


Fig. 9.9

Example 9.22 Solve the following system of linear inequations graphically :

$x - y < 2, \ 2x + y < 6; x \geq 0, y \geq 0.$

Solution : The given inequations are

$x - y < 2$ ... (i)
$2x + y < 6$ ... (ii)
$x \geq 0 ; y \geq 0$ ... (iii)

After representing the inequations $x - y < 2$, $2x + y < 6$, $x \geq 0$ and $y \geq 0$ on the graph we find the common region which is the bounded region $OABC$ as shown in Fig. 9.10

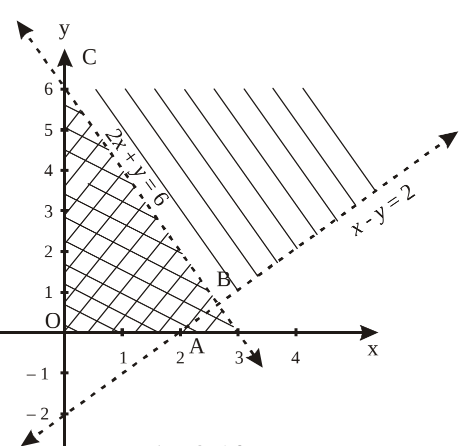


Fig. 9.10

CHECK YOUR PROGRESS 9.6

Solve each of the follwing systems of linear inequations in two variables graphically :

1. $x \geq 3,\ y \geq 1.$
2. $y \geq 2x, y \leq 2.$
3. $2x + y - 3 \geq 0, x - 2y + 1 \leq 0.$
4. $3x + 4y \leq 12,\ 4x+3y \leq 12,\ x \geq 0,\ y \geq 0$
5. $2x + 3y \geq 3,\ 3x + 4y \leq 18,\ 7x - 4y + 14 \geq 0,\ x - 6y \leq 3,\ x \geq 0, y \geq 0$
6. $x + y \geq 9,\ 3x + y \geq 12,\ x \geq 0,\ y \geq 0$
7. $x + y \geq 1;\ 2x + 3y \leq 6,\ x \geq 0,\ y \geq 0.$
8. $x + 3y \geq 10;\ x + 2y \leq 3,\ x - 2y \leq 2,\ x \geq 0\ ;\ y \geq 0$

LET US SUM UP

- Roots of the quadratic equation $ax^2 + bx + c = 0$ are complex and conjugate of each other, when $D < 0$. and $a, b, c \in R$.
- If α , β be the roots of the quadratic equation

  $ax^2 + bx + c = 0$ then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$
- If α and β are the roots of a quadratic equation. then the equation is:

  $x^2 - (\alpha + \beta)x + \alpha\beta = 0$
- The maximum number of roots of an equation is equal to the degree of the equation.
  A statement involving a sign of inequality like, <, >, $\leq$, $\geq$, is called an inequation.
- The equation $ax + by + c = 0$ is a straight line which divides the xy-plane into two half planes which are represented by $ax + by + c \geq 0$ and $ax + by + c \leq 0$
- By the term, solution of a system of simultaneous linear inequalities we mean, finding all values of the ordered pairs (x,y) for which each linear inequalities of the system are satisfied.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=EoCeL4SPIcA
http://www.youtube.com/watch?v=FnrqBgot3jM
http://www.youtube.com/watch?v=-aTy1ED1m5I
http://www.youtube.com/watch?v=YBYu5aZPLeg
http://www.youtube.com/watch?v=2oGsLdAWxlk

TERMINAL EXERCISE

1. Show that the roots of the equation
$2(a^2+b^2)x^2+2(a+b)x+1=0$ are imaginary, when $a \neq b$

2. Show that the roots of the equation
$bx^2+(b-c)x=c+a-b$ are always real if those of
$ax^2+b(2x+1)=0$ are imaginary.

3. If α, β be the roots of the equation $2x^2-6x+5=0$, find the equation whose roots are:

(i) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$ (ii) $\alpha+\dfrac{1}{\beta}, \beta+\dfrac{1}{\alpha}$ (iii) $\alpha^2+\beta^2, \dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$

Solve the following inequalities graphically.

4. $x \geq -2$ 5. $y \leq 2.$ 6. $x < 3$ 7. $y \geq -3$

8. $5-3y \geq -4$ 9. $2x-5 \leq 3.$ 10. $3x-2y \leq 12$ 11. $\dfrac{x}{3}+\dfrac{y}{5} \geq 1.$

12. $2x-3y \geq 0$ 13. $x+2y \leq 0.$

Solve each of the following systems of linear inequalities in two variables graphically.

14. $-1 \leq x \leq 3, 1 \leq y \leq 4.$

15. $2x+3y \leq 6, \; 3x+2y \leq 6.$

16. $6x+5y \leq 150,$
$x+4y \leq 80$
$x \leq 15, x \geq 0, y \geq 0.$

17. $3x+2y \leq 24, x+2y \leq 16$
$x+y \leq 10, x \geq 0, y \geq 0$

18. $x+y \geq 3, 7x+6y \leq 42$
$x \leq 5, \quad y \leq 4$
$x \geq 0, \quad y \geq 0$

Solve that inequalities:

19. $\dfrac{3(x-2)}{5} \leq \dfrac{5(2-x)}{3}$

20. $37-(3x+5) \geq 9x-8(x-3)$

21. $\dfrac{(2x-1)}{3} \geq \dfrac{(3x-2)}{4} - \dfrac{(2-x)}{5}$

22. $5x+1 > -24, 5x-1 < 24$

23. $3x-7 > 2(x-6), 6-x > 11-2x$

24. $5(2x-7)-3(2x+3) \leq 0, 2x+19 \leq 6x+47.$

25. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be move than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?

26. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?

ANSWERS

CHECK YOUR PROGRESS 9.1

1. (i) $-2\sqrt{3}, \frac{-4}{\sqrt{3}}$ (ii) $a-\sqrt{b}, a+\sqrt{b}$ (iii) $-\frac{ab}{c}, \frac{c}{ab}$ (iv) $3\sqrt{2}, \sqrt{2}$

CHECK YOUR PROGRESS 9.2

1. (i) $\frac{3\pm\sqrt{15}\,i}{4}$ (ii) $\frac{1\pm i}{\sqrt{2}}$ (iii) $\frac{\sqrt{5}\pm\sqrt{43}\,i}{8}$ (iv) $\frac{-\sqrt{2}\pm\sqrt{58}\,i}{6}$

2. $-1, \frac{1}{2}$

CHECK YOUR PROGRESS 9.3

1. (i) $\frac{b^2-2ac}{c^2}$ (ii) $\frac{(b^2-2ac)^2-2a^2c^2}{c^4}$

2. (i) $25x^2-6x+9=0$ (ii) $625x^2-90x+81=0$

4. $q^2-5p^2=0$

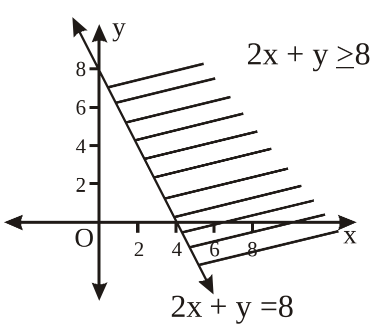
CHECK YOUR PROGRESS 9.4

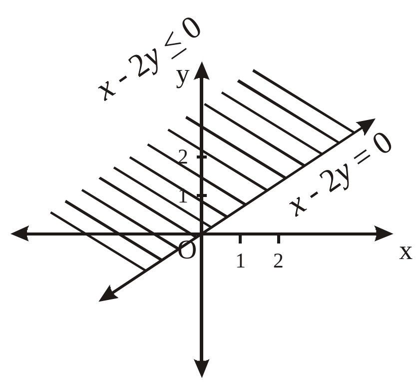
1. $\frac{-1\pm\sqrt{7}i}{2}$ 2. $\frac{\sqrt{2}\pm\sqrt{34}i}{2}$

3. $\frac{-1\pm\sqrt{7}i}{2\sqrt{2}}$ 4. $\frac{-1\pm\sqrt{19}i}{2}$

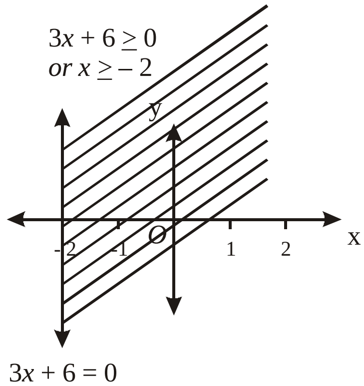
5. $\frac{-3\pm\sqrt{11}i}{2}$

CHECK YOUR PROGRESS 9.5

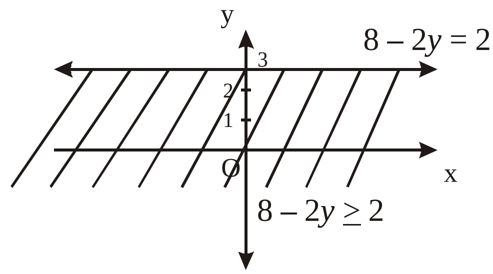
1. 

2. 

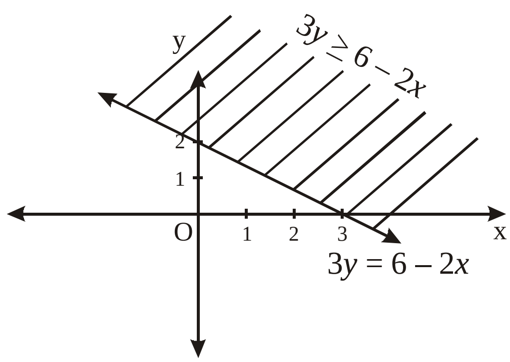
3.


3x + 6 ≥ 0
or x ≥ – 2
y
x
-2
-1
O
1
2
3x + 6 = 0

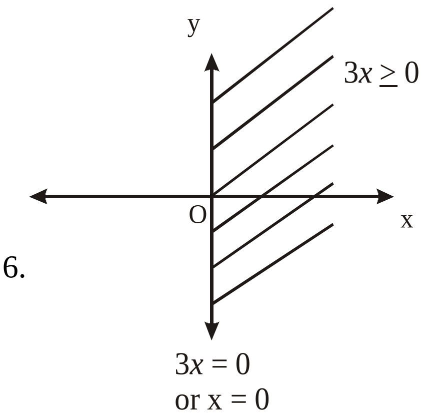
4.


y
8 – 2y = 2
3
2
1
O
x
8 – 2y ≥ 2

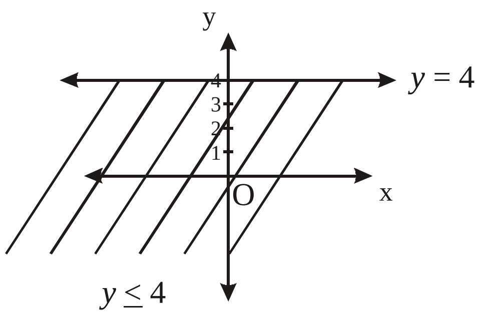
5.


y
3y ≥ 6 – 2x
2
1
O
1
2
3
x
3y = 6 – 2x

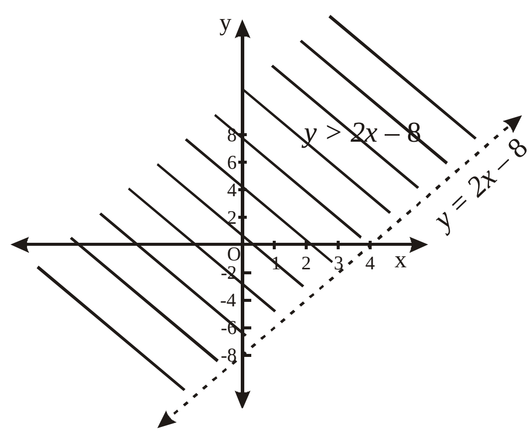
6.


y
3x ≥ 0
O
x
3x = 0
or x = 0

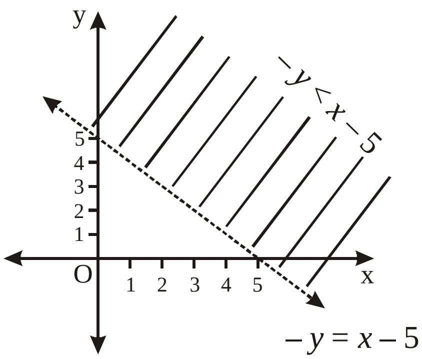
7.


y
y = 4
4
3
2
1
O
x
y ≤ 4

8.


y
8
6
4
2
O
-2
-4
-6
-8
1
2
3
4
x
y = 2x – 8

9.



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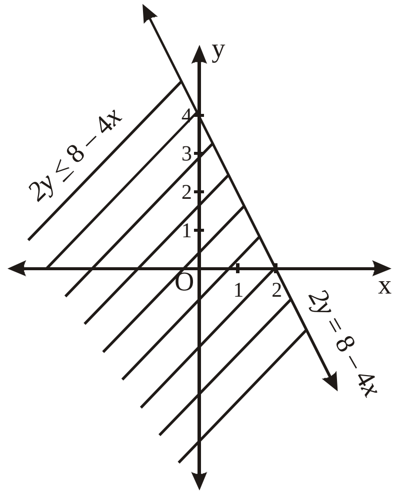
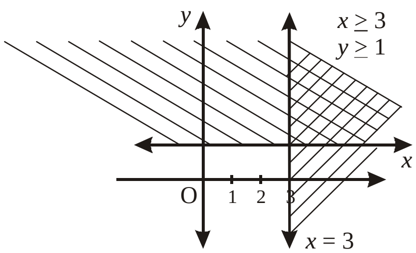


CHECK YOUR PROGRESS 9.6

1.



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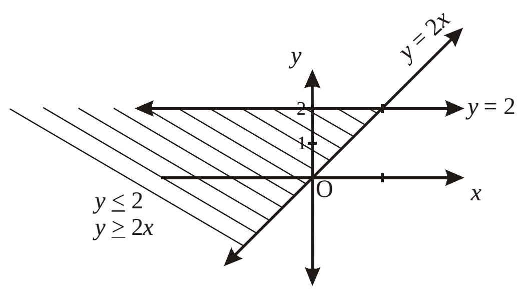
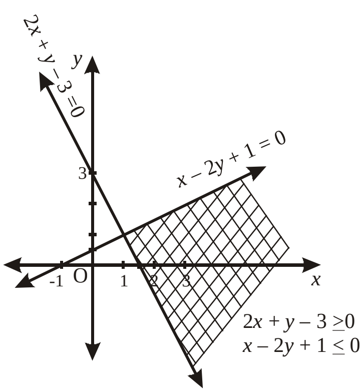
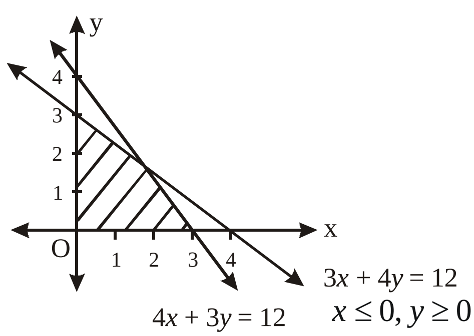


(3)



(4)



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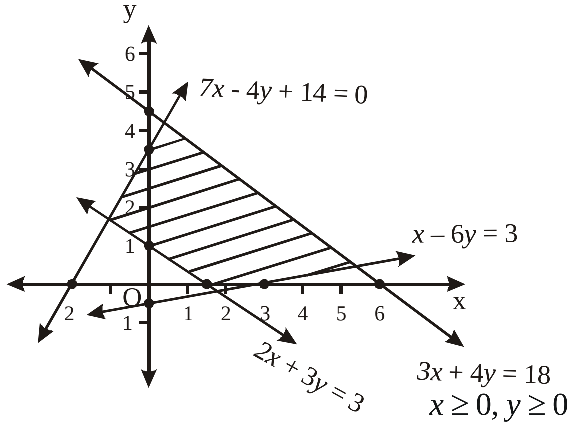

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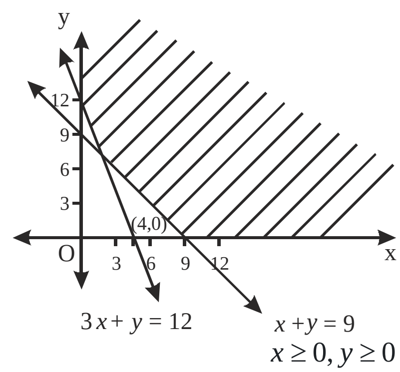
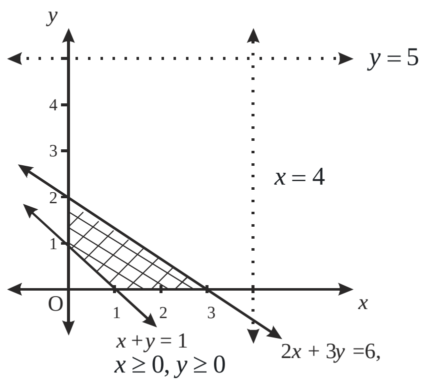


7 .



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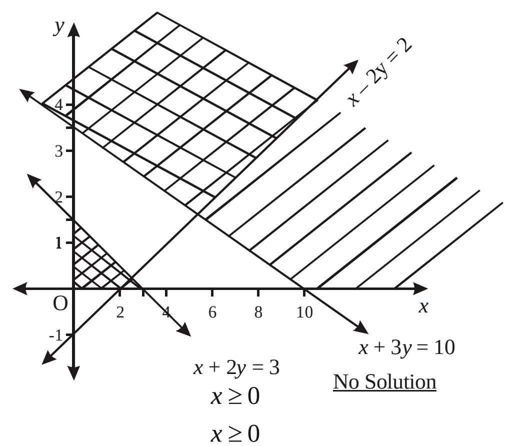
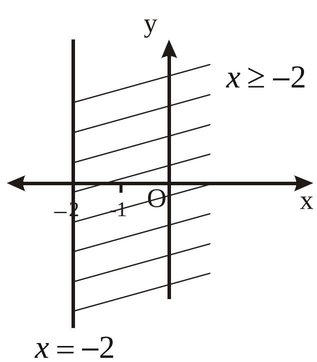


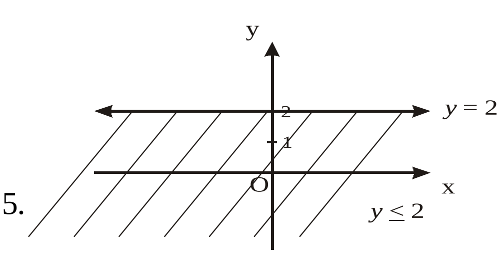
TERMINAL EXERCISE

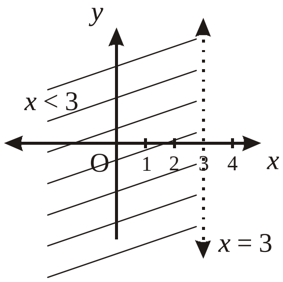
3. (i) $5x^2 - 8x + 5 = 0$ (ii) $10x^2 - 42x + 49 = 0$ (iii) $25x^2 - 116x + 64 = 0$

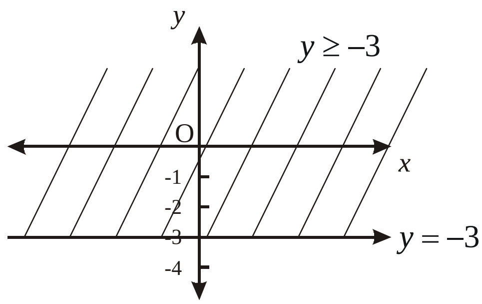
4.

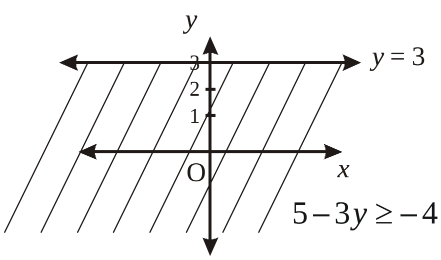


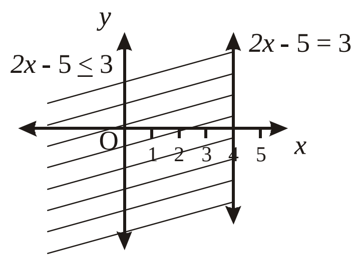
5.

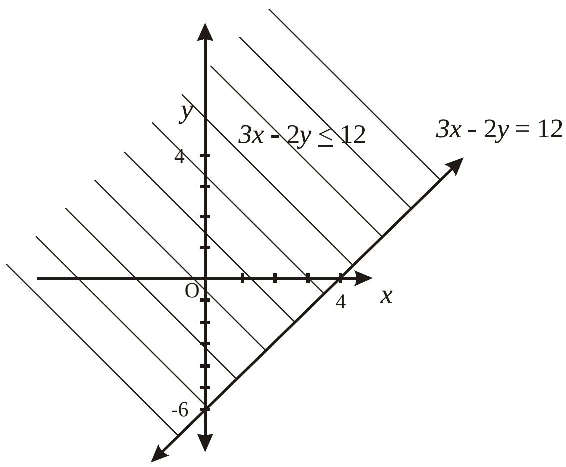


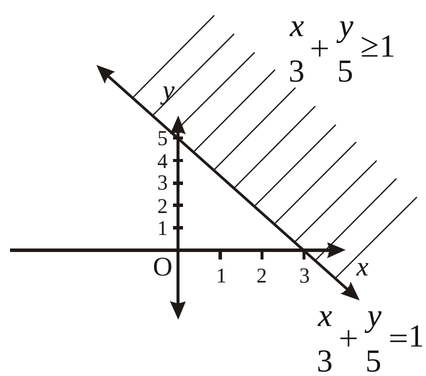
6. 

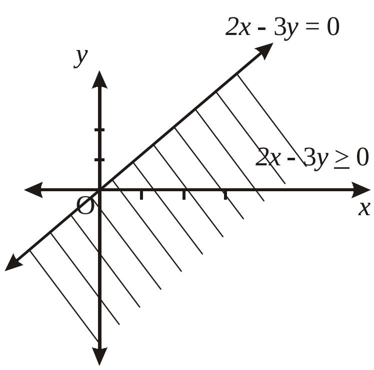
7. 

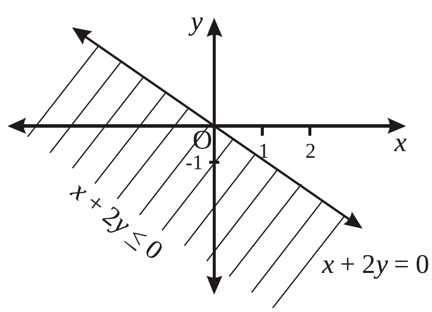
8. 

9. 

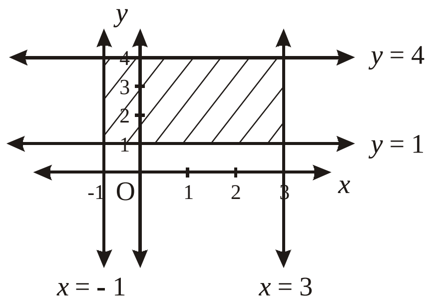
10. 

11. 

12. 

13. 

11.



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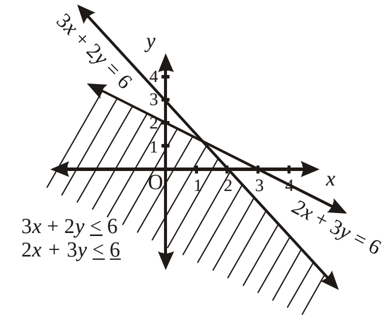
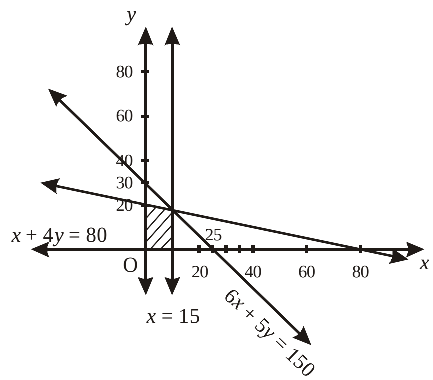


13.



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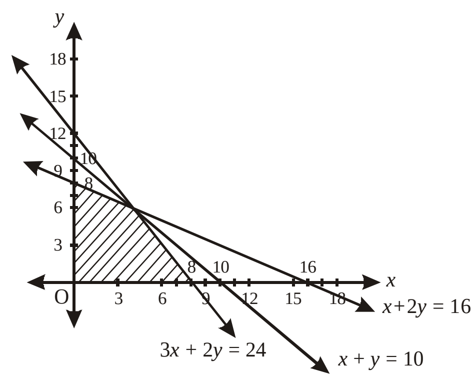

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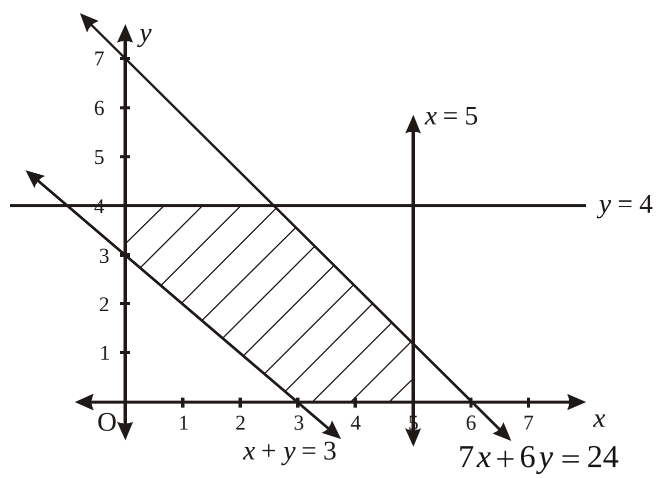


19. $(-\infty, 2]$ 20. $(-\infty, 2]$

21. $(-\infty, 2]$ 22. $(-5, 5)$ 23. $(5, \infty)$ 24. $[-7, 11]$

25. More then 320 litre but less then 1280 litres

26. More then 562.5 litres but less then 900 litres

10

PRINCIPLE OF MATHEMATICAL INDUCTION

In your daily life, you must be using various kinds of reasoning depending on the situation you are faced with. For instance, if you are told that your friend just has a child, you would know that it is either a girl or a boy. In this case, you would be applying general principles to a particular case. This form of reasoning is an example of **deductive** logic.

Now let us consider another situation. When you look around, you find students who study regularly, do well in examinations, you may formulate the general rule (rightly or wrongly) that "any one who studies regularly will do well in examinations". In this case, you would be formulating a general principle (or rule) based on several particular instances. Such reasoning is inductive, a process of reasoning by which general rules are discovered by the observation and consideration of several individual cases. Such reasoning is used in all the sciences, as well as in Mathematics.

Mathematical induction is a more precise form of this process. This precision is required because a statement is accepted to be true mathematically only if it can be shown to be true for each and every case that it refers to.

In the present chapter, first of all we shall introduce you with a statement and then we shall introduce the concept of principale of Mathematical induction, which we shall be using in proving some statements.

OBJECTIVES

After studying this lesson, you will be able to:

- To check whether the given sentence is a statement or not.
- state the Principle of Mathematical Induction;
- verify the truth or otherwise of the statement P(n) for n = 1;
- verify P(k+1) is true, assuming that P(k) is true;
- use principle of mathematical induction to establish the truth or otherwise of mathematical statements;

EXPECTED BACKGROUND KNOWLEDGE

- Number System
- Four fundamental operations on numbers and expressions.

10.1 WHAT IS A STATEMENT ?

In your daily interactions, you must have made several assertions in the form of sentences. Of these assertions, the ones that are **either true** or **false** are called **statement** or **propositions**. For instance,

"I am 20 years old" and "If $x = 3$, then $x^2 = 9$" are statements, but 'When will you leave ?' And 'How wonderful!' are not statements.

Notice that a statement has to be a definite assertion which can be true or false, but not both. For example, '$x - 5 = 7$' is not a statement, because we don't know what x, is. If $x = 12$, it is true, but if $x = 5$, 'it is not true. Therefore, '$x - 5 = 7$' is not accepted by mathematicians as a statement.

But both '$x - 5 = 7 \Rightarrow x = 12$' and $x - 5 = 7$ for any real number x' are statements, the first one true and the second one false.

Example 10.1 Which of the following sentences is a statement ?

(i) India has never had a woman President. , (ii) 5 is an even number.

(iii) $x^n > 1$, (iv) $(a + b)^2 = a^2 + 2ab + b^2$

Solution : (i) and (ii) are statements, (i) being true and (ii) being false. (iii) is not a statement, since we can not determine whether it is true or false, unless we know the range of values that x and n can take.

Now look at (iv). At first glance , you may say that it is not a statement, for the very same reasons that (iii) is not. But look at (iv) carefully. It is true for any value of a and b. It is an identity. Therefore, in this case, even though we have not specified the range of values for a and b, (iv) is a statement.

Some statements, like the one given below are about natural numbers in general. Let us look at the statement :

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

This involves a general natural number n. Let us call this statement P (n) [P stands for proposition].

Then P (1) would be $1 = \frac{1(1+1)}{2}$

Similarly, P (2) would be the statement , $1 + 2 = \frac{2(2+1)}{2}$ and so on.

Let us look at some examples to help you get used to this notation.

Example 10.2 If P (n) denotes $2^n > n - 1$, write P (1), P (k) and P (k+1), where $k \in N$.

Solution : Replacing n by 1, k and $k + 1$, respectively in P (n), we get

$P(1) : 2^1 > 2 - 1$, i.e., $2 > 1$, $P(k) : 2^k > k - 1$

$P(k+1) : 2^{k+1} > (k+1) - 1$, i.e., $2^{k+1} > k$

Example 10.3 If P (n) is the statement, '$1 + 4 + 7 + (3n - 2) = \dfrac{n(3n-1)}{2}$

write $P(1)$, $P(k)$ and $P(k+1)$.

Solution : To write $P(1)$, the terms on the left hand side (LHS) of $P(n)$ continue till $3 \times 1 - 2$, i.e., 1. So, P (1) will have only one term in its LHS, i.e., the first term.

Also, the right hand side (RHS)of $P(1) = \dfrac{1 \times (3 \times 1 - 1)}{2} = 1$, Therefore, $P(1)$ is $1 = 1$.

Replacing n by 2, we get

$$P(2) : 1 + 4 = \frac{2 \times (3 \times 2 - 1)}{2}, \text{ i.e., } 5 = 5.$$

Replacing n by k and $k + 1$, respectively, we get

$$P(k) : 1 + 4 + 7 + ... + (3k - 2) = \frac{k(3k-1)}{2}$$

$$P(k+1) : 1 + 4 + 7 + ... + (3k - 2) + [3(k+1) - 2] = \frac{(k+1)[3(k+1)-1]}{2}$$

$$\text{i.e.}, 1 + 4 + 7 + ... + (3k + 1) = \frac{(k+1)[(3k+2)}{2}$$

CHECK YOUR PROGRESS 10.1

1. Determine which of the following are statements :

   (a) $1 + 2 + 4 \quad .... + 2^n > 20$ (b) $1 + 2 + 3 + ... + 10 = 99$

   (c) Chennai is much nicer than Mumbai. (d) Where is Timbuktu ?

   (e) $\dfrac{1}{1 \times 2} + ... + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$ for $n = 5$ (f) $\operatorname{cosec}\theta < 1$

2. Given that $P(n)$: 6 is a factor of $n^3 + 5n$, write $P(1)$, $P(2)$, $P(k)$ and $P(k+1)$ where k is a natural number.

3. Write $P(1)$, $P(k)$ and $P(k+1)$, if $P(n)$ is:

   (a) $2^n \geq n + 1$ (b) $(1 + x)^n \geq 1 + nx$

   (c) $n(n+1)(n+2)$ is divisible by 6. (d) $(x^n - y^n)$ is divisible by $(x - y)$.

(e) $(ab)^n = a^n b^n$ (f) is a natural number.

4. Write $P(1)$, $P(2)$, $P(k)$ and $P(k+1)$, if $P(n)$ is :

(a) $\frac{1}{1\times 2}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$, (b) $1 + 3 + 5 + \ldots\ldots + (2n-1) = n^2$

(c) $(1\times 2)+(2\times 3)+\ldots+n(n+1) < n(n+1)^2$

(d) $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\ldots\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

Now ,when you are given a statement like the ones given in Examples 10.2 and 10.3, how would you check whether it is true or false ? One effective method is mathematical induction, which we shall now discuss.

10.2 The Principle of Mathematical Induction:

Let $P(n)$ be a statement involving a natural number n. If

(i) it is true for $n = 1$, i.e., $P(1)$ is true; and

(ii) assuming $k \geq 1$ and $P(k)$ to be true, it can be proved that $P(k+1)$ is true; then $P(n)$ must be true for every natural number n.

***Note** that condition (ii) above **does not** say that P(k) **is** true. It says that **whenever** P(k) is true, then P(k + 1) is true'.*

Let us see, for example, how the principle of mathematical induction allows us to conclude that $P(n)$ is true for $n = 11$.

By (i) $P(1)$ is true. As $P(1)$ is true, we can put $k = 1$ in (ii), So $P(1+1)$, i.e., $P(2)$ is true. As $P(2)$ is true, we can put $k = 2$ in (ii) and conclude that $P(2+1)$, i.e., $P(3)$ is true. Now put $k = 3$ in (ii), so we get that $P(4)$ is true. It is now clear that if we continue like this, we shall get that $P(11)$ is true.

It is also clear that in the above argument, 11 does not play any special role. We can prove that $P(137)$ is true in the same way. Indeed, it is clear that $P(n)$ is true for all $n > 1$.

Example 10.4 Prove that, $1+2+3+\cdots+n=\frac{n}{2}(n+1)$, where n is a natural number.

Solution: We have , $P(n) : 1 + 2 + 3 + \ldots + n = \frac{n}{2}(n+1)$

Therefore, $P(1)$ is '$1 = \frac{1}{2}(1+1)$', which is true,. Therefore, $P(1)$ is true.

Let us now see, is $P(k+1)$ true whenever $P(k)$ is true.

Let us, therefore, assume that $P(k)$ is true, i.e., $1 + 2 + 3 \ldots + k = \frac{k}{2}(k+1)$ ....(i)

Now, $P(k + 1)$ is $1 + 2 + 3 + \ldots + k + (k + 1) = \frac{(k+1)(k+2)}{2}$

It will be true, if we can show that LHS = RHS

The LHS of $P(k + 1) = (1 + 2 + 3 \ldots + k) + (k + 1) = \frac{k}{2}(k+1) + (k + 1)$ ....[From (i)]

$$= (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2} = \text{RHS of } P(k+1)$$

So, $P(k + 1)$ is true, if we assume that $P(k)$ is true.

Since $P(1)$ is also true, both the conditions of the principle of mathematical induction are fulfilled, we conclude that the given statement is true for every natural number n.

As you can see, we have proved the result in three steps – the **basic step** [i.e., checking (i)], the **Induction step** [i.e., checking (ii)], and hence arriving at the end result.

Example 10.5 For every natural number n, prove that $\left(x^{2n-1} + y^{2n-1}\right)$ is divisible by $(x + y)$, where $x, y \in N$.

Solution: Let us see if we can apply the principle of induction here. Let us call $P(n)$ the statement '$\left(x^{2n-1} + y^{2n-1}\right)$ is divisible by $(x + y)$',

Then $P(1)$ is '$\left(x^{2-1} + y^{2-1}\right)$ is divisible by $(x + y)$', i.e., '$(x + y)$ is divisible by $(x + y)$', which is true.

Therefore, $P(1)$ is true.

Let us now assume that $P(k)$ is true for some natural number k, i.e., $(x^{2k-1} + y^{2k-1})$ is divisible by $(x + y)$.

This means that for some natural number t, $x^{2k-1} + y^{2k-1} = (x + y)t$

Then, $x^{2k-1} = (x + y)t - y^{2k-1}$

We wish to prove that P $(k+1)$ is true, i.e., '$[x^{2(k+1)-1} + y^{2(k+1)-1}]$ is divisible by $(x+y)$' is true.

Now,

$$x^{2(k+1)-1} + y^{2(k+1)-1} = x^{2k+1} + y^{2k+1}$$

$$= x^{2k-1+2} + y^{2k+1}$$

$$= x^{2} \cdot x^{2k-1} + y^{2k+1}$$

$$= x^{2\cdot}[(x+y)t - y^{2k-1}] + y$$
$$= x^2(x+y)t - x^2 y^{2k-1} + y^{2k+1}$$
$$= x^2(x+y)t - x^2 y^{2k-1} + y^2 y^{2k-1}$$
$$= x^2(x+y)t - y^{2k-1}(x^2 - y^2)$$
$$= (x+y)[x^2 t - (x-y)\, y^{2k-1}]$$

which is divisible by $(x + y)$.
Thus, $P\,(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Example 10.6 Prove that $2^n > n$ for every natural number n.

Solution: We have $P(n) : 2^n > n$.

Therefore, $P(1) : 2^1 > 1$, i.e., $2 > 1$, which is true.

We assume $P(k)$ to be true, that is,

$2^k > k$ ... (i)

We wish to prove that $P(k + 1)$ is true, i.e. $2^{k+1} > k + 1$.

Now, multiplying both sides of (i) by 2, we get, $2^{k+1} > 2k$

$\Rightarrow 2^{k+1} > k + 1$, since $k > 1$. Therefore, $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Sometimes, we need to prove a statement for all natural numbers greater than a particular natural number, say a (as in Example 10.7 below). In such a situation, we replace $P(1)$ by $P(a + 1)$ in the statement of the principle.

Example 10.7 Prove that

$n^2 > 2(n + 1)$ for all $n \geq 3$, where n is a natural number.

Solution: For $n \geq 3$, let us call the given statement, $P(n) : n^2 > 2\,(n + 1)$

Since we have to prove the given statement for $n \geq 3$, the first relevant statement is $P(3)$.
We, therefore, see whether $P(3)$ is true.
$P(3) : 3^2 > 2 \times 4$, i.e. $9 > 8$. So, $P\,(3)$ is true.

Let us assume that $P(k)$ is true, where $k \geq 3$, that is , $k^2 > 2(k + 1)$ .... (i)

We wish to prove that $P\,(k + 1)$ is true.

$P\,(k + 1) : (k + 1)^2 > 2(k + 2)$

LHS of $P\,(k + 1) = (k + 1)^2 = k^2 + 2k + 1$

$> 2(k+1) + 2k + 1$ ... [By (i)]

$> 3 + 2k + 1$, since $2(k+1) > 3$. $= 2(k+2)$,

Thus, $(k+1)^2 > 2(k+2)$. Therefore, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n \geq 3$.

Example 10.8 Using principle of mathematical induction, prove that

$$\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right) \text{ is a natural number for all natural numbers } n.$$

Solution : Let $P(n) : \left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ be a natural number.

$\therefore P(1) : \left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$ is a natural number.

or, $\frac{1}{5}+\frac{1}{3}+\frac{7}{15}=\frac{3+5+7}{15}=\frac{15}{15}=1$, which is a natural number $\therefore$ $P(1)$ is true.

Let $P(k) : \left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)$ is a natural number be true ... (i)

Now $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$

$$=\frac{1}{5}\left[k^5+5k^4+10k^3+10k^2+5k+1\right]+\frac{1}{3}\left[k^3+3k^2+3k+1\right]+\left(\frac{7}{15}k+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+\left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+1 \quad \text{...(ii)}$$

By (i), $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

also $k^4+2k^3+3k^2+2k$ is a natural number and 1 is also a natural number.

$\therefore$ (ii) being sum of natural numbers is a natural number.

$\therefore$ $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(n)$ is true for all natural numbers n.

Hence, $\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ is a natural number for all natural numbers n.

CHECK YOUR PROGRESS 10.2

1. Using the principle of mathematical induction, prove that the following statements hold for any natural number n:

   (a) $1^2+2^2+3^2+........+n^2=\frac{n}{6}(n+1)(2n+1)$

   (b) $1^3+2^3+3^3+........+n^3=(1+2+.....+n)^2$

   (c) $1+3+5+........+(2n-1)=n^2$

   (d) $1+4+7+........+(3n-2)=\frac{n}{2}(3n-1)$

2. Using principle of mathematical induction, prove the following equalities for any natural number n:

   (a) $\frac{1}{1\times 2}+\frac{1}{2\times 3}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$

   (b) $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

   (c) $(1\times 2)+(2\times 3)+....+n(n+1)=\frac{n(n+1)(n+2)}{3}$

3. For every natural number n, prove that

   (a) n^3+5n is divisible by 6. (b) (x^n-1) is divisible by $(x-1)$.

   (c) (n^3+2n) is divisible by 3. (d) 4 divides $(n^4+2n^3+n^2)$.

4. Prove the following inequalities for any natural number n:

   (a) $3^n \geq 2n+1$ (b) $4^{2n} > 15n$ (c) $1+2+....+n<\frac{1}{8}(2n+1)^2$

5. Prove the following statements using induction:

   (a) $2^n > n^2$ for $n \geq 5$, where n *is any natural number.*

   (b) $\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{2n}>\frac{13}{24}$ for any natural number n greater than 1.

6. Prove that $n(n^2 - 1)$ is divisible by 3 for every natural number n greater than 1.

To prove that a statement $P(n)$ is true for every $n \in N$, both the basic as well as the induction steps must hold.

If even one of these conditions does not hold, then the proof is invalid. For instance, if $P(n)$ is' $(a+b)^n \le a^n + b^n$ ' for all reals a and b, then $P(1)$ is certainly true. But, $P(k)$ being true does not imply the truth of $P(k + 1)$. So, the statement is not true for every natural number n. (For instance, $(2+3)^2 \le 2^2 + 3^2$).

As another example, take $P(n)$ to be $n > \frac{n}{2} + 20$.

In this case, $P(1)$ is not true. But the induction step is true. Since $P(k)$ being true.

$$\Rightarrow k > \frac{k}{2} + 20 \quad \Rightarrow k+1 > \frac{k}{2} + 20 + 1 > \frac{k}{2} + 20 + \frac{1}{2} = \frac{k+1}{2} + 20 \quad \Rightarrow P(k+1) \text{ is true.}$$

LET US SUM UP

- Sentences which are either true or false are called statement or propasitions.
- The word induction means, formulating a general principle (or rule) based on several particular instances.
- The statement of the principle of mathematical indction.

  P (n), a statement involving a natural number n, is true for all $n \ge 1$, where n is a fixed natural number, if

  (i) P (1) is true, and

  (ii) whenever P (k) is true, then P (K+1) is true for $k \in n$

SUPPORTIVE WEB SITES

http://www.bbc.co.uk/education/asguru/maths/13pure/01proof/01proof/05induction/index.shtml

www.mathguru.com/result/**principle-of-mathematical-induction**.aspx

http://en.wikipedia.org/wiki/**Mathematical_induction**

TERMINAL EXERCISE

1. Verify each of the following statements, using the principle of mathematical induction :

   (1) The number of subsets of a set with n elements is 2^n.

   (2) $(a + b)^n > a^n + b^n \ \forall n \ge 2$, where a and b are positive real numbers.

(3) $a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r-1}$,where r > 1 and a is a real number.

(4) $(x^{2n} - 1)$ is divisible by $(x+1) \ \forall x \in N$.

(5) $(10^{2n-1} + 1)$ is a multiple of 11, where $n \in N$

(6) $(4.10^{2n} + 9.10^{2n-1} + 5)$ is a multiple of 99. where $n \in N$

(7) $(1 + x)^n > 1 + nx$, where $x > 0$ and $n \in N$

(8) $1.2 + 2.2^2 + 3.2^3 + 4.2^4 + \cdots + n.2^n = (n-1).2^{n+1}$, where $n \in N$

(9) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2, n \in N$

(10) $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$ where $n \in N$

ANSWERS

CHECK YOUR PROGRESS 10.1

1. (b), (e) and (f) are statements; (a) is not, since we have not given the range of values of n, and therefore we are not in a position to decide, if it is true or not. (c) is subjective and hence not a mathematical statement. (d) is a question, not a statement.

   Note that (*f*)is universally false.

2. P(1) : 6 is a factor of $1^3 + 5.1$, P(2) : 6 is a factor of $2^3 + 5.2$

   P(k) : 6 is a factor of $k^3 + 5k$, P(k+1) : 6 is a factor of $(k + 1)^3 + 5(k + 1)$

3. (a) $P(1) : 2 \geq 2$, $P(k) : 2^k \geq k + 1$, $P(k + 1) : 2^{k+1} \geq k + 2$

   (b) $P(1) : 1 + x \geq 1 + x$, $P(k) : (1 + x)^k \geq 1 + kx$

   $P(k + 1): (1 + x)^{k+1} \geq 1 + (k + 1)x$

   (c) $P(1)$: 6 is divisible by 6. $P(k) : k(k + 1)(k + 2)$ is divisible by 6.

   $P(k + 1): (k + 1)(k + 2)(k + 3)$ is divisible by 6

   (d) $P(1): (x - y)$ is divisible by $(x - y)$. $P(k): (x^k - y^k)$ is divisible by $(x - y)$

   $P(k + 1): (x^{k+1} - y^{k+1})$ is divisible by $(x - y)$

   (e) $P(1): ab = ab$, $P(k): (ab)^k = a^k b^k$

   $P(k+1): (ab)^{k+1} = a^{k+1} . b^{k+1}$

   (f) $P(1) : \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$ is a natural number. $P(k)$: $\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}$ is a natural number.

   $P(k + 1)$: $\frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$ is a natural number.

4. (a) $P(1)$: $\frac{1}{1 \times 2} = \frac{1}{2}$, $P(2) : \frac{1}{1 \times 2} + \frac{1}{2 \times 3} = \frac{2}{3}$

$$P(k): \frac{1}{1\times 2}+\frac{1}{2\times 3}+...+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

$$P(k+1): \frac{1}{1\times 2}+...+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

(b) $P(1):\ 1=1^2,\ \ P(2):\ 1+3=2^2$

$$P(k):\ 1+3+5+...+(2k-1)=k^2$$

$$P(k+1):\ 1+3+5+...+(2k-1)+\left[2(k+1)-1\right]=(k+1)^2$$

(c) $P(1): 1\times 2<1(2)^2,\ P(2): (1\times 2)+(2\times 3)<2(3)^2$

$$P(k): (1\times 2)+(2\times 3)+...+k(k+1)<k(k+1)^2.$$

$$P(x+1): (1\times 2)+(2\times 3)+...+(k+1)(k+2)<(k+1)(k+2)^2$$

(d) $P(1):\ \frac{1}{1\times 3}=\frac{1}{3},\ P(2):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}=\frac{2}{5}$

$$P(k):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+...+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

$$P(k+1): \frac{1}{1\times 3}+\frac{1}{3\times 5}+...+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

11

PERMUTATIONS AND COMBINATIONS

One day, I wanted to travel from Bangalore to Allahabad by train. There is no direct train from Bangalore to Allahabad, but there are trains from Bangalore to Itarsi and from Itarsi to Allahabad. From the railway timetable I found that there are two trains from Bangalore to Itarsi and three trains from Itarsi to Allahabad. Now, in how many ways can I travel from Bangalore to Allahabad?

There are **counting problems** which come under the branch of Mathematics called **combinatorics.**

Suppose you have five jars of spices that you want to arrange on a shelf in your kitchen. You would like to arrange the jars, say three of them, that you will be using often in a more accessible position and the remaining two jars in a less accessible position. In this situation the order of jars is important. In how many ways can you do it?

In another situation suppose you are painting your house. If a particular shade or colour is not available, you may be able to create it by mixing different colours and shades. While creating new colours this way, the order of mixing is not important. It is the combination or choice of colours that determine the new colours; but not the order of mixing.

To give another similar example, when you go for a journey, you may not take all your dresses with you. You may have 4 sets of shirts and trousers, but you may take only 2 sets. In such a case you are choosing 2 out of 4 sets and the order of choosing the sets doesn't matter. In these examples, we need to find out the number of choices in which it can be done.

In this lesson we shall consider simple counting methods and use them in solving such simple counting problems.

OBJECTIVES

After studying this lesson, you will be able to :

- find out the number of ways in which a given number of objects can be arranged;
- state the Fundamental Principle of Counting;
- define n! and evaluate it for defferent values of n;
- state that permutation is an arrangement and write the meaning of ${}^nP_r$;
- state that ${}^nP_r = \dfrac{n!}{(n-r)!}$ and apply this to solve problems;

- show that (i) $(n+1)\,{}^nP_n = {}^{n+1}P_n$ (ii) ${}^nP_{r+1} = (n-r)\,{}^nP_r$;
- state that a combination is a selection and write the meaning of ${}^nC_r$;
- distinguish between permutations and combinations;
- derive ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ and apply the result to solve problems;
- derive the relation ${}^nP_r = r!\,{}^nC_r$;
- verify that ${}^nC_r = {}^nC_{n-r}$ and give its interpretation; and
- derive ${}^nC_r + {}^nC_{n-r} = {}^{n+1}C_r$ and apply the result to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Number Systems
- Four Fundamental Operations

11.1 FUNDAMENTAL PRINCIPLE OF COUNTING

Let us now solve the problem mentioned in the introduction. We will write t_1, t_2 to denote trains from Bangalore to Itarsi and T_1, T_2, T_3, for the trains from Itarsi to Allahabad. Suppose I take t_1 to travel from Bangalore to Itarsi. Then from Itarsi I can take T_1 or T_2 or T_3. So the possibilities are t_1T_1, t_1T_2 and t_1T_3 where t_1T_1 denotes travel from Bangalore to Itarsi by t_1 and travel from Itarsi to Allahabad by T_1. Similarly, if I take t_2 to travel from Bangalore to Itarsi, then the possibilities are t_2T_1, t_2T_2 and t_2T_3. Thus, in all there are $6(2\times 3)$ possible ways of travelling from Bangalore to Allahabad.

Here we had a small number of trains and thus could list all possibilities. Had there been 10 trains from Bangalore to Itarsi and 15 trains from Itarsi to Allahabad, the task would have been very tedious. Here the **Fundamental Principle of Counting** or simply the **Counting Principle** comes in use :

If any event can occur in *m* ways and after it happens in any one of these ways, a second event can occur in *n* ways, then both the events together can occur in $m\times n$ ways.

Example 11.1 How many multiples of 5 are there from 10 to 95 ?

Solution : As you know, multiples of 5 are integers having 0 or 5 in the digit to the extreme right (i.e. the unit's place).

The first digit from the right can be chosen in 2 ways.

The second digit can be any one of 1,2,3,4,5,6,7,8,9.

i.e. There are 9 choices for the second digit.

Thus, there are $2\times 9 = 18$ multiples of 5 from 10 to 95.

Example 11.2 In a city, the bus route numbers consist of a natural number less than 100, followed by one of the letters A,B,C,D,E and F. How many different bus routes are possible?

Solution : The number can be any one of the natural numbers from 1 to 99.
There are 99 choices for the number.

The letter can be chosen in 6 ways.

$\therefore$ Number of possible bus routes are $99 \times 6 = 594$.

CHECK YOUR PROGRESS 11.1

1. (a) How many 3 digit numbers are multiples of 5?

   (b) A coin is tossed thrice. How many possible outcomes are there?

   (c) If you have 3 shirts and 4 trousers and any shirt can be worn with any of trousers, in how many ways can you wear your shirts and trousers?

2. (a) In how many ways can two vacancies be filled from among 4 men and 12 women if one vacancy is filled by a man and the other by a woman?

   (b) A tourist wants to go to another country by ship and return by air. She has a choice of 5 different ships to go by and 4 airlines to return by. In how many ways can she perform the journey?

So far, we have applied the counting principle for two events. But it can be extended to three or more, as you can see from the following examples :

Example 11.3 There are 3 questions in a question paper. If the questions have 4,3 and 2 solutions respectively, find the total number of solutions.

Solution : Here question 1 has 4 solutions, question 2 has 3 solutions and question 3 has 2 solutions.

$\therefore$ By the multiplication (counting) rule,

total number of solutions = $4 \times 3 \times 2$ = 24

Example 11.4 Consider the word ROTOR. Whichever way you read it, from left to right or from right to left, you get the same word. Such a word is known as *palindrome*. Find the maximum possible number of 5-letter palindromes.

Solution : The first letter from the right can be chosen in 26 ways because there are 26 alphabets. Having chosen this, the second letter can be chosen in 26 ways

$\therefore$ The first two letters can be chosen in $26 \times 26 = 676$ ways

Having chosen the first two letters, the third letter can be chosen in 26 ways.
$\therefore$ All the three letters can be chosen in $676 \times 26 = 17576$ ways.

It implies that the maximum possible number of five letter palindromes is 17576 because the fourth letter is the same as the second letter and the fifth letter is the same as the first letter.

Note : *In Example 11.4 we found the maximum possible number of five letter palindromes. There cannot be more than 17576. But this does not mean that there are 17576 palindromes. Because some of the choices like CCCCC may not be meaningful words in the English language.*

Example 11.5 How many 3-digit numbers can be formed with the digits 1,4,7,8 and 9 if the digits are not repeated.

Solution : Three digit number will have unit's, ten's and hundred's place.

Out of 5 given digits any one can take the unit's place.

This can be done in 5 ways. ... (*i*)

After filli ng the unit's place, any of the four remaining digits can take the ten's place.

This can be done in 4 ways. ... (*ii*)

After filling in ten's place, hundred's place can be filled from any of the three remaining digits.

This can be done in 3 ways. ... (*iii*)

$\therefore$ By counting principle, the number of 3 digit numbers = $5 \times 4 \times 3$ = 60

Let us now state the **General Counting Principle**

If there are *n* events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occured, the third event can occur in m_3 ways after the second event has ocurred, and so on, then all the *n* events can occur in $m_1 \times m_2 \times \ldots \times m_{n-1} \times m_n$ ways.

Example 11.6 Suppose you can travel from a place *A* to a place *B* by 3 buses, from place *B* to place *C* by 4 buses, from place *C* to place *D* by 2 buses and from place *D* to place *E* by 3 buses. In how many ways can you travel from *A* to *E*?

Solution : The bus from *A* to *B* can be selected in 3 ways.

The bus from *B* to *C* can be selected in 4 ways.

The bus from *C* to *D* can be selected in 2 ways.

The bus from *D* to *E* can be selected in 3 ways.

So, by the General Counting Principle, one can travel from *A* to *E* in $3 \times 4 \times 2 \times 3$ ways = 72 ways.

CHECK YOUR PROGRESS 11.2

1. (a) What is the maximum number of 6-letter palindromes?

   (b) What is the number of 6-digit palindromic numbers which do not have 0 in the first digit?

2. (a) In a school there are 5 English teachers, 7 Hindi teachers and 3 French teachers. A three member committee is to be formed with one teacher representing each language. In how many ways can this be done?

   (b) In a college students union election, 4 students are contesting for the post of President. 5 students are contesting for the post of Vice-president and 3 students are contesting for the post of Secretary. Find the number of possible results.

3. (a) How many three digit numbers greater than 600 can be formed using the digits 1,2,5,6,8 without repeating the digits?

   (b) A person wants to make a time table for 4 periods. He has to fix one period each for English, Mathematics, Economics and Commerce. How many different time tables can he make?

11.2 PERMUTATIONS

Suppose you want to arrange your books on a shelf. If you have only one book, there is only one way of arranging it. Suppose you have two books, one of History and one of Geography.

You can arrange the Geography and History books in two ways. Geography book first and the History book next, *GH* or History book first and Geography book next; *HG*. In other words, there are two arrangements of the two books.

Now, suppose you want to add a Mathematics book also to the shelf. After arranging History and Geography books in one of the two ways, say *GH*, you can put Mathematics book in one of the following ways: *MGH*, *GMH* or *GHM*. Similarly, corresponding to *HG*, you have three other ways of arranging the books. So, by the Counting Principle, you can arrange Mathematics, Geography and History books in 3×2 ways = 6 ways.

By permutation we mean an arrangement of objects in a particular order. In the above example, we were discussing the number of permutations of one book or two books.

In general, if you want to find the number of permutations of n objects $n \geq 1$, how can you do it? Let us see if we can find an answer to this.

Similar to what we saw in the case of books, there is one permutation of 1 object, 2×1 permutations of two objects and $3 \times 2 \times 1$ permutations of 3 objects. It may be that, there are $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$ permutations of n objects. In fact, it is so, as you will see when we prove the following result.

Theorem 11.1 The total number of permutations of n objects is $n(n-1)$ ....2.1.

Proof : We have to find the number of possible arrangements of n different objects.

The first place in an arrangement can be filled in n different ways. Once it has been done, the second place can be filled by any of the remaining (n–1) objects and so this can be done in (n–1) ways. Similarly, once the first two places have been filled, the third can be filled in (n–2) ways and so on. The last place in the arrangement can be filled only in one way, because in this case we are left with only one object.

Using the counting principle, the total number of arrangements of n different objects is $n(n-1)(n-2)$ .... 2.1. **.....(11.1)**

The product $n(n-1)$ ... 2.1 occurs so often in Mathematics that it deserves a name and notation. It is usually denoted by $\boldsymbol{n!}$ (or by $\underline{|n}$ read as ***n*** **factorial**).

$$\boldsymbol{n! = n(n-1) \ldots 3.2.1}$$

Here is an example to help you familiarise yourself with this notation.

Example 11.7 Evaluate (a) 3! (b) 2! + 4! (c) 2!×3!

Solution : (a) $3! = 3 \times 2 \times 1 = 6$

(b) $2! = 2 \times 1 = 2$, $4! = 4 \times 3 \times 2 \times 1 = 24$

Therefore, $2! + 4! = 2 + 24 = 26$

(c) $2! \times 3! = 2 \times 6 = 12$

Notice that $n!$ satisfies the relation, $n! = n \times (n-1)!$ **... (11.2)**

This is because, $n(n-1)! = n[(n-1).(n-2)... 2.1]$

$$= n.(n-1).(n-2)...2.1 = n!$$

Of course, the above relation is valid only for $n \geq 2$ because 0! has not been defined so far. Let us see if we can define 0! to be consistent with the relation. In fact, if we define

$$0! = 1$$

then the relation 11.2 holds for $n = 1$ also.

Example 11.8 Suppose you want to arrange your English, Hindi, Mathematics, History, Geography and Science books on a shelf. In how many ways can you do it?

Solution : We have to arrange 6 books.

The number of permutations of n objects is $n! = n.(n-1).(n-2) \ldots 2.1$

Here $n = 6$ and therefore, number of permutations is 6.5.4.3.2.1 = 720

CHECK YOUR PROGRESS 11.3

1. (a) Evaluate : (i) 6! (ii) 7! (iii) 7! + 3! (iv) $6! \times 4!$ (v) $\frac{5!}{3!.2!}$

   (b) Which of the following statements are true?

   (i) $2! \times 3! = 6!$ (ii) $2! + 4! = 6!$

   (iii) 3! divides 4! (iv) 4! - 2! = 2!

2. (a) 5 students are staying in a dormitory. In how many ways can you allot 5 beds to them?

   (b) In how many ways can the letters of the word 'TRIANGLE' be arranged?

   (c) How many four digit numbers can be formed with digits 1, 2, 3 and 4 and with distinct digits?

11.3 PERMUTATION OF *r* OBJECTS OUT OF *n* OBJECTS

Suppose you have five story books and you want to distribute one each to Asha, Akhtar and Jasvinder. In how many ways can you do it? You can give any one of the five books to Asha and after that you can give any one of the remaining four books to Akhtar. After that, you can give one of the remaining three books to Jasvinder. So, by the Counting Principle, you can distribute the books in $5 \times 4 \times 3$ *ie.*60 ways.

More generally, suppose you have to arrange *r* objects out of *n* objects. In how many ways can you do it? Let us view this in the following way. Suppose you have *n* objects and you have to arrange *r* of these in *r* boxes, one object in each box.

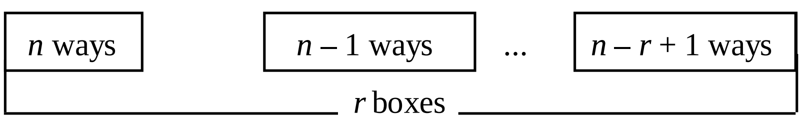


Fig. 11.1

Suppose there is one box. $r = 1$. You can put any of the *n* objects in it and this can be done in *n* ways. Suppose there are two boxes. $r = 2$. You can put any of the objects in the first box and after that the second box can be filled with any of the remaining $n - 1$ objects. So, by the counting principle, the two boxes can be filled in $n(n-1)$ ways. Similarly, 3 boxes can be filled in $n(n-1)(n-2)$ ways.

In general, we have the following theorem.

Theorem 11.2 The number of permutations of *r* objects out of *n* objects is

$$n(n-1)\cdots(n-r+1).$$

The number of permutations of *r* objects out of *n* objects is usually denoted by ${}^nP_r$.

Thus, $${}^nP_r = n(n-1)(n-2)...(n-r+1) \qquad \textbf{... (11.3)}$$

Proof : Suppose we have to arrange *r* objects out of *n* different objects. In fact it is equivalent to filling *r* places, each with one of the objects out of the given *n* objects.

The first place can be filled in *n* different ways. Once this has been done, the second place can be filled by any one of the remaining (n–1) objects, in (n–1) ways. Similarly, the third place can be filled in (n – 2) ways and so on. The last place, the *r*th place can be filled in [n–(r–1)] i.e. (n–r+1) different ways. You may easily see, as to why this is so.

Using the Counting Principle, we get the required number of arrangements of *r* out of *n* objects is $n(n-1)(n-2)............(n-r+1)$

Example 11.9 Evaluate : (a) ${}^4P_2$ (b) ${}^6P_3$ (c) $\dfrac{{}^4P_3}{{}^3P_2}$ (d) ${}^6P_3 \times {}^5P_2$

Solution : (a) ${}^4P_2 = 4(4-1) = 4\times3 = 12.$

(b) ${}^6P_3 = 6(6-1)\ (6-2) = 6\times5\times4 = 120.$

(c) $\dfrac{{}^4P_3}{{}^3P_2} = \dfrac{4(4-1)(4-2)}{3(3-1)} = \dfrac{4\times3\times2}{3\times2} = 4$

(d) ${}^6P_3 \times {}^5P_2 = 6\ (6-1)\ (6-2)\times5\ (5-1),\ = 6\times5\times4\times5\times4\ =\ 2400$

Example 11.10 If you have 6 New Year greeting cards and you want to send them to 4 of your friends, in how many ways can this be done?

Solution : We have to find number of permutations of 4 objects out of 6 objects.

This number is ${}^6P_4 = 6(6-1)(6-2)(6-3) = 6.5.4.3 = 360$

Therefore, cards can be sent in 360 ways.

Consider the formula for ${}^nP_r$, namely, ${}^nP_r = n\,(n-1)\ ...\ (n-r+1)$. This can be obtained by removing the terms $n-r,\ n-r-1,...,2,1$ from the product for n!. The product of these terms is $(n-r)\ (n-r-1)$ ...2.1, i.e., $(n-r)!$.

$$\text{Now, } \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)...(n-r+1)(n-r)...2.1}{(n-r)\ (n-r-1)...2.1}$$

$$= n(n-1)(n-2)...(n-r+1) = {}^nP_r$$

So, using the factorial notation, this formula can be written as follows : ${}^nP_r = \dfrac{n!}{(n-r)!}$ .....**(11.4)**

Example 11.11 Find the value of ${}^nP_0$.

Solution : Here $r = 0$. Using relation 11.4, we get ${}^nP_0 = \dfrac{n!}{n!} = 1$

Example 11.12 Show that $(n + 1)\,{}^nP_r = {}^{n+1}P_{r+1}$

Solution : $(n + 1)\,{}^nP_r = (n+1)\dfrac{n!}{(n-r)!} = \dfrac{(n+1)\,n!}{(n-r)!}$

$$= \frac{(n+1)!}{[(n+1)-(r+1)]\,!} \quad \text{[writing } n - r \text{ as } [(n + 1) - (r + 1)]$$

$$= {}^{n+1}P_{r+1} \qquad \text{(By definition)}$$

CHECK YOUR PROGRESS 11.4

1. (a) Evaluate : (i) ${}^4P_2$ (ii) ${}^6P_3$ (iii) $\dfrac{{}^4P_3}{{}^3P_2}$ (iv) ${}^6P_3 \times {}^5P_2$ (v) ${}^nP_n$

   (b) Verify each of the following statements :

   (i) $6 \times {}^5P_2 = {}^6P_2$ (ii) $4 \times {}^7P_3 = {}^7P_4$

   (iii) ${}^3P_2 \times {}^4P_2 = {}^{12}P_4$ (iv) ${}^3P_2 + {}^4P_2 = {}^7P_4$

2. (a) (i) What is the maximum possible number of 3- letter words in English that do not contain any vowel?

   (ii) What is the maximum possible number of 3- letter words in English which do not have any vowel other than 'a'?

   (b) Suppose you have 2 cots and 5 bedspreads in your house. In how many ways can you put the bedspreads on your cots?

   (c) You want to send Diwali Greetings to 4 friends and you have 7 greeting cards with you. In how many ways can you do it?

3. Show that ${}^nP_{n-1} = {}^nP_n$.

4. Show that $(n-r)\,{}^nP_r = {}^nP_{r+1}$.

11.4 PERMUTATIONS UNDER SOME CONDITIONS

We will now see examples involving permutations with some extra conditions.

Example 11.13 Suppose 7 students are staying in a hall in a hostel and they are allotted 7 beds. Among them, Parvin does not want a bed next to Anju because Anju snores. Then, in how many ways can you allot the beds?

Solution : Let the beds be numbered 1 to 7.

Case 1 : Suppose Anju is allotted bed number 1.

Then, Parvin cannot be allotted bed number 2.

So Parvin can be allotted a bed in 5 ways.

After alloting a bed to Parvin, the remaining 5 students can be allotted beds in 5! ways.

So, in this case the beds can be allotted in $5 \times 5! \, ways = 600$ ways.

Case 2 : Anju is allotted bed number 7.

Then, Parvin cannot be allotted bed number 6

As in Case 1, the beds can be allotted in 600 ways.

Case 3 : Anju is allotted one of the beds numbered 2,3,4,5 or 6.

Parvin cannot be allotted the beds on the right hand side and left hand side of Anju's bed. For example, if Anju is allotted bed number 2, beds numbered 1 or 3 cannot be allotted to Parvin.

Therefore, Parvin can be allotted a bed in 4 ways in all these cases.

After allotting a bed to Parvin, the other 5 can be allotted a bed in 5! ways.

Therefore, in each of these cases, the beds can be allotted in $4 \times 5! = 480$ ways.

$\therefore$ The beds can be allotted in

$$(2 \times 600 + 5 \times 480) \, ways = (1200 + 2400) \, ways = 3600 \text{ ways.}$$

Example 11.14 In how many ways can an animal trainer arrange 5 lions and 4 tigers in a row so that no two lions are together?

Solution : They have to be arranged in the following way :

L	T	L	T	L	T	L	T	L

The 5 lions should be arranged in the 5 places marked 'L'. This can be done in 5! ways.

The 4 tigers should be in the 4 places marked 'T'.This can be done in 4! ways.

Therefore, the lions and the tigers can be arranged in $5! \times 4! \, ways = 2880$ ways.

Example 11.15 There are 4 books on fairy tales, 5 novels and 3 plays. In how many ways can you arrange these so that books on fairy tales are together, novels are together and plays are together and in the order, books on fairytales, novels and plays.

Solution : There are 4 books on fairy tales and they have to be put together.

They can be arranged in 4! ways. Similarly, there are 5 novels.

They can be arranged in 5! ways. And there are 3 plays.

They can be arranged in 3! ways.

So, by the counting principle all of them together can be arranged in $4! \times 5! \times 3!$ ways = 17280 ways.

Example 11.16 Suppose there are 4 books on fairy tales, 5 novels and 3 plays as in Example 11.15. They have to be arranged so that the books on fairy tales are together, novels are together and plays are together, but we no longer require that they should be in a specific order. In how many ways can this be done?

Solution : First, we consider the books on fairy tales, novels and plays as single objects.

These three objects can be arranged in 3!ways = 6 ways.

Let us fix one of these 6 arrangements.

This may give us a specific order, say, novels $\rightarrow$ fairy tales $\rightarrow$ plays.

Given this order, the books on the same subject can be arranged as follows.

The 4 books on fairy tales can be arranged among themselves in 4! = 24 ways.

The 5 novels can be arranged in 5! = 120 ways.

The 3 plays can be arranged in 3! = 6 ways.

For a given order, the books can be arranged in $24 \times 120 \times 6 = 17280$ ways.

Therefore, for all the 6 possible orders the books can be arranged in $6 \times 17280 = 103680$ ways.

Example 11.17 In how many ways can 4 girls and 5 boys be arranged in a row so that all the four girls are together?

Solution : Let 4 girls be one unit and now there are 6 units in all.

They can be arranged in 6! ways.

In each of these arrangements 4 girls can be arranged in 4! ways.

$\therefore$ Total number of arrangements in which girls are always together

$= 6! \times 4! = 720 \times 24 = 17280$

Example 11.18 How many arrangements of the letters of the word 'BENGALI' can be made if the vowels are always together.

Solution : There are 7 letters in the word 'Bengali; of these 3 are vowels and 4 consonants. Considering vowels *a*, *e*, *i* as one letter, we can arrange 4+1 letters in 5! ways in each of which vowels are together. These 3 vowels can be arranged among themselves in 3! ways.

$\therefore$ Total number of words = $5! \times 3! = 120 \times 6 = 720$

CHECK YOUR PROGRESS 11.5

1. Mr. Gupta with Ms. Gupta and their four children are travelling by train. Two lower berths, two middle berths and 2 upper berths have been allotted to them. Mr. Gupta has

undergone a knee surgery and needs a lower berth while Ms. Gupta wants to rest during the journey and needs an upper berth. In how many ways can the berths be shared by the family?

2. Consider the word UNBIASED. How many words can be formed with the letters of the word in which no two vowels are together?

3. There are 4 books on Mathematics, 5 books on English and 6 books on Science. In how many ways can you arrange them so that books on the same subject are together and they are arranged in the order Mathematics $\rightarrow$ English $\rightarrow$ Science.

4. There are 3 Physics books, 4 Chemistry books, 5 Botany books and 3 Zoology books. In how many ways can you arrange them so that the books on the same subject are together?

5. 4 boys and 3 girls are to be seated in 7 chairs such that no two boys are together. In how many ways can this be done?

6. Find the number of permutations of the letters of the word 'TENDULKAR', in each of the following cases :

   (i) beginning with T and ending with R. (ii) vowels are always together.

   (iii) vowels are never together.

11.5 COMBINATIONS

Let us consider the example of shirts and trousers as stated in the introduction. There you have 4 sets of shirts and trousers and you want to take 2 sets with you while going on a trip. In how many ways can you do it?

Let us denote the sets by S_1, S_2, S_3, S_4. Then you can choose two pairs in the following ways :

1. $\{S_1, S_2\}$ 2. $\{S_1, S_3\}$ 3. $\{S_1, S_4\}$

4. $\{S_2, S_3\}$ 5. $\{S_2, S_4\}$ 6. $\{S_3, S_4\}$

[Observe that $\{S_1, S_2\}$ is the same as $\{S_2, S_1\}$]. So, there are 6 ways of choosing the two sets that you want to take with you. Of course, if you had 10 pairs and you wanted to take 7 pairs, it will be much more difficult to work out the number of pairs in this way.

Now as you may want to know the number of ways of wearing 2 out of 4 sets for two days, say Monday and Tuesday, and the order of wearing is also important to you. We know from section 11.3, that it can be done in ${}^4P_2 = 12$ ways. But note that each choice of 2 sets gives us two ways of wearing 2 sets out of 4 sets as shown below :

1. $\{S_1,S_2\} \rightarrow$ S_1 on Monday and S_2 on Tuesday or S_2 on Monday and S_1 on Tuesday
2. $\{S_1,S_3\} \rightarrow$ S_1 on Monday and S_3 on Tuesday or S_3 on Monday and S_1 on Tuesday
3. $\{S_1,S_4\} \rightarrow$ S_1 on Monday and S_4 on Tuesday or S_4 on Monday and S_1 on Tuesday

4. $\{S_2, S_3\} \rightarrow S_2$ on Monday and S_3 on Tuesday or S_3 on Monday and S_2 on Tuesday
5. $\{S_2, S_4\} \rightarrow S_2$ on Monday and S_4 on Tuesday or S_4 on Monday and S_2 on Tuesday
6. $\{S_3, S_4\} \rightarrow S_3$ on Monday and S_4 on Tuesday or S_4 on Monday and S_3 on Tuesday

Thus, there are 12 ways of wearing 2 out of 4 pairs.

This argument holds good in general as we can see from the following theorem.

Theorem 11.3 Let $n \geq 1$ be an integer and $r \leq n$. Let us denote the number of ways of choosing *r* objects out of *n* objects by ${}^nC_r$. Then

$$^nC_r = \frac{^nP_r}{r!} \qquad \text{... (11.5)}$$

Proof : We can choose *r* objects out of *n* objects in ${}^nC_r$ ways. Each of the *r* objects chosen can be arranged in *r*! ways. The number of ways of arranging *r* objects is *r*!. Thus, by the counting principle, the number of ways of choosing *r* objects and arranging the *r* objects chosen can be done in ${}^nC_r r!$ ways. But, this is precisely ${}^nP_r$. In other words, we have

$$^nP_r = r!.{}^nC_r \qquad \text{... (11.6)}$$

Dividing both sides by *r*!, we get the result in the theorem.

Here is an example to help you to familiarise yourself with ${}^nC_r$.

Example 11.19 Evaluate each of the following :

(a) ${}^5C_2$ (b) ${}^5C_3$ (c) ${}^4C_3 + {}^4C_2$ (d) $\frac{{}^6C_3}{{}^4C_2}$

Solution : (a) ${}^5C_2 = \frac{{}^5P_2}{2!} = \frac{5.4}{1.2} = 10$. (b) ${}^5C_3 = \frac{{}^5P_3}{3!} = \frac{5.4.3}{1.2.3} = 10$.

(c) $${}^4C_3 + {}^4C_2 = \frac{{}^4P_3}{3!} + \frac{{}^4P_2}{2!} = \frac{4.3.2}{1.2.3} + \frac{4.3}{1.2} = 4 + 6 = 10$$

(d) $${}^6C_3 = \frac{{}^6P_3}{3!} = \frac{6.5.4}{1.2.3} = 20 \text{ and } {}^4C_2 = \frac{4.3}{1.2} = 6$$

$$\therefore \frac{{}^6C_3}{{}^4C_2} = \frac{20}{6} = \frac{10}{3}.$$

Example 11.20 Find the number of subsets of the set {1,2,3,4,5,6,7,8,9,10,11} having 4 elements.

Solution : Here the order of choosing the elements doesn't matter and this is a problem in combinations.

We have to find the number of ways of choosing 4 elements of this set which has 11 elements.

By relation (11.5), this can be done in $^{11}C_4 = \dfrac{11.10.9.8}{1.2.3.4} = 330$ ways.

Example 11.21 12 points lie on a circle. How many cyclic quadrilaterals can be drawn by using these points?

Solution : For any set of 4 points we get a cyclic quadrilateral. Number of ways of choosing 4 points out of 12 points is $^{12}C_4 = 495$. Therefore, we can draw 495 quadrilaterals.

Example 11.22 In a box, there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen?

Solution : Number of ways of choosing 2 black pens from 5 black pens

$$= {}^5C_2 = \frac{^5P_2}{2!} = \frac{5.4}{1.2} = 10.$$

Number of ways of choosing 2 white pens from 3 white pens , $= {}^3C_2 = \dfrac{^3P_2}{2!} = \dfrac{3.2}{1.2} = 3$.

Number of ways of choosing 2 red pens from 4 red pens , $= {}^4C_2 = \dfrac{^4P_2}{2!} = \dfrac{4.3}{1.2} = 6$.

$\therefore$ By the Counting Principle, 2 black pens, 2 white pens, and 2 red pens can be chosen in $10 \times 3 \times 6 = 180$ ways.

Example 11.23 A question paper consists of 10 questions divided into two parts *A* and *B*. Each part contains five questions. A candidate is required to attempt six questions in all of which at least 2 should be from part *A* and at least 2 from part *B*. In how many ways can the candidate select the questions if he can answer all questions equally well?

Solution : The candidate has to select six questions in all of which at least two should be from Part *A* and two should be from Part *B*. He can select questions in any of the following ways :

	Part *A*	Part *B*
(i)	2	4
(ii)	3	3
(iii)	4	2

If the candidate follows choice (i), the number of ways in which he can do so is

$$^5C_2 \times {}^5C_4 = 10 \times 5 = 50$$

If the candidate follows choice (ii), the number of ways in which he can do so is

$$^5C_3 \times {}^5C_3 = 10 \times 10 = 100.$$

Similarly, if the candidate follows choice (iii), then the number of ways in which he can do so is

$$^5C_4 \times {}^5C_2 = 50.$$

Therefore, the candidate can select the question in 50 + 100 + 50 = 200 ways.

Example 11.24 A committee of 5 persons is to be formed from 6 men and 4 women. In how many ways can this be done when

(i) at least 2 women are included?

(ii) atmost 2 women are included?

Solution : (i) When at least 2 women are included.

The committee may consist of

2 women, 3 men : It can be done in $^4C_2 \times {}^6C_3$ ways.

or, 3 women, 2 men : It can be done in $^4C_3 \times {}^6C_2$ ways.

or, 4 women, 1 man : It can be done in $^4C_4 \times {}^6C_1$ ways.

$\therefore$ Total number of ways of forming the committee

$$= {}^4C_2.{}^6C_3 + {}^4C_3.{}^6C_2 + {}^4C_4.{}^6C_1 = 6 \times 20 + 4 \times 15 + 1 \times 6 = 120 + 60 + 6 = 186$$

(ii) When atmost 2 women are included

The committee may consist of

2 women, 3 men : It can be done in $^4C_2.{}^6C_3$ ways

or, 1 woman, 4 men : It can be done in $^4C_1.{}^6C_4$ ways

or, 5 men : It can be done in $^6C_5$ ways

$\therefore$ Total number of ways of forming the committee

$$= {}^4C_2.{}^6C_3 + {}^4C_1.{}^6C_4 + {}^6C_5 = 6 \times 20 + 4 \times 15 + 6 = 120 + 60 + 6 = 186$$

Example 9.25 The Indian Cricket team consists of 16 players. It includes 2 wicket keepers and 5 bowlers. In how many ways can a cricket eleven be selected if we have to select 1 wicket keeper and atleast 4 bowlers?

Solution : We are to choose 11 players including 1 wicket keeper and 4 bowlers

or, 1 wicket keeper and 5 bowlers.

MODULE-III
Algebra-I

Notes

Number of ways of selecting 1 wicket keeper, 4 bowlers and 6 other players

$$= {}^{2}C_{1}.{}^{5}C_{4}.{}^{9}C_{6}$$

$$= 2\times\frac{5\times4\times3\times2.1}{4.3.2.1.}\times\frac{9\times8\times7\times6\times5\times4}{6\times5\times4\times3\times2\times1} = 2\times5\times\frac{9\times8\times7}{3\times2\times1} = 840$$

Number of ways of selecting 1 wicket keeper, 5 bowlers and 5 other players

$$= {}^{2}C_{1}.{}^{5}C_{5}.{}^{9}C_{5} = 2\times1\times\frac{9\times8\times7\times6\times5}{5\times4\times3\times2\times1} = 2\times1\times\frac{9\times8\times7\times6}{4\times3\times2\times1} = 252$$

$\therefore$ Total number of ways of selecting the team $= 840 + 252 = 1092$

CHECK YOUR PROGRESS 11.6

1. (a) Evaluate :

(i) ${}^{13}C_3$ (ii) ${}^{9}C_5$ (iii) ${}^{8}C_2 + {}^{8}C_3$ (iv) $\frac{{}^{9}C_3}{{}^{6}C_3}$

(b) Verify each of the following statement :

(i) ${}^{5}C_2 = {}^{5}C_3$ (ii) ${}^{4}C_3 \times {}^{3}C_2 = {}^{12}C_6$

(iii) ${}^{4}C_2 + {}^{4}C_3 = {}^{8}C_5$ (iv) ${}^{10}C_2 + {}^{10}C_3 = {}^{11}C_3$

2. Find the number of subsets of the set {1, 3, 5, 7, 9, 11, 13, ..., 23}each having 3 elements.

3. There are 14 points lying on a circle. How many pentagons can be drawn using these points?

4. In a fruit basket there are 5 apples, 7 plums and 11 oranges. You have to pick 3 fruits of each type. In how many ways can you make your choice?

5. A question paper consists of 12 questions divided into two parts *A* and *B*, containing 5 and 7 questions repectively. A student is required to attempt 6 questions in all, selecting at least 2 from each part. In how many ways can a student select the questions?

6. Out of 5 men and 3 women, a committee of 3 persons is to be formed. In how many ways can it be formed selecting (i) exactly 1 woman. (ii) atleast 1 woman.

7. A cricket team consists of 17 players. It includes 2 wicket keepers and 4 bowlers. In how many ways can a playing eleven be selected if we have to select 1 wicket keeper and atleast 3 bowlers?

8. To fill up 5 vacancies, 25 applications were recieved. There were 7 S.C. and 8 O.B.C. candidates among the applicants. If 2 posts were reserved for S.C. and 1 for O.B.C. candidates, find the number of ways in which selection could be made?

11.6 SOME SIMPLE PROPERTIES OF ^nC_r

In this section we will prove some simple properties of ^nC_r which will make the computations of these coefficients simpler. Let us go back again to Theorem 11.3. Using relation 11.6 we can rewrite the formula for ^nC_r as : $^nC_r = \dfrac{n!}{r!(n-r)!}$ **....(11.7)**

Example 11.26 Find the value of $^nC_0$.

Solution : Here $r = 0$. Therefore, $^nC_0 = \dfrac{n!}{0!\,n!} = \dfrac{1}{0!} = 1$,

since we have defined 0! = 1.

The formula given in Theorem 11.3 was used in the previous section. As we will see shortly, the formula given in Equation 11.7 will be useful for proving certain properties of ^nC_r.

$$^nC_r = {}^nC_{n-r} \qquad \textbf{...(11.8)}$$

This means just that the number of ways of choosing *r* objects out of *n* objects is the same as the number of ways of not choosing *(n – r)* objects out of *n* objects. In the example described in the introduction, it just means that the number of ways of selecting 2 sets of dresses is the same as the number of ways of rejecting 4 – 2 = 2 dresses. In Example 11.20, this means that the number of ways of choosing subsets with 4 elements is the same as the number of ways of rejecting 8 elements since choosing a particular subset of 4 elements is equivalent to rejecting its complement, which has 8 elements.

Let us now prove this relation using Equation 11.7. The denominator of the right hand side of this equation is $r!\,(n-r)!$. This does not change when we replace *r* by $n-r$.

$$(n-r)!.[n-(n-r)]! = (n-r)!.r!$$

The numerator is independent of *r*. Therefore, replacing *r* by $n-r$ in Equation 11.7 we get result.

How is the relation 11.8 useful? Using this formula, we get, for example, $^{100}C_{98}$ is the same as $^{100}C_2$. The second value is much more easier to calculate than the first one.

Example 11.27 Evaluate :

(a) $^7C_5$ (b) $^{11}C_9$ (c) $^{10}C_9$ (d) $^{12}C_9$

Solution : (a) From relation 11.8, we have

$$^7C_5 = {}^7C_{7-5} = {}^7C_2 = \frac{7.6}{1.2} = 21$$

(b) Similarly $\quad {}^{10}C_9 = {}^{10}C_{10-9} = {}^{10}C_1 = 10$

(c) $\quad {}^{11}C_9 = {}^{11}C_{11-9} = {}^{11}C_2 = \dfrac{11.10}{1.2} = 55$

(d) $\quad {}^{12}C_{10} = {}^{12}C_{12-10} = {}^{12}C_2 = \dfrac{12.11}{1.2} = 66$

There is another relation satisfied by ${}^nC_r$ which is also useful. We have the following relation:

$$ {}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r \qquad \textbf{....(11.9)} $$

$$ {}^{n-1}C_{r-1} + {}^{n-1}C_r = \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!} $$

$$ = \frac{(n-1)!}{(n-r)(n-r-1)!(r-1)!} + \frac{(n-1)!}{r(n-r-1)!(r-1)!} $$

$$ = \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right] $$

$$ = \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{n}{(n-r)r}\right] $$

$$ = \frac{n(n-1)!}{(n-r)(n-r-1)!r(r-1)!} $$

$$ = \frac{n!}{(n-r)!r!} = {}^nC_r $$

Example 11.28 Evaluate :

(a) ${}^6C_2 + {}^6C_1$ (b) ${}^8C_2 + {}^8C_1$ (c) ${}^5C_3 + {}^5C_2$ (d) ${}^{10}C_2 + {}^{10}C_3$

Solution : (a) Let us use relation (11.9) with $n = 7$, $r = 2$. So, ${}^6C_2 + {}^6C_1 = {}^7C_2 = 21$

(b) Here $n = 9$, $r = 2$. Therefore, ${}^8C_2 + {}^8C_1 = {}^9C_2 = 36$

(c) Here $n = 6$, $r = 3$. Therefore, ${}^5C_3 + {}^5C_2 = {}^6C_3 = 20$

(d) Here $n = 11$, $r = 3$. Therefore, ${}^{10}C_2 + {}^{10}C_3 = {}^{11}C_3 = 165$

Example 11.29 If ${}^nC_{10} = {}^nC_{12}$ find n,

Solution : Using ${}^nC_r = {}^nC_{n-r}$ we get $n - 10 = 12$ or, $n = 12 + 10 = 22$

CHECK YOUR PROGRESS 11.7

1. (a) Find the value of ${}^nC_{n-1}$. Is ${}^nC_{n-1} = {}^nC_n$? (b) Show that ${}^nC_n = {}^nC_0$

2. Evaluate :

   (a) ${}^9C_5$ (b) ${}^{14}C_{10}$

   (c) ${}^{13}C_9$ (d) ${}^{15}C_{12}$

3. Evaluate :

   (a) ${}^7C_3 + {}^7C_2$ (b) ${}^8C_4 + {}^8C_5$

   (c) ${}^9C_3 + {}^9C_2$ (d) ${}^{12}C_3 + {}^{12}C_2$

4. If ${}^{10}C_r = {}^{10}C_{2r+1}$, find the value of r.

5. If ${}^{18}C_r = {}^{18}C_{r+2}$ find ${}^rC_5$

11.7 PROBLEMS INVOLVING BOTH PERMUTATIONS AND COMBINATIONS

So far, we have studied problems that involve either permutation alone or combination alone. In this section, we will consider some examples that need both of these concepts.

Example 11.30 There are 5 novels and 4 biographies. In how many ways can 4 novels and 2 biographies can be arranged on a shelf ?

Soluton : 4 novels can be selected out of 5 in ${}^5C_4$ ways. 2 biographies can be selected out of 4 in ${}^4C_2$ ways.

Number of ways of arranging novels and biographies = ${}^5C_4 \times {}^4C_2 = 5 \times 6 = 30$

After selecting any 6 books (4 novels and 2 biographies) in one of the 30 ways, they can be arranged on the shelf in 6! = 720 ways.

By the Counting Principle, the total number of arrangements = $30 \times 720 = 21600$

Example 11.31 From 5 consonants and 4 vowels, how many words can be formed using 3 consonants and 2 vowels ?

Solution : From 5 consonants, 3 consonants can be selected in ${}^5C_3$ ways.

From 4 vowels, 2 vowels can be selected in ${}^4C_2$ ways.

Now with every selection, number of ways of arranging 5 letters is ${}^5P_5$

$\therefore$ Total number of words $= {}^5C_3 \times {}^4C_2 \times {}^5P_5 = \frac{5 \times 4}{2 \times 1} \times \frac{4 \times 3}{2 \times 1} \times 5!$

$= 10 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7200$

CHECK YOUR PROGRESS 11.8

1. There are 5 Mathematics, 4 Physics and 5 Chemistry books. In how many ways can you arrange 4 Mathematics, 3 Physics and 4 Chemistry books.

   (a) if the books on the same subjects are arranged together, but the order in which the books are arranged within a subject doesn't matter ?

   (b) if books on the same subjects are arranged together and the order in which books are arranged within subject matters ?

2. There are 9 consonants and 5 vowels. How many words of 7 letters can be formed using 4 consonents and 3 vowels ?

3. In how many ways can you invite at least one of your six friends to a dinner?

4. In an examination, an examinee is required to pass in four different subjects. In how many ways can he fail?

LET US SUM UP

- Fundamental principle of counting states.

  If there are n events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occurred, the third event can occur in m_3 ways after the second event has occurred and so on, then all the n events can occur in

  $m_1 \times m_2 \times m_3 \times \ldots\ldots \times m_{n-1} \times m_n$ ways.
- The number of permutations of n objects taken all at a time is $n!$
- $${}^{n}P_{r} = \frac{n!}{(n-r)!}$$
- $${}^{n}P_{n} = n!$$
- The number of ways of selecting r objects out of n objects is ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$
- $${}^{n}C_{r} = {}^{n}C_{n-r}$$
- $${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=XqQTXW7XfYA

http://www.youtube.com/watch?v=bCxMhncR7PU

www.mathsisfun.com/combinatorics/**combinations-permutations**.html

TERMINAL EXERCISE

1. There are 8 true - false questions in an examination. How many responses are possible ?
2. The six faces of a die are numbered 1,2,3,4,5 and 6. Two such dice are thrown simultaneously. In how many ways can they turn up ?
3. A restaurant has 3 vegetables, 2 salads and 2 types of bread. If a customer wants 1 vegetable, 1 salad and 1 bread, how many choices does he have ?
4. Suppose you want to paper your walls. Wall papers are available in 4 diffrent backgrounds colours with 7 different designs of 5 different colours on them. In how many ways can you select your wall paper ?
5. In how many ways can 7 students be seated in a row on 7 seats ?
6. Determine the number of 8 letter words that can be formed from the letters of the word *ALTRUISM*.
7. If you have 5 windows and 8 curtains in your house, in how many ways can you put the curtains on the windows ?
8. Determine the maximum number of 3- letter words that can be formed from the letters of the word *POLICY*.
9. There are 10 athletes participating in a race and there are three prizes, 1st, 2nd and 3rd to be awarded. In how many ways can these be awarded ?
10. In how many ways can you arrange the letters of the word *ATTAIN* so that the *T*s and As are together?
11. A group of 12 friends meet at a party. Each person shake hands once with all others. How many hand shakes will be there. ?
12. Suppose that you own a shop which sells televisions. You are selling 5 different kinds of television sets, but your show case has enough space for display of 3 televison sets only. In how many ways can you select the television sets for the display ?
13. A contractor needs 4 carpenters. Five equally qualified carpenters apply for the job. In how many ways can the contractor make the selection ?
14. In how many ways can a committe of 9 can be selected from a group of 13?
15. In how many ways can a committee of 3 men and 2 women be selected from a group of 15 men and 12 women ?
16. In how ways can 6 persons be selected from 4 grade 1 and 7 grade II officers, so as to include at least two officers from each category ?
17. Out of 6 boys and 4 girls, a committee of 5 has to be formed. In how many ways can this be done if we take :

    (a) 2 girls. (b) at least 2 girls.
18. The English alphabet has 5 vowels and 21 consonants. What is the maximum number of words, that can be formed from the alphabet with 2 different vowels and 2 different consonants?

19. From 5 consonants and 5 vowels, how many words can be formed using 3 consonants and 2 vowels?

20. In a school annual day function a variety programme was organised. It was planned that there would be 3 short plays, 6 recitals and 4 dance programmes. However, the chief guest invited for the function took much longer time than expected to finish his speech. To finish in time, it was decided that only 2 short plays, 4 recitals and 3 dance programmes would be perfomed, How many choices were available to them ?

    (a) if the programmes can be perfomed in any order ?

    (b) if the programmes of the same kind were perfomed at a stretch?

    (c) if the programmes of the same kind were perfomed at a strech and considering the order of performance of the programmes of the same kind ?

ANSWERS

CHECK YOUR PROGRESS 11.1

1. (a) 180 (b) 8 (c) 12
2. (a) 48 (b) 20

CHECK YOUR PROGRESS 11.2

1. (a) 17576 (b) 900
2. (a) 105 (b) 60
3. (a) 24 (b) 24

CHECK YOUR PROGRESS 11.3

1. (a) (i) 720 (ii) 5040 (iii) 5046 (iv) 17280 (v) 10
   (b) (i) False (ii) False (iii) True (iv) False
2. (a) 120 (b) 40320 (c) 24

CHECK YOUR PROGRESS 11.4

1. (a) (i) 12 (ii) 120 (iii) 4 (iv) 7200 (v) $n!$
   (b) (i) False (ii) True (iii) False (iv) False
2. (a) (i) 7980 (ii) 9240 (b) 20 (c) 840

CHECK YOUR PROGRESS 11.5

1. 96 2. 1152 3. 2073600 4. 2488320
5. 144 6. (i) 5040 (ii) 30240 (iii) 332640

CHECK YOUR PROGRESS 11.6

1. (a) (i) 286 (ii) 126 (iii) 84 (iv) $\frac{21}{5}$
   (b) (i) True (ii) False (iii) False (iv) True
2. 1771 3. 2002 4. 57750 5. 805
6. (i) 30 (ii) 46
7. 3564 8. 7560

CHECK YOUR PROGRESS 11.7

1.	(a) n , No			
2.	(a) 126	(b) 1001	(c) 715	(d) 455
3.	(a) 56	(b) 126	(c) 120	(d) 286
4.	3	5. 56		

CHECK YOUR PROGRESS 11.8

1	(a) 600	(b) 207 3600
2.	6350400	
3.	63	4. 15

TERMINAL EXERCISE

1.	256	2.	36	3.	12	4.	140
5.	5040	6.	40320	7.	6720	8.	120
9.	720	10.	24	11.	66	12.	10
13.	5	14.	715	15.	30030	16.	371
17.	(a) 120	(b) 186					
18.	50400	19. 12000					
20.	(a) 65318400	(b) 1080	(c) 311040				

12

BINOMIAL THEOREM

Suppose you need to calculate the amount of interest you will get after 5 years on a sum of money that you have invested at the rate of 15% compound interest per year. Or suppose we need to find the size of the population of a country after 10 years if we know the annual growth rate. A result that will help in finding these quantities is the **binomial theorem.** This theorem, as you will see, helps us to calculate positive integral powers of any real binomial expression, that is, any expression involving two terms.

The binomial theorem, was known to Indian and Greek mathematicians in the 3rd century B.C. for some cases. The credit for the result for natural exponents goes to the Arab poet and mathematician Omar Khayyam (A.D. 1048-1122). Further generalisation to rational exponents was done by the British mathematician Newton (A.D. 1642-1727).

There was a reason for looking for further generalisation, apart from mathematical interest. The reason was its many applications. Apart from the ones we mentioned at the beginning, the binomial theorem has several applications in probability theory, calculus, and in approximating numbers like $(1.02)^7$, etc. We shall discuss them in this lesson.

OBJECTIVES

After studying this lesson, you will be able to:

- state the binomial theorem for a positive integral index and prove it using the principle of mathematical induction;
- write the binomial expansion for expressions like $(x+y)^n$ for different values of x and y using binomial theorem;
- write the general term and middle term (s) of a binomial expansion;

EXPECTED BACKGROUND KNOWLEDGE

- Number System
- Four fundamental operations on numbers and expressions.
- Algebraic expressions and their simplifications.
- Indices and exponents.

12.1 THE BINOMIAL THEOREM FOR A NATURAL EXPONENT

You must have multiplied a binomial by itself, or by another binomial. Let us use this knowledge to do some expansions. Consider the binomial $(x + y)$. Now,

$(x + y)^1 = x + y$

$(x + y)^2 = (x + y)(x + y) = x^2 + 2xy + y^2$

$(x + y)^3 = (x + y)(x + y)^2 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = (x + y)(x + y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x + y)^5 = (x + y)(x + y)^4 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ and so on.

In each of the equations above, the right hand side is called the binomial expansion of the left hand side.

Note that in each of the above expansions, we have written the power of a binomial in the expanded form in such a way that the terms are in descending powers of the first term of the binomial (which is x in the above examples). If you look closely at these expansions, you would also observe the following:

1. The number of terms in the expansion is one more than the exponent of the binomial. For example, in the expansion of $(x + y)^4$, the number of terms is 5.
2. The exponent of x in the first term is the same as the exponent of the binomial, and the exponent decreases by 1 in each successive term of the expansion.
3. The exponent of y in the first term is zero (as $y^0 = 1$). The exponent of y in the second term is 1, and it increases by 1 in each successive term till it becomes the exponent of the binomial in the last term of the expansion.
4. The sum of the exponents of x and y in each term is equal to the exponent of the binomial. For example, in the expansion of $(x + y)^5$, the sum of the exponents of x and y in each term is 5.

   If we use the combinatorial co-efficients, we can write the expansion as

   $(x + y)^3 = {}^3C_0\, x^3 + {}^3C_1 x^2y + {}^3C_2 x\, y^2 + {}^3C_3 y^3$

   $(x + y)^4 = {}^4C_0 x^4 + {}^4C_1 x^3y + {}^4C_2 x^2y^2 + {}^4C_3 xy^3 + {}^4C_4 y^4$

   $(x + y)^5 = {}^5C_0 x^5 + {}^5C_1 x^4y + {}^5C_2 x^3y^2 + {}^5C_3 x^2y^3 + {}^5C_4 xy^4 + {}^5C_5 y^5$, and so on.

   More generally, we can write the binomial expansion of $(x + y)^n$, where n is a positive integer, as given in the following theorem. This statement is called the **binomial theorem for a natural** (or **positive integral**) **exponent**.

Theorem

$$(x + y)^n = {}^nC_0\, x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_{n-1} xy^{n-1} + {}^nC_n y^n \quad \ldots(A)$$

where $n \in N$ and $x, y \in R$.

Proof : Let us try to prove this theorem, using the principle of mathematical induction.

Let statement (A) be denoted by $P(n)$, i.e.,

$$P(n):\ (x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2} y^2 + {}^nC_3 x^{n-3} y^3 + \ldots$$

$$+ {}^nC_{n-1}\, xy^{\,n-1} + {}^nC_n y^{\,n} \qquad \ldots(i)$$

Let us examine whether $P(1)$ is true or not.

From (i), we have $P(1) : (x + y)^1 = {}^1C_0 x + {}^1C_1 y = 1\times x + 1\times y$

i.e., $(x + y)^1 = x + y$ Thus, $P(1)$ holds.

Now, let us assume that $P(k)$ is true, i.e.,

$$P(k):\ (x + y)^k = {}^kC_0 x^{\,k} + {}^kC_1 x^{\,k-1} y + {}^kC_2 x^{\,k-2} y^{\,2} + {}^kC_3 x^{\,k-3} y^{\,3} + \ldots + {}^kC_{k-1} xy^{\,k-1} + {}^kC_k\, y^{\,k} \qquad \ldots(ii)$$

Assuming that $P(k)$ is true, if we prove that $P(k + 1)$ is true, then $P(n)$ holds, for all n. Now,

$$(x+y)^{k+1} = (x+y)(x+y)^k = (x+y)\left({}^kC_0x^k + {}^kC_1x^{k-1}y + {}^kC_2x^{k-2}y^2 + \ldots + {}^kC_{k-1}xy^{k-1} + {}^kC_ky^k\right)$$

$$= {}^kC_0x^{k+1} + {}^kC_0x^ky + {}^kC_1x^ky + {}^kC_1x^{k-1}y^2 + {}^kC_2x^{k-1}y^2 + {}^kC_2x^{k-2}y^3 + \ldots + {}^kC_{k-1}x^2y^{k-1} + {}^kC_{k-1}xy^k + {}^kC_kxy^k + {}^kC_ky^{k+1}$$

i.e. $$(x+y)^{k+1} = {}^kC_0x^{k+1} + ({}^kC_0 + {}^kC_1)x^ky + ({}^kC_1 + {}^kC_2)x^{k-1}y^2 + \ldots + ({}^kC_{k-1} + {}^kC_k)xy^k + {}^kC_ky^{k+1} \qquad \ldots(iii)$$

From Lesson 11, you know that ${}^kC_0 = 1 = {}^{k+1}C_0$, and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$ ....(iv)

Also, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$

Therefore, ${}^kC_0 + {}^kC_1 = {}^{k+1}C_1$, ${}^kC_1 + {}^kC_2 = {}^{k+1}C_2$, ${}^kC_2 + {}^kC_3 = {}^{k+1}C_3$ .....(v)

....

.... and so on

Using (iv) and (v), we can write (iii) as

$$(x+y)^{k+1} = {}^{k+1}C_0x^{k+1} + {}^{k+1}C_1x^ky + {}^{k+1}C_2x^{k-1}y^2 + \ ,\ \ldots + {}^{k+1}C_kxy^k + {}^{k+1}C_{k+1}y^{k+1}$$

which shows that P$(k+1)$ is true.

Thus, we have shown that (a) $P(1)$ is true, and (b) if $P(k)$ is true, then $P(k+1)$ is also true.

Therefore, by the principle of mathematical induction, $P(n)$ holds for any value of n. So, we have proved the binomial theorem for any natural exponent.

This result is supported to have been proved first by the famous Arab poet Omar Khayyam, though no one has been able to trace his proof so far.

We will now take some examples to illustrate the theorem.

Example 12.1 Write the binomial expansion of $(x + 3y)^5$.

Solution : Here the first term in the binomial is x and the second term is $3y$. Using the binomial theorem, we have

$$(x+3y)^5 = {}^5C_0x^5 + {}^5C_1x^4(3y)^1 + {}^5C_2x^3(3y)^2 + {}^5C_3x^2(3y)^3 + {}^5C_4x(3y)^4 + {}^5C_5(3y)^5$$

$$= 1\times x^5 + 5x^4\times 3y + 10x^3\times(9y^2) + 10x^2\times(27y^3) + 5x\times(81y^4) + 1\times 243y^5$$

$$= x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$$

Thus, $(x+3y)^5 = x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$

Example 12.2 Expand $(1+a)^n$ in terms of powers of a, where a is a real number.

Solution : Taking $x = 1$ and $y = a$ in the statement of the binomial theorem, we have

$$(1+a)^n = {}^nC_0(1)^n + {}^nC_1(1)^{n-1}a + {}^nC_2(1)^{n-2}a^2 + \ldots + {}^nC_{n-1}(1)a^{n-1} + {}^nC_na^n$$

i.e., $(1+a)^n = 1 + {}^nC_1a + {}^nC_2a^2 + \ldots + {}^nC_{n-1}a^{n-1} + {}^nC_na^n$ ... (B)

(B) is another form of the statement of the binomial theorem.

The theorem can also be used in obtaining the expansions of expressions of the type

$$\left(x+\frac{1}{x}\right)^5, \left(\frac{y}{x}+\frac{1}{y}\right)^5, \left(\frac{a}{4}+\frac{2}{a}\right)^5, \left(\frac{2t}{3}-\frac{3}{2t}\right)^6, \text{ etc.}$$

Let us illustrate it through an example.

Example 12.3 Write the expansion of $\left(\frac{y}{x}+\frac{1}{y}\right)^4$, where $x, y \neq 0$.

Solution : We have :

$$\left(\frac{y}{x}+\frac{1}{y}\right)^4 = {}^4C_0\left(\frac{y}{x}\right)^4 + {}^4C_1\left(\frac{y}{x}\right)^3\left(\frac{1}{y}\right) + {}^4C_2\left(\frac{y}{x}\right)^2\left(\frac{1}{y}\right)^2 + {}^4C_3\left(\frac{y}{x}\right)\left(\frac{1}{y}\right)^3 + {}^4C_4\left(\frac{1}{y}\right)^4$$

$$= 1\times\frac{y^4}{x^4} + 4\times\frac{y^3}{x^3}\times\frac{1}{y} + 6\times\frac{y^2}{x^2}\times\frac{1}{y^2} + 4\times\left(\frac{y}{x}\right)\times\frac{1}{y^3} + 1\times\frac{1}{y^4}$$

$$= \frac{y^4}{x^4} + 4\frac{y^2}{x^3} + \frac{6}{x^2} + \frac{4}{xy^2} + \frac{1}{y^4}$$

Example 12.4 The population of a city grows at the annual rate of 3%. What percentage increase is expected in 5 years ? Give the answer up to 2 decimal places.

Solution : Suppose the population is a at present. After 1 year it will be

$$a+\frac{3}{100}a=a\left(1+\frac{3}{100}\right)$$

After 2 years, it will be $a\left(1+\frac{3}{100}\right)+\frac{3}{100}\left[a\left(1+\frac{3}{100}\right)\right]$

$$=a\left(1+\frac{3}{100}\right)\left(1+\frac{3}{100}\right) = a\left(1+\frac{3}{100}\right)^2$$

Similarly, after 5 years, it will be $a\left(1+\frac{3}{100}\right)^5$

Using the binomial theorem, and ignoring terms involving more than 3 decimal places,

we get , $a\left(1+\frac{3}{100}\right)^5 \approx a\left[1+5(0.03)+10(0.03)^2\right] = a\times 1.159$

So, the increase is $0.159\times 100\% = \frac{159}{1000}\times 100\times\frac{1}{100} = 15.9\%$ in 5 years.

Example 12.5 Using binomial theorem, evaluate , (*i*) 102^4 (*ii*) 97^3

Solution : (*i*) $102^4 = (100+2)^4$

$= {}^4C_0\,(100)^4 + {}^4C_1(100)^3.2 + {}^4C_2(100)^2.2^2 + {}^4C_3\,(100).2^3 + {}^4C_4\,.2^4$

$= 100000000 + 8000000 + 240000 + 3200 + 16 = 108243216$

(*ii*) $(97)^3 = (100-3)^3 = {}^3C_0(100)^3 - {}^3C_1(100)^2.3 + {}^3C_2(100).3^2 - {}^3C_3.3^3$

$= 1000000 - 90000 + 2700 - 27 = 1002700 - 90027 = 912673$

CHECK YOUR PROGRESS 12.1

1. Write the expansion of each of the following :

   (a) $(2a+b)^3$ (b) $(x^2-3y)^6$ (c) $(4a-5b)^4$ (d) $(ax+by)^n$

2. Write the expansions of :

   (a) $(1-x)^7$ (b) $\left(1+\frac{x}{y}\right)^7$ (c) $(1+2x)^5$

3. Write the expansions of :

(a) $\left(\frac{a}{3}+\frac{b}{2}\right)^5$ (b) $\left(3x-\frac{5}{x^2}\right)^7$ (c) $\left(x+\frac{1}{x}\right)^4$ (d) $\left(\frac{x}{y}+\frac{y}{x}\right)^5$

4. Suppose I invest Rs. 1 lakh at 18% per year compound interest. What sum will I get back after 10 years? Give your answer up to 2 decimal places.

5. The population of bacteria increases at the rate of 2% per hour. If the count of bacteria at 9 a.m. is 1.5×10^5, find the number at 1 p.m. on the same day.

6. Using binomial theorem, evaluate each of the following :

(*i*) $(101)^4$ (*ii*) $(99)^4$ (*iii*) $(1.02)^3$ (iv) $(0.98)^3$

12.2 GENERAL TERM IN A BINOMIAL EXPANSION

Let us examine various terms in the expansion (A) of $(x + y)^n$, i.e., in

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + {}^nC_3x^{n-3}y^3 + \ldots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$$

We observe that , the first term is ${}^nC_0x^n$, i.e., ${}^nC_{1-1}x^ny^0$;

the second term is ${}^nC_1x^{n-1}y$, i.e., ${}^nC_{2-1}x^{n-1}y^1$;

the third term is ${}^nC_2x^{n-2}y^2$, i.e., ${}^nC_{3-1}x^{n-2}y^2$; and so on.

From the above, we can generalise that

the $(r + 1)^{\text{th}}$ term is ${}^nC_{(r+1)-1}x^{n-r}y^r$, i.e., ${}^nC_rx^{n-r}y^r$.

If we denote this term by T_{r+1}, we have , $T_{r+1} = {}^nC_rx^{n-r}y^r$

T_{r+1} is generally referred to as the **general term** of the binomial expansion.

Let us now consider some examples and find the general terms of some expansions.

Example 12.6 Find the $(r + 1)^{\text{th}}$ term in the expansion of $\left(x^2+\frac{1}{x}\right)^n$, where n is a natural number. Verify your answer for the first term of the expansion.

Solution : The general term of the expansion is given by :

$$T_{r+1} = {}^nC_r(x^2)^{(n-r)}\left(\frac{1}{x}\right)^r$$

$$= {}^nC_rx^{2n-2r}\frac{1}{x^r} = {}^nC_rx^{2n-3r} \qquad \ldots(i)$$

Hence, the $(r + 1)$th term in the expansion is ${}^nC_r x^{2n-3r}$.

On expanding $\left(x^2 + \frac{1}{x}\right)^n$, we note that the first term is $(x^2)^n$ or x^{2n}.

Using (i), we find the first term by putting $r = 0$.

Since $T_1 = T_{0+1}$ $\therefore$ $\quad T_1 = {}^nC_0 x^{2n-0} = x^{2n}$

This verifies that the expression for T_{r+1} is correct for $r + 1 = 1$.

Example 12.7 Find the fifth term in the expansion of

$$\left(1 - \frac{2}{3}x^3\right)^6$$

Solution : Using here $T_{r+1} = T_5$ which gives $r+1 = 5$, i.e., $r = 4$.

Also $n = 6$ and let $x = 1$ and $y = \frac{-2}{3}x^3$.

$$T_5 = {}^6C_4\left(-\frac{2}{3}x^3\right)^4 = {}^6C_2\left(\frac{16}{81}x^{12}\right) = \frac{6\times 5}{2}\times\frac{16}{81}\times x^{12} = \frac{80}{27}x^{12}$$

Thus, the fifth term in the expansion is $\frac{80}{27}x^{12}$.

CHECK YOUR PROGRESS 12.2

1. For a natural number n, write the $(r + 1)^{th}$ term in the expansion of each of the following:

   (a) $(2x+y)^n$ (b) $(2a^2-1)^n$ (c) $(1-a)^n$ (d) $\left(3+\frac{1}{x^2}\right)^n$

2. Find the specified terms in each of the following expansions:

   (a) $(1+2y)^8$; 6th term (b) $(2x+3)^7$; 4th term (c) $(2a-b)^{11}$; 7th term

   (d) $\left(x+\frac{1}{x}\right)^6$; 4th term (e) $\left(x^3-\frac{1}{x^2}\right)^7$; 5th term

12.3 MIDDLE TERMS IN A BINOMIAL EXPANSION

Now you are familiar with the general term of an expansion, let us see how we can obtain the **middle term** (or terms) of a binomial expansion. Recall that the number of terms in a binomial expansion is always one more than the exponent of the binomial. This implies that if the exponent is even, the number of terms is odd, and if the exponent is odd, the number of terms is even.

Thus, while finding the middle term in a binomial expansion, we come across two cases:

Case 1 : When n is even. To study such a situation, let us look at a particular value of n, say $n = 6$. Then the number of terms in the expansion will be 7. From Fig. 12.1, you can see that there are three terms on either side of the fourth term.

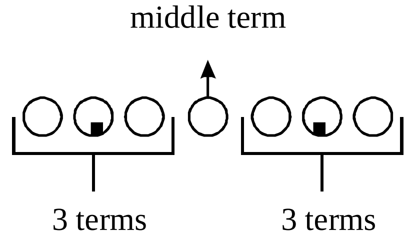


Fig. 12.1

In general, when the exponent n of the binomial is even, there are $\frac{n}{2}$ terms on either side of the $\left(\frac{n}{2}+1\right)$th term. Therefore, the $\left(\frac{n}{2}+1\right)$th term is the middle term.

Case 2: When n is odd , Let us take $n = 7$ as an example to see what happens in this case. The number of terms in the expansion will be 8. Looking at Fig. 12.2, do you find any one middle term in it? There is not. But we can partition the terms into two equal parts by a line as shown in the figure. We call the terms on either side of the partitioning line taken together, the middle terms. This is because there are an equal number of terms on either side of the two, taken together.

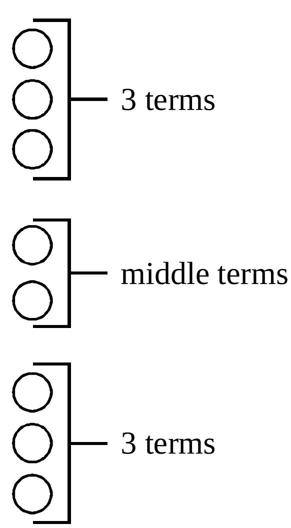


Fig. 12.2

Thus, in this case, there are two middle terms, namely, the fourth,

i.e., $\left(\frac{7+1}{2}\right)$ and the fifth, i.e., $\left(\frac{7+3}{2}\right)$ terms

Similarly, if $n = 13$, then the $\left(\frac{13+1}{2}\right)$ th and the $\left(\frac{13+3}{2}\right)$ th terms, i.e., the 7th and 8th terms are two middle terms, as is evident from Fig. 12.3.

From the above, we conclude that

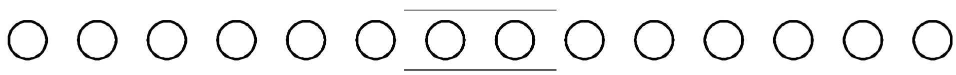


Fig. 12.3

When the exponent n of a binomial is an odd natural number, then the $\left(\frac{n+1}{2}\right)$ th and $\left(\frac{n+3}{2}\right)$ th terms are two middle terms in the corresponding binomial expansion.

Let us now consider some examples.

Example 12.8 Find the middle term in the expansion of $(x^2 + y^2)^8$.

Solutuion : Here $n = 8$ (an even number).

Therefore, the $\left(\frac{8}{2}+1\right)$ th, i.e., the 5th term is the middle term.

Putting $r = 4$ in the general term $T_{r+1} = {}^8C_r(x^2)^{8-r}y^r$, $T_5 = {}^8C_4(x^2)^{8-4}(y^2)^4 = 70x^8y^8$

Example 12.9 Find the middle term(s) in the expansion of $\left(2x^2 + \frac{1}{x}\right)^9$.

Solution : Here $n = 9$ (an odd number). Therefore, the $\left(\frac{9+1}{2}\right)th$ and $\left(\frac{9+3}{2}\right)th$ are middle terms. i.e. T_5 and T_6 are middle terms.

For finding T_5 and T_6, putting $r = 4$ and $r = 5$ in the general term , $T_{r+1} = {}^9C_r(2x^2)^{9-r}\left(\frac{1}{x}\right)^6$,

$$T_5 = {}^9C_4(2x^2)^{9-4}\left(\frac{1}{x}\right)^4 \quad = \frac{9\times8\times7\times6}{4\times3\times2}\times(32x^{10})\times\left(\frac{1}{x}\right)^4 = 4032\,x^6$$

and $T_6 = {}^9C_5\,(2x^2)^{9-5}\left(\frac{1}{x}\right)^5 = 2016\,x^3$

Thus, the two middle terms are $4032\,x^6$ and $2016\,x^3$.

CHECK YOUR PROGRESS 12.3

1. Find the middle term(s) in the expansion of each of the following :

(a) $(2x+y)^{10}$ (b) $\left(1+\frac{2}{3}x^3\right)^8$ (c) $\left(x+\frac{1}{x}\right)^6$ (d) $\left(1-x^2\right)^{10}$

2. Find the middle term(s) in the expansion of each of the following :

(a) $(a+b)^7$ (b) $(2a-b)^9$ (c) $\left(\frac{3x}{4}-\frac{4y}{3}\right)^7$ (d) $\left(x+\frac{1}{x^2}\right)^{11}$

LET US SUM UP

- For a natural number n,

  $$(x+y)^n = {}^nC_o\,x^n + {}^nC_1\,x^{n-1}y + {}^nC_2\,x^{n-2}\,y^2 + \ldots + {}^nC_{n-1}\,xy^{n-1} + {}^nC_n\,y^n$$

  This is called the **Binomial Theorem for a positive integral (or natural) exponent.**

- Another form of the Binomical Theorem for a positive integral exponent is $(1+a)^n = {}^nC_o + {}^nC_1\,a + {}^nC_2\,a^2 + \ldots . + {}^nC_{n-1}\,a^{n-1} + {}^nC_n\,a^n$

- The general term in the expansion of $(x+y)^n$ is ${}^nC_r\,x^{n-r}y^r$ and in the expansion of $(1+a)^n$ is ${}^nC_r a^r$, where n is a natural number and $0 \le r \le n$.
- If n is an even natural number, there is only one middle term in the expansion of $(x+y)^n$. If n is odd, there are two middle trems in the expansion.

- The formula for the general term can be used for finding the middle term(s) and some other specific terms in an expansion.

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=Cv4YhIMfbeM

http://www.youtube.com/watch?v=-fFWWt1m9k0

http://www.youtube.com/watch?v=xF_hJaXUNfE

TERMINAL EXERCISE

1. Write the expansion of each of the following :

(a) $(3x + 2y)^5$ (b) $(p - q)^8$ (c) $(1 - x)^8$

(d) $\left(1+\frac{2}{3}x\right)^6$ (e) $\left(x+\frac{1}{2x}\right)^6$ (f) $\left(3x-y^2\right)^5$

(g) $\left(\frac{x^2}{4}+\frac{2}{x}\right)^4$ (h) $\left(x^2-\frac{1}{x^3}\right)^7$ (i) $\left(x^3+\frac{1}{x^2}\right)^5$ (j) $\left(\frac{1}{x^2}-x^3\right)^4$

2. Write the (r + 1)th term in the expansion of each of the following, where $n \in N$:

(a) $(3x - y^2)^n$ (b) $\left(x^3+\frac{1}{x}\right)^n$

3. Find the specified terms in the expansion of each of the following :

(a) $(1 - 2x)^7$: 3rd term [Hint : Here $r = 2$] (b) $\left(x+\frac{1}{2x}\right)^6$: middle term (s)

(c) $(3x - 4y)^6$: 4th term (d) $\left(y^2-\frac{1}{y}\right)^{11}$: middle term (s)

(e) $(x^3 - y^3)^{12}$: 4th term (f) $(1 - 3x^2)^{10}$: middle term (s)

(g) $(-3x-4y)^6$: 5th term (h) Write the rth term in the expansion of $(x - 2y)^6$.

4. If T_r, denotes the rth term in the expansion of $(1 + x)^n$ in ascending powers of x (n being a natural number) , prove that

$r(r + 1)T_{r+2} = (n - r + 1)(n - r)\, x^2\, T_r$. [**Hint :** $T_r = {}^nC_{r-1}\, x^{r-1}$ and $T_{r+2} = {}^nC_{r+1}\, x^{r+1}$]

5. k_r is the coefficient of x^{r-1} in the expansion of $(1 + 2x)^{10}$ in ascending powers of x and $k_{r+2} = 4k_r$. Find the value of r. [**Hint :** $k_r = {}^{10}C_{r-1}\, 2^{r-1}$ and $k_{r+2} = {}^{10}C_{r+1}2^{r+1}$]

6. The coefficients of the 5th, 6th and 7th terms in the expansion of $(1+a)^n$ (n being a natural number) are in A.P. Find n. [**Hint :** ${}^nC_5 - {}^nC_4 = {}^nC_6 - {}^nC_5$]

7. Expand $(1 + y + y^2)^4$. $\left[\textbf{Hint :} \left(1+y+y^2\right)^4 = \left\{(1+y)+y^2\right\}^4\right]$

ANSWERS

CHECK YOUR PROGRESS 12.1

1. (a) $8a^3 + 12a^2b + 6ab^2 + b^3$

   (b) $x^{12} - 18x^{10}y + 135x^8y^2 - 540x^6y^3 + 1215x^4y^4 - 1458x^2y^5 + 729y^6$

   (c) $256a^4 - 1280a^3b + 2400a^2b^2 - 2000ab^3 + 625b^4$

   (d) $a^nx^n + na^{n-1}x^{n-1}by + \frac{n(n-1)}{2!}a^{n-2}x^{n-2}b^2y^2 + ... + b^ny^n$

2. (a) $1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$

   (b) $1 + \frac{7x}{y} + \frac{21x^2}{y^2} + \frac{35x^3}{y^3} + \frac{35x^4}{y^4} + \frac{21x^5}{y^5} + \frac{7x^6}{y^6} + \frac{x^7}{y^7}$

   (c) $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

3. (a) $\frac{a^5}{243} + \frac{5a^4b}{162} + \frac{5a^3b^2}{54} + \frac{5a^2b^3}{36} + \frac{5ab^4}{48} + \frac{b^5}{32}$

   (b) $2187x^7 - 25515x^4 + 127575x - \frac{354375}{x^2} + \frac{590625}{x^5} - \frac{590625}{x^8}$

   $+ \frac{328125}{x^{11}} - \frac{78125}{x^{14}}$

   (c) $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$ (d) $\frac{x^5}{y^5} + 5\frac{x^3}{y^3} + 10\frac{x}{y} + 10\frac{y}{x} + 5\frac{y^3}{x^3} + \frac{y^5}{x^5}$

4. Rs 4.96 lakh   5. 162360

6. (i) 104060401   (ii) 96059601

   (iii) 1.061208   (iv) 0.941192

CHECK YOUR PROGRESS 12.2

1. (a) ${}^nC_r 2^{n-r} x^{n-r} y^r$ (b) ${}^nC_r 2^{n-r} a^{2n-2r} (-1)^r$

   (c) ${}^nC_r (-1)^r a^r$ (d) ${}^nC_r 3^{n-r}.x^{-2r}$

2. (a) $1792y^5$ (b) $15120x^4$

(c) $14784a^5b^6$ (d) 20

(e) $35x$

CHECK YOUR PROGRESS 12.3

1. (a) $8064x^5y^5$ (b) $\frac{1120}{81}x^{12}$

(c) 20 (d) $-252x^{10}$

2. (a) $35a^4b^3,\ 35a^3b^4$

(b) $4032a^5b^4, -2016a^4b^5$

(c) $\frac{-105}{4}x^4y^3, \frac{140}{3}x^3y^4$

(d) $\frac{462}{x^4}, \frac{462}{x^7}$

TERMINAL EXERCISE

1. (a) $243x^5+810x^4y+1080x^3y^2+720x^2y^3+240xy^4+32y^5$

(b) $p^8-8p^7q+28p^6q^2-56p^5q^3+70p^4q^4-56p^3q^5+28p^2q^6-8pq^7+q^8$

(c) $1-8x+28x^2-56x^3+70x^4-56x^5+28x^6-8x^7+x^8$

(d) $1+4x+\frac{20}{3}x^2+\frac{160}{27}x^3+\frac{80}{27}x^4+\frac{64}{81}x^5+\frac{64}{729}x^6$

(e) $x^6+3x^4+\frac{15}{4}x^2+\frac{5}{2}+\frac{15}{16x^2}+\frac{3}{16x^4}+\frac{1}{64x^6}$

(f) $243x^5-405x^4y^2+270x^3y^4-90x^2y^6+15xy^8-y^{10}$

(g) $\frac{x^8}{256}+\frac{x^5}{8}+\frac{3}{2}x^2+\frac{8}{x}+\frac{16}{x^4}$

(h) $x^{14}-7x^9+21x^4-\frac{35}{x}+\frac{35}{x^6}-\frac{21}{x^{11}}+\frac{7}{x^{16}}-\frac{1}{x^{21}}$

(i) $x^{15}+5x^{10}+10x^5+10+\frac{5}{x^5}+\frac{1}{x^{10}}$

(j) $\frac{1}{x^8}-\frac{4}{x^3}+6x^2-4x^7+x^{12}$

2. (a) $(-1)^r\ {}^nC_r 3^{n-r} x^{n-r} y^{2r}$

   (b) ${}^nC_r x^{3n-4r}$

3. (a) $84x^2$

   (b) $\frac{5}{2}$

   (c) $-34560\ x^3y^3$

   (d) $-462y^7$, $462y^4$

   (e) $-220x^{27}y^9$

   (f) $-61236\ x^{10}$

   (g) $34560x^2y^4$

   (h) $(-2)^{r-1}\ {}^6C_{r-1}\ x^{7-r} y^{r-1}$

5. 5

6. 7, 14

7. $1 + 4y + 10y^2 + 16y^3 + 19y^4 + 16y^5 + 10y^6 + 4y^7 + y^8$

13

CARTESIAN SYSTEM OF RECTANGULAR CO-ORDINATES

You must have searched for your seat in a cinema hall, a stadium, or a train. For example, seat *H*-4 means the fourth seat in the H^{th} row. In other words, *H* and 4 are the coordinates of your seat. Thus, the geometrical concept of location is represented by numbers and alphabets (an algebraic concept).

Also a road map gives us the location of various houses (again numbered in a particular sequence), roads and parks in a colony, thus representing algebraic concepts by geometrical figures like straight lines, circles and polygons.

The study of that branch of Mathematics which deals with the interrelationship between geometrical and algebraic concepts is called Coordinate Geometry or Cartesian Geometry in honour of the famous French mathematician **Rene Descartes.**

In this lesson we shall study the basics of coordinate geometry and relationship between concept of straight line in geometry and its algebraic representation.

OBJECTIVES

After studying this lesson, you will be able to:

- define Cartesian System of Coordinates including the origin, coordinate axes, quadrants, etc;
- derive distance formula and section formula;
- derive the formula for area of a triangle with given vertices;
- verify the collinearity of three given points;
- state the meaning of the terms : inclination and slope of a line;
- find the formula for the slope of a line through two given points;
- state the condition for parallelism and perpendicularity of lines with given slopes;
- find the intercepts made by a line on coordinate axes;
- find the angle between two lines when their slopes are given;
- find the coordinates of a point when origin is shifted to some other point;
- find transformed equation of curve when oregin is shifted to another point.

EXPECTED BACKGROUND KNOWLEDGE

- Number system .
- Plotting of points in a coordinate plane.
- Drawing graphs of linear equations .
- Solving systems of linear equations .

13.1 RECTANGULAR COORDINATE AXES

Recall that in previous classes, you have learnt to fix the position of a point in a plane by drawing two mutually perpendicular lines. The fixed point O,where these lines intersect each other is called the **origin** O as shown in Fig. 13.1 These mutually perpencular lines are called the **coordinate axes**. The horizontal line XOX' is the **x-axis** or **axis** of x and the vertical line YOY' is the y- **axis** or **axis** of y.

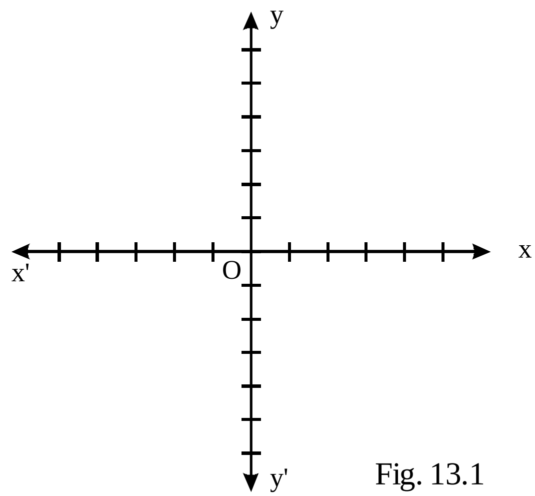


Fig. 13.1

9.1.1 CARTESIAN COORDINATES OF A POINT

To find the coordinates of a point we proceed as follows. Take X'OX and YOY' as coordinate axes. Let P be any point in this plane. From point P draw $PA \perp XOX'$ and $PB \perp YOY'$. Then the distance OA = x measured along x-axis and the distance OB = y measured along y-axis determine the position of the point P with reference to these axes. The distance OA measured along the axis of x is called the **abscissa** or x-coordinate and the distance OB (=PA) measured along y – axis is called the **ordinate** or y-coordinate of the point P. The abscissa and the ordinate taken together are called the **coordinates** of the point P. Thus, the coordinates of the point P are (x and y) which represent the position of the point P point in a plane. These two numbers are to form an **ordered pair** beacuse the order in which we write these numbers is important.

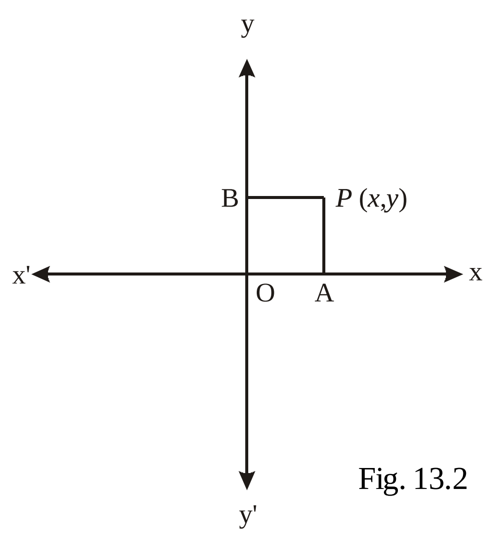


Fig. 13.2

In Fig. 13.3 you may note that the position of the ordered pair (3,2) is different from that of (2,3). Thus, we can say that (x,y) and (y,x) are two different ordered pairs representing two different points in a plane.

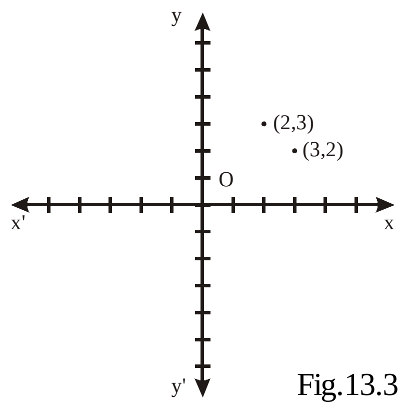


Fig.13.3

13.1.2 QUARDRANTS

We know that coordinate axes XOX' and YOY' divide the region of the plane into four regions. These regions are called the quardrants as shown in Fig. 13.4. In accordance with the convention of signs, for a point P (x,y) in different quadrants, we have

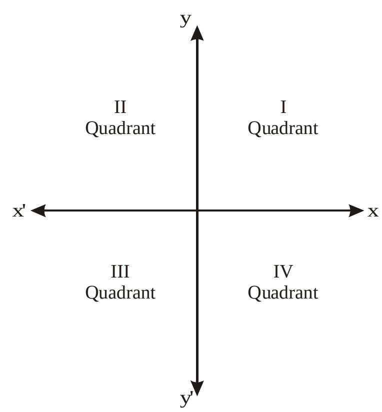


Fig.13.4

I	quadrant :	$x > 0,\ y > 0$
II	quadrant :	$x < 0,\ y > 0$
III	quadrant :	$x < 0,\ y < 0$
IV	quadrant :	$x > 0,\ y < 0$

13.2 DISTANCE BETWEEN TWO POINTS

Recall that you have derived the distance formula between two points P (x_1,y_1) and Q (x_2,y_2) in the following manner:

Let us draw a line $l \parallel XX'$ through P. Let R be the point of intersection of the perpendicular from Q to the line l. Then ΔPQR is a right-angled triangle.

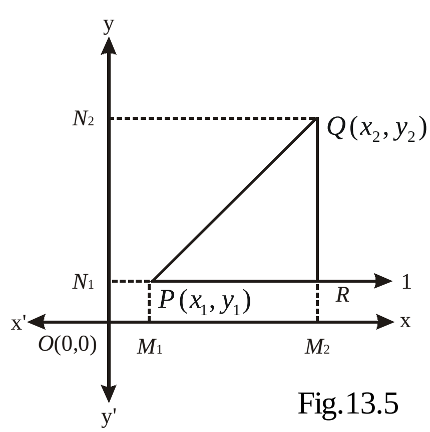


Fig.13.5

$$\text{Also } PR = M_1M_2$$

$$= OM_2 - OM_1$$

$$= x_2 - x_1$$

$$\text{and } QR = QM_2 - RM_2$$

$$= QM_2 - PM_1$$
$$= ON_2 - ON_1$$
$$= y_2 - y_1$$

Now $PQ^2 = PR^2 + QR^2$ (Pythagoras theorem)

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note : This formula holds for points in all quadrants.

Also the distance of a point P(x,y) from the origin O(0,0)

is OP $= \sqrt{x^2 + y^2}$.

Let us illustrate the use of these formulae with some examples.

Example 13.1 Find the distance between the following pairs of points :

(*i*) $A(14,3)$ and $B(10,6)$ (*ii*) $M(-1,2)$ and $N(0,-6)$

Solution :

(*i*) Distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here $x_1 = 14$, $y_1 = 3$, $x_2 = 10$, $y_2 = 6$

$\therefore$ Distance between A and B $= \sqrt{(10-14)^2 + (6-3)^2}$

$$= \sqrt{(-4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

Distance between A and B is 5 units.

(*ii*) Here $x_1 = -1$, $y_1 = 2$, $x_2 = 0$ and $y_2 = -6$

Distance between A and B $= \sqrt{(0-(-1))^2 + (-6-2)^2} = \sqrt{1+(-8)^2}$

$$= \sqrt{1+64} = \sqrt{65}$$

Distance between M and N $= \sqrt{65}$ units

Example 13.2 Show that the points P(–1, –1), Q(2, 3) and R (–2, 6) are the vertices of a right-angled triangle.

Solution: $PQ^2 = (2 + 1)^2 + (3 + 1)^2 = 3^2 + 4^2 = 9 + 16 = 25$

$QR^2 = (-4)^2 + (3)^2 = 16 + 9 = 25$

and $RP^2 = 1^2 + (-7)^2 = 1 + 49 = 50$

$\therefore$ $PQ^2 + QR^2 = 25+25=50= RP^2$

$\Rightarrow$ ΔPQR is a right-angled triangle (by converse of Pythagoras Theorem)

Example 13.3 Show that the points A(1, 2), B(4, 5) and C(–1, 0) lie on a straight line.

Solution: Here,

$$AB = \sqrt{(4-1)^2 + (5-2)^2} \text{ units} = \sqrt{18} \text{ units} = 3\sqrt{2} \text{ units}$$

$$BC = \sqrt{(-1-4)^2 + (0-5)^2} \text{ units} = \sqrt{50} \text{ units} = 5\sqrt{2} \text{ units}$$

and $$AC = \sqrt{(-1-1)^2 + (0-2)^2} \text{ units} = \sqrt{4+4} \text{ units} = 2\sqrt{2} \text{ units}$$

Now $$AB + AC = \left(3\sqrt{2} + 2\sqrt{2}\right) units = 5\sqrt{2}\, units = BC$$

i.e. $BA + AC = BC$

Hence, A, B, C lie on a straight line. In other words, A,B,C are collinear.

Example 13.4 Prove that the points (2a, 4a), (2a, 6a) and $\left(2a+\sqrt{3}a, 5a\right)$ are the vertices of an equilateral triangle whose side is 2a.

Solution: Let the points be A (2a, 4a), B (2a, 6a) and C $\left(2a+\sqrt{3}a, 5a\right)$

$$AB = \sqrt{0 + (2a)^2} = 2a \text{ units}$$

$$BC = \sqrt{(\sqrt{3}a)^2 + (-a)^2} \text{ units} = \sqrt{3a^2 + a^2} = 2a \text{ units}$$

and $$AC = \sqrt{(\sqrt{3}a)^2 + (+a)^2} = 2a \text{ units}$$

$\Rightarrow$ AB + BC > AC, BC + AC > AB and

AB + AC > BC and AB = BC = AC = 2a

$\Rightarrow$ A, B, C form the vertices of an equilateral triangle of side 2a.

CHECK YOUR PROGRESS 13.1

1. Find the distance between the following pairs of points.

   (a) (5, 4) and (2, –3) (b) (a, –a) and (b, b)

2. Prove that each of the following sets of points are the vertices of a right angled-trangle.

   (a) (4, 4), (3, 5), (–1,–1) (b) (2, 1), (0, 3), (–2, 1)

3. Show that the following sets of points form the vertices of a triangle:

   (a) (3, 3), (–3, 3) and (0, 0) (b) (0, a), (a, b) and (0, 0) (if ab = 0)

4. Show that the following sets of points are collinear :

   (a) (3, –6), (2, –4) and (–4, 8) (b) (0,3), (0, –4) and (0, 6)

5. (a) Show that the points (0, –1), (–2, 3), (6, 7) and (8, 3) are the vertices of a rectangle.

   (b) Show that the points (3, –2), (6, 1), (3, 4) and (0, 1) are the vertices of a square.

13.3 SECTION FORMULA

13.3.1 INTERNAL DIVISION

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points on a line *l* and $R(x, y)$ divide *PQ* internally in the ratio $m_1 : m_2$.

To find : The coordinates *x* and *y* of point *R*.

Construction : Draw *PL*, *QN* and *RM* perpendiculars to *XX'* from *P*, *Q* and *R* respectively and *L*, *M* and *N* lie on *XX'*. Also draw $RT \perp QN$ and $PV \perp QN$.

Method : *R* divides *PQ* internally in the ratio $m_1 : m_2$.

$\Rightarrow$ *R* lies on *PQ* and $\dfrac{PR}{RQ} = \dfrac{m_1}{m_2}$

Also, in triangles, *RPS* and *QRT*,

$\angle RPS = \angle QRT$ (Corresponding angles as $PS \parallel RT$)

and $\angle RSP = \angle QTR = 90^0$

$\therefore$ $\Delta RPS \sim \Delta QRT$ (*AAA* similarity)

$\Rightarrow$ $\dfrac{PR}{RQ} = \dfrac{RS}{QT} = \dfrac{PS}{RT}$ ... (*i*)

Also, $PS = LM = OM - OL = x - x_1$

$RT = MN = ON - OM = x_2 - x$

$RS = RM - SM = y - y_1$

$QT = QN - TN = y_2 - y$.

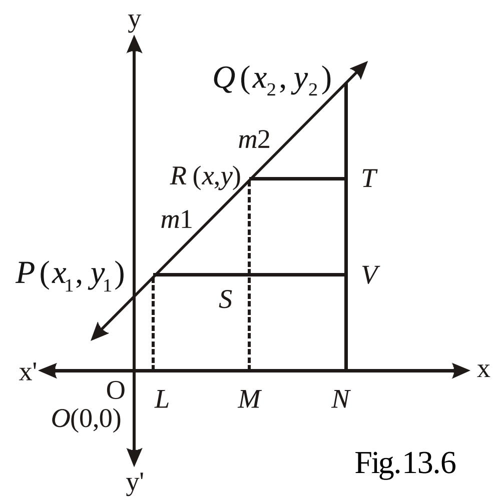


Fig.13.6

From (i), we have

$$\therefore \frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

$$\Rightarrow m_1(x_2 - x) = m_2(x - x_1)$$

and $\quad m_1(y_2 - y) = m_2(y - y_1)$

$$\Rightarrow \quad x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2} \text{ and } y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

Thus, the coordinates of R are:

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right)$$

Coordinates of the mid-point of a line segment

If R is the mid point of PQ, then,

$m_1 = m_2 = 1$ (as R divides PQ in the ratio 1:1

Coordinates of the mid point are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

13.3.2 EXTERNAL DIVISION

Let R divide PQ externally in the ratio $m_1 : m_2$

To find : The coordinates of R.

Construction : Draw PL, QN and RM perpendiculars to XX' from P, Q and R respectively and $PS \perp RM$ and $QT \perp RM$.

Clearly, $\triangle RPS \sim \triangle RQT$.

$$\therefore \frac{RP}{RQ} = \frac{PS}{QT} = \frac{RS}{RT}$$

or $\quad \dfrac{m_1}{m_2} = \dfrac{x - x_1}{x - x_2} = \dfrac{y - y_1}{y - y_2}$

$$\Rightarrow m_1(x - x_2) = m_2(x - x_1)$$

and $m_1(y - y_2) = m_2(y - y_1)$

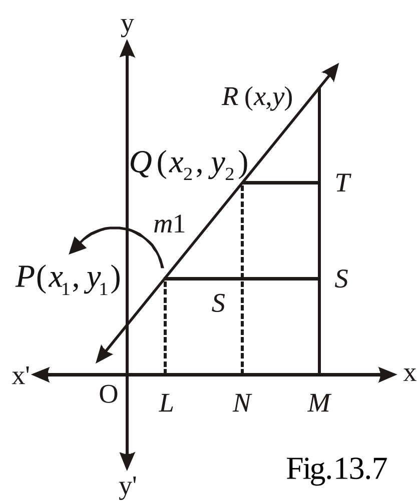


Fig.13.7

These give:

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \quad and \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

Hence, the coordinates of the point of external division are

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2} \right)$$

Let us now take some examples.

Example 13.5 Find the coordinates of the point which divides the line segment joining the points (4, –2) and (–3, 5) internally and externally in the ratio 2:3.

Solution:

(i) Let $P\,(x, y)$ be the point of internal division.

$$\therefore\ x = \frac{2(-3)+3(4)}{2+3} = \frac{6}{5} \text{ and } y = \frac{2(5)+3(-2)}{2+3} = \frac{4}{5}$$

$\therefore$ P has coordinates $\left(\frac{6}{5}, \frac{4}{5}\right)$

If $Q\,(x', y')$ is the point of external division, then

$$x' = \frac{(2)(-3)-3(4)}{2-3} = 18 \text{ and } y' = \frac{(2)(5)-3(-2)}{2-3} = -16$$

Thus, the coordinates of the point of external division are (18, –16).

Example 13.6 In what ratio does the point (3, –2) divide the line segment joining the points (1,4) and (–3, 16) ?

Solution : Let the point $P(3, -2)$ divide the line segement in the ratio $k : 1$.

Then the coordinates of P are $\left(\frac{-3k+1}{k+1}, \frac{16k+4}{k+1} \right)$

But the given coordinates of P are $(3, -2)$

$$\therefore\ \frac{-3k+1}{k+1} = 3 \Rightarrow -3k+1 = 3k+3 \quad \Rightarrow k = -\frac{1}{3}$$

$\Rightarrow P$ divides the line segement externally in the ratio 1:3.

Example 13.7 The vertices of a quadrilateral *ABCD* are respectively (1, 4), (–2,1), (0, –1) and (3, 2). If *E, F, G, H* are respectively the midpoints of *AB, BC, CD* and *DA*, prove that the quadrilateral *EFGH* is a parallelogram.

Solution : Since *E, F, G,* and *H*, are the midpoints of the sides *AB, BC, CD* and *DA*, therefore, the coordinates of *E, F, G,* and *H* respectively are :

$$\left(\frac{1-2}{2},\frac{4+1}{2}\right),\left(\frac{-2+0}{2},\frac{1-1}{2}\right),\left(\frac{0+3}{2},\frac{-1+2}{2}\right) \textit{ and } \left(\frac{1+3}{2},\frac{4+2}{2}\right)$$

$$\Rightarrow E\left(\frac{-1}{2},\frac{5}{2}\right), F(-1,0), G\left(\frac{3}{2},\frac{1}{2}\right) \textit{ and } H(2,\ 3)$$ are the required points.

Also, the mid point of diagonal *EG* has coordinates

$$\left(\frac{\frac{-1}{2}+\frac{3}{2}}{2},\frac{\frac{5}{2}+\frac{1}{2}}{2}\right)=\left(\frac{1}{2},\frac{3}{2}\right)$$

Coordinates of midpoint of *FH* are $\left(\frac{-1+2}{2},\frac{0+3}{2}\right)=\left(\frac{1}{2},\frac{3}{2}\right)$

Since, the midpoints of the diagonals are the same, therefore, the diagonals bisect each other.

Hence *EFGH* is a parallelogram.

CHECK YOUR PROGRESS 13.2

1. Find the midpoint of each of the line segements whose end points are given below:

   (a) (–2, 3) and (3, 5) (b) (6,0) and (–2,10)

2. Find the coordinates of the point dividing the line segment joining

   (–5, –2) and (3, 6) internally in the ratio 3:1.

3. (a) Three vertices of a parallelogram are (0,3), (0,6) and (2,9). Find the fourth vertex.

   (b) (4, 0), (– 4, 0), (0,– 4) and (0, 4) are the vertices of a square. Show that the quadrilateral formed by joining the midpoints of the sides is also a square.

4. The line segement joining (2, 3) and (5, –1) is trisected. Find the points of trisection.

5. Show that the figure formed by joining the midpoints of the sides of a rectangle is a rhombus.

MODULE-IV
Co-ordinate Geometry

Notes

13.4 AREA OF A TRIANGLE

Let us find the area of a triangle whose

vertices are $A(x_1, y_1),\ B(x_2, y_2)\ and\ C(x_3, y_3)$

Draw AL, BM and CN perpendiculars to XX'.

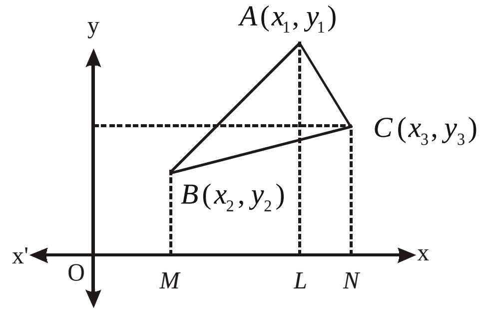


Fig.13.8

area of Δ ABC

= Area of trapzium. *BMLA* + Area of trapzium. ALNC – Area of trapzium. BMNC

$$= \frac{1}{2}(BM + AL)ML + \frac{1}{2}(AL + CN)LN - \frac{1}{2}(BM + CN)MN$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_3y_3)\right]$$

$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

This can be stated in the determinant form as follows :

$$\text{Area of } \Delta \text{ ABC} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example 13.8 Find the area of the triangle whose vertices are *A*(3, 4), *B*(6, –2) and *C*(– 4, –5).

Solution: The area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 3 & 4 & 1 \\ 6 & -2 & 1 \\ -4 & -5 & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[3(-2+5) - 4(6+4) + 1(-30-8)\right] = \frac{1}{2}\left[9 - 40 - 38\right] = \frac{-69}{2}$$

As the area is to be positive

$$\therefore \text{Area of } \Delta ABC = \frac{69}{2} \text{ square units}$$

Example 13.9 If the vertices of a triangle are (1, k), (4, –3) and (–9, 7) and its area is 15 square units, find the value(s) of k.

Solution : Area of triangle $= \frac{1}{2}\begin{vmatrix} 1 & k & 1 \\ 4 & -3 & 1 \\ -9 & 7 & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[-3-7-k(4+9)+1(28-27)\right] = \frac{1}{2}\left[-10-13k+1\right] = \frac{1}{2}\left[-9-13k\right]$$

Since the area of the triangle is given to be15,

$\therefore \quad \frac{-9-13k}{2} = 15$ or, $-9-13k = 30$, $-13k = 39$, or, $k = -3$

CHECK YOUR PROGRESS 13.3

1. Find the area of each of the following triangles whose vertices are given below :

   (1) (0, 5), (5, –5), and (0,0) (b) (2, 3), (–2,–3) and (–2, 3)

   (c) (a, 0), (0, – a) and (0, 0)

2. The area of a triangle ABC, whose vertices are A (2, –3), B(3, –2) and C$\left(\frac{5}{2}, k\right)$ is $\frac{3}{2}$ sq unit. Find the value of k

3. Find the area of a rectangle whose vertices are (5, 4), (5,– 4), (–5, 4) and (–5, – 4)

4. Find the area of a quadrilateral whose vertices are (5, –2), (4, –7), (1, 1) and (3, 4)

13.5 CONDITION FOR COLLINEARITY OF THREE POINTS

The three points $A(x_1, y_1)$, $B(x_2, y_2)$ *and* $C(x_3, y_3)$ are collinear if and only if the area of the triangle ABC becomes zero.

i.e. $\frac{1}{2}\left[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\right] = 0$

i.e. $x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0$

In short, we can write this result as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Let us illustrate this with the help of examples:

Example 13.10 Show that the points $A(a,\ b+c),\ B(b, c+a)\ and\ C(c, a+b)$ are collinear.

Solution : Area of triangle ABC $= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$ (Applying $C_1 \to C_1 + C_2$)

$$= \frac{1}{2}\begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix} = \frac{1}{2}(a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = 0$$

Hence the points are collinear.

Example 13.11 For what value of *k*, are the points (1, 5) , (k, 1) and (4, 11) collinear ?

Solution : Area of the triangle formed by the given points is

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ k & 1 & 1 \\ 4 & 11 & 1 \end{vmatrix} = \frac{1}{2}[-10 - 5k + 20 + 11k - 4] = \frac{1}{2}[6k + 6] = 3k + 3$$

Since the given points are collinear, therefore

$3k + 3 = 0 \Rightarrow k = -1$

Hence, for $k = -1$, the given points are collinear.

CHECK YOUR PROGRESS 13.4

1. Show that the points (–1,–1), (5, 7) and (8, 11) are collinear.
2. Show that the points (3, 1), (5, 3) and (6, 4) are collinear.
3. Prove that the points (*a*, 0), (0, *b*) and (1, 1) are collinear if $\frac{1}{a} + \frac{1}{b} = 1$.
4. If the points $(a,\ b), (a_1, b_1)\ and\ (a - a_1,\ b - b_1)$ are collinear, show that $a_1 b = ab_1$

5. Find the value of k for which the points (5, 7), (k, 5) and (0, 2) are collinear.

6. Find the values of k for which the point (k, 2–2k), (– k+1, 2k) and (– 4 – k, 6 – 2k) are collinear.

13.6 INCLINATION AND SLOPE OF A LINE

Look at the Fig. 13.9. The line AB makes an angle or $\pi+\alpha$ with the x-axis (measured in anticlockwise direction).

The *inclination* of the given line is represented by the measure of angle made by the line with the positive direction of x-axis (measured in anticlockwise direction)

In a special case when the line is parallel to x-axis or it coincides with the x-*axis*, the inclination of the line is defined to be 0^0.

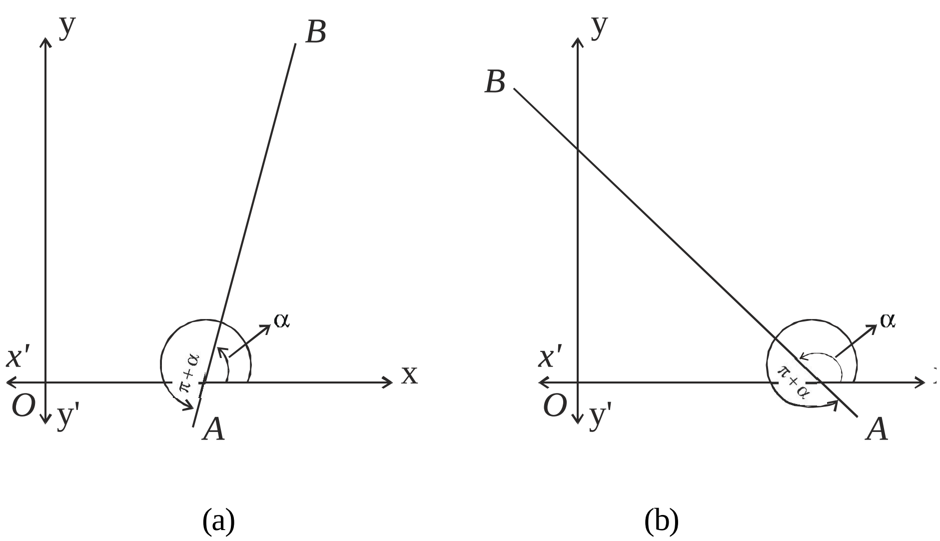
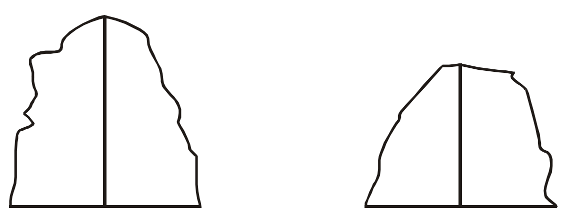


Fig. 13.9

Again look at the pictures of two mountains given below. Here we notice that the mountain in Fig. 13.10 (a) is more steep compaired to mountain in Fig. 13.10 (b).



(a) Fig.13.10 (b)

How can we quantify this steepness ? Here we say that the angle of inclination of mountain (a) is more than the angle of inclination of mountain (b) with the ground.

Try to see the difference between the ratios of the maximum height from the ground to the base in each case.

Naturally, you will find that the ratio in case (a) is more as compared to the ratio in case (b). That means we are concerned with height and base and their ratio is linked with tangent of an angle, so mathematically this ratio or the tangent of the inclination is termed as *slope*. We define the slope as tangent of an angle.

The *slope* of a line is the the tangent of the angle θ (say) which the line makes with the positive direction of x-axis. Generally, it is denoted by m $(= \tan\theta)$

Note : If a line makes an angle of 90^0 or 270^0 with the x–axis, the slope of the line can not be defined.

Example 13.12 In Fig. 13.9 find the slope of lines *AB* and *BA*.

Solution : Slope of line *AB* = tan α

Slope of line *BA* = tan $(\pi + \alpha) = \tan\alpha$.

Note : From this example, we can observe that "slope is independent of the direction of the line segement".

Example 13.13 Find the slope of a line which makes an angle of 30^0 with the negative direction of *x*-axis.

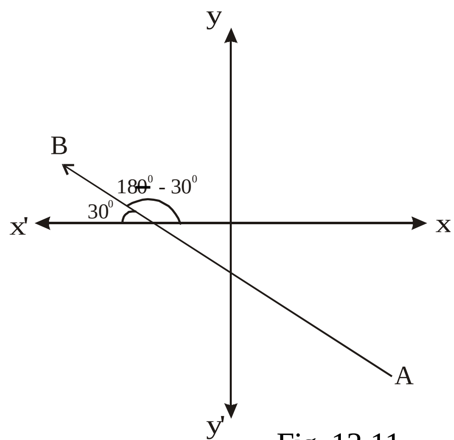


Fig. 13.11

Solution : Here $\theta = 180^0 - 30^0 = 150^0$

$\therefore$ m = slope of the line = tan $(180^0 - 30^0)$

$= -\tan 30^0$

$= -\dfrac{1}{\sqrt{3}}$

Example 13.14 Find the slope of a line which makes an angle of 60^0 with the positive direction of *y*-axis.

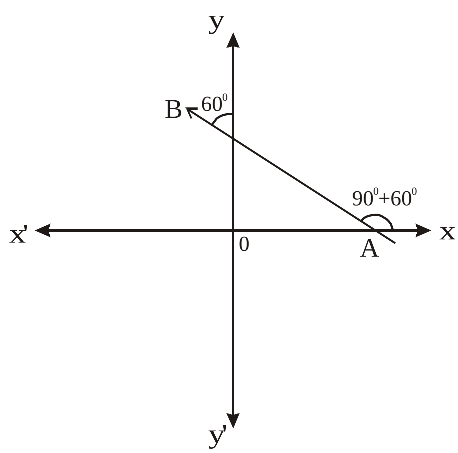


Fig. 13.12

Solution : Here $\theta = 90^0 + 60^0$

$\therefore$ m = slope of the line

$= \tan(90^0 + 60^0)$

$= -\cot 60^0$

$= -\tan 30^0 \qquad = -\dfrac{1}{\sqrt{3}}$

Example 13.15 If a line is equally inclined to the axes, show that its slope is ± 1.

Solution : Let a line *AB* be equally inclined to the axes and meeting axes at points *A* and *B* as shown in the Fig. 13.13

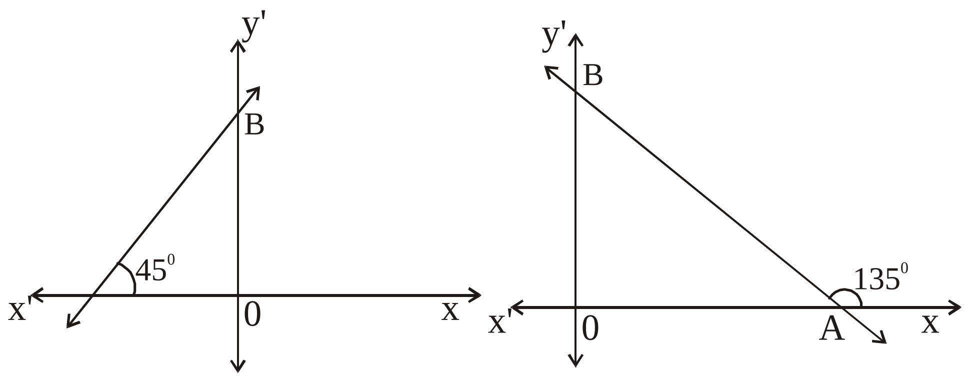


Fig. 13.13

In Fig 13.13(a), inclination of line *AB* = $\angle XAB = 45^0$

$\therefore$ Slope of the line *AB* = $\tan 45^0 = 1$

In Fig. 13.13 (b) inclination of line *AB* = $\angle XAB = 180^0 - 45^0 = 135^0$

$\therefore$ Slope of the line *AB* = $\tan 135^0 = \tan(180^0 - 45^0) = -\tan 45^0 = -1$

Thus, if a line is equally inclined to the axes, then the slope of the line will be ± 1.

CHECK YOUR PROGRESS 13.5

1. Find the Slope of a line which makes an angle of (i) 60^0, (ii) 150^0 with the positive direction of *x*-axis.
2. Find the slope of a line which makes an angle of 30^0 with the positive direction of *y*-axis.
3. Find the slope of a line which makes an angle of 60^0 with the negative direction of *x*-axis.

13.7 SLOPE OF A LINE JOINING TWO DISTINCT POINTS

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two distinct points. Draw a line through *A* and *B* and let the inclination of this line be θ. Let the point of intersection of a horizontal line through *A* and a vertical line through *B* be *M*, then the coordinates of *M* are as shown in the Fig. 13.14

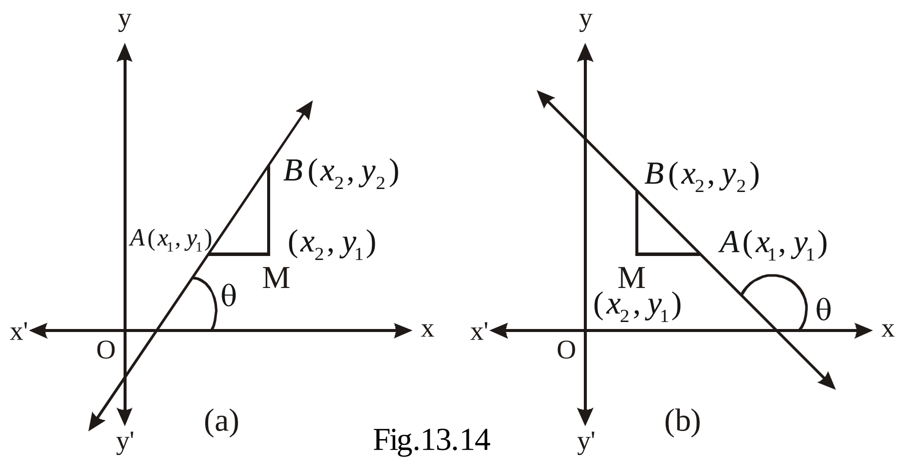


Fig.13.14

(A) In Fig 13.14 (a), angle of inclination *MAB* is equal to θ (acute). Consequently.

$$\tan\theta = \tan(\angle MAB) = \frac{MB}{AM} = \frac{y_2 - y_1}{x_2 - x_1}$$

(B) In Fig. 13.14 (b), angle of inclination θ is obtuse, and since θ and $\angle MAB$ are supplementary, consequently,

$$\tan\theta = -\tan(\angle MAB) = -\frac{MB}{MA} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

Hence in both the cases, the slope *m* of a line through $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Note : *if* $x_1 = x_2$*, then m is not defined. In that case the line is parallel to y-axis.*

Is there a line whose slope is 1? Yes, when a line is inclined at 45^0 with the positive direction of *x*-axis.

Is there a line whose slope is $\sqrt{3}$? Yes, when a line is inclined at 60^0 with the positive direction of *x*-axis.

From the answers to these questions, you must have realised that given any real number *m*, there will be a line whose slope is *m* (because we can always find an angle α such that $\tan\alpha = m$).

Example 13.16 Find the slope of the line joining the points *A*(6, 3) and *B*(4, 10).

Solution : The slope of the line passing through the points (x_1, y_1) and $(x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$

Here, $x_1 = 6,\ y_1 = 3;\ x_2 = 4,\ y_2 = 10.$

Now substituting these values, we have slope $= \frac{10-3}{4-6} = -\frac{7}{2}$

Example 13.17 Determine x, so that the slope of the line passing through the points (3, 6) and (x, 4) is 2.

Solution :

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-6}{x-3} = \frac{-2}{x-3}$$

$$\therefore \frac{-2}{x-3} = 2 \qquad \text{.............. (Given)}$$

$$\therefore \quad 2x - 6 = -2 \quad or \quad x = 2$$

CHECK YOUR PROGRESS 13.6

1. What is the slope of the line joining the points $A(6, 8)$ and $B(4, 14)$?
2. Determine x so that 4 is the slope of the line through the points $A(6,12)$ and $B(x, 8)$.
3. Determine y, if the slope of the line joining the points $A(-8, 11)$ and $B(2, y)$ is $-\frac{4}{3}$.
4. $A(2, 3)$ $B(0, 4)$ and $C(-5, 0)$ are the vertices of a triangle ABC. Find the slope of the line passing through *the point B* and the mid point of AC
5. $A(-2, 7)$, B(1, 0), C(4, 3) and D(1, 2) are the vertices of a quadrilateral $ABCD$. Show that

   (i) slope of AB = slope of CD (ii) slope of BC = slope of AD

13.8 CONDITIONS FOR PARALLELISM AND PERPENDI CULARITY OF LINES.

9.8.1 Slope of Parallel Lines

Let l_1, l_2, be two (non-vertical) lines with their slopes m_1 and m_2 respectively.

Let θ_1 and θ_2 be the angles of inclination of these lines respectively.

Case I : Let the lines *l_1 and l_2 be parallel*

Then $\theta_1 = \theta_2 \Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 = m_2$

Thus, if two lines are parallel then their slopes are equal.

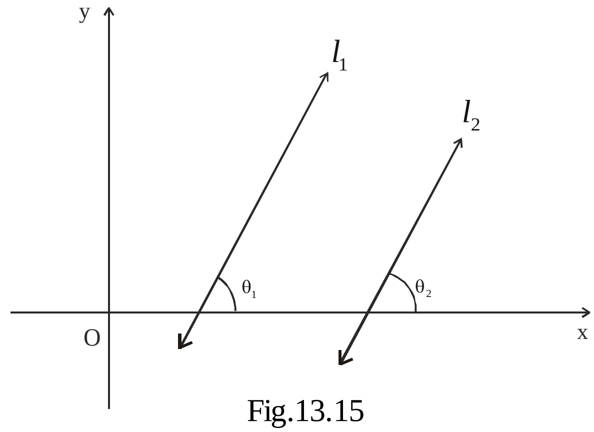


Fig.13.15

Case II : Let the lines l_1, and l_2 have equal slopes.

i.e. $m_1 = m_2 \Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow \theta_1 = \theta_2 \;\; (0^\circ \le \theta \le 180^\circ)$

$\Rightarrow l_1 \parallel l_2$

Hence, two (non-vertical) lines are parallel if and only if $m_1 = m_2$

13.8.2 SLOPES OF PERPENDICULAR LINES

Let l_1 and l_2 be two (non-vertical)lines with their slopes m_1 and m_2 respectively. Also let θ_1 and θ_2 be their inclinations respectively.

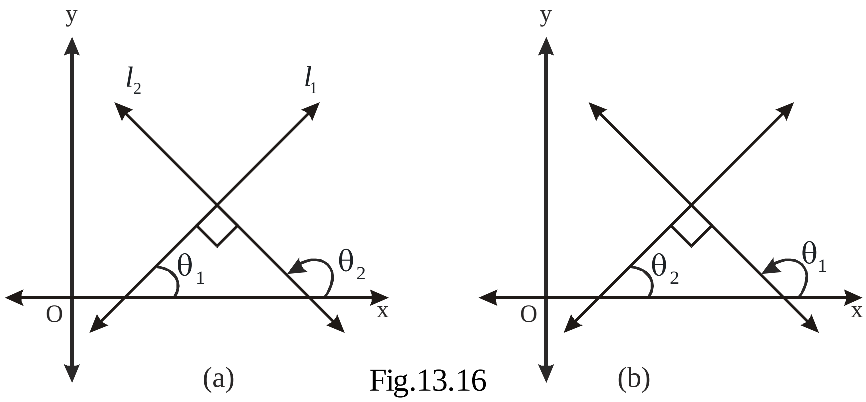


Fig.13.16

Case-I : Let $l_1 \perp l_2$

$\Rightarrow \theta_2 = 90^0 + \theta_1$ or $\theta_1 = 90^0 + \theta_2$

$\Rightarrow \tan\theta_2 = \tan\left(90^0 + \theta_1\right)$ or $\tan\theta_1 = \tan\left(90^0 + \theta_2\right)$

$\Rightarrow \tan\theta_2 = -\cot(\theta_1)$ or $\tan\theta_1 = -\cot(\theta_2)$

$\Rightarrow \tan\theta_2 = -\dfrac{1}{\tan\theta_1}$ or $\Rightarrow \tan\theta_1 = -\dfrac{1}{\tan\theta_2}$

$\Rightarrow$ In both the cases, we have

$\tan\theta_1 \tan\theta_2 = -1$

or $m_1 . m_2 = -1$

Thus, if two lines are perpendicular then the product of their slopes is equal to –1.

Case II : Let the two lines l_1 and l_2 be such that the product of their slopes is –1.

i.e. $m_1.m_2 = -1$

$$\Rightarrow \tan\theta_1 \tan\theta_2 = -1$$

$$\Rightarrow \tan\theta_1 = -\frac{1}{\tan\theta_2} = -\cot\theta_2 = \tan\left(90^0 + \theta_2\right)$$

or

$$\tan\theta_2 = \frac{-1}{\tan\theta_1} = -\cot\theta_1 = \tan\left(90 + \theta_1\right)$$

$\Rightarrow$ Either $\theta_1 = 90^0 + \theta_2$ or $\theta_2 = 90^0 + \theta_1$ $\Rightarrow$ In both cases $l_1 \perp l_2$.

Hence, two (non-vertical) lines are perpendicular if and only if $m_1.m_2 = -1$.

Example 13.18 Show that the line passing through the points A(5,6) and B(2,3) is parallel to the line passing, through the points C(9,–2) and D(6,–5).

Solution : Slope of the line AB $= \dfrac{3-6}{2-5} = \dfrac{-3}{-3} = 1$

and slope of the line CD $= \dfrac{-5+2}{6-9} = \dfrac{-3}{-3} = 1$

As the slopes are equal $\therefore$ AB $\parallel$ CD.

Example 13.19 Show that the line passing through the points A(2,–5) and B(–2,5) is perpendicular to the line passing through the points L(6,3) and M(1,1).

Solution : Here

$$m_1 = \text{slope of the line AB} = \frac{5+5}{-2-2} = \frac{10}{-4} = \frac{-5}{2}$$

$$\text{and } m_2 = \text{slope of the line LM} = \frac{1-3}{1-6} = \frac{2}{5}$$

$$\text{Now } m_1.m_2 = \frac{-5}{2} \times \frac{2}{5} = -1$$

Hence, the lines are perpendicular to each other.

Example 13.20 Using the concept of slope, show that A(4,4), B(3,5) and C (-1,-1) are the vertices of a right triangle.

Solution : Slope of line AB = $m_1 = \dfrac{5-4}{3-4} = -1$

Slope of line BC = $m_2 = \dfrac{-1-5}{-1-3} = \dfrac{3}{2}$

and slope of line AC = $m_3 = \dfrac{-1-4}{-1-4} = 1$

Now $m_1 \times m_3 = -1 \quad \Rightarrow AB \perp AC$

$\Rightarrow \triangle ABC$ is a right-angled triangle.

Hence, A(4,4), B(3,5) and C(– 1, – 1) are the vertices of right triangle.

Example 13.21 What is the value of y so that the line passing through the points A(3,y) and B(2,7) is perpendicular to the line passing through the point C (– 1,4) and D (0,6)?

Solution : Slope of the line AB = $m_1 = \dfrac{7-y}{2-3} = y-7$

Slope of the line CD = $m_2 = \dfrac{6-4}{0+1} = 2$

Since the lines are perpendicular,

$\therefore m_1 \times m_2 = -1$ or $(y-7) \times 2 = -1$

or $2y - 14 = -1$ or $2y = 13$ or $y = \dfrac{13}{2}$

CHECK YOUR PROGRESS 13.7

1. Show that the line joining the points (2,–3) and (– 4,1) is

   (i) parallel to the line joining the points (7,–1) and (0,3).

   (ii) perpendicular to the line joining the points (4,5) and (0,–2).

2. Find the slope of a line parallel to the line joining the points (– 4,1) and (2,3).

3. The line joining the points (–5,7) and (0,–2) is perpendicular to the line joining the points (1,3) and (4,x). Find x.

4. A(–2,7), B(1,0), C(4,3) and D(1,2) are the vertices of quadrilateral ABCD. Show that the sides of ABCD are parallel.

5. Using the concept of the slope of a line, show that the points A(6, – 1), B(5,0) and C(2,3) are collinear.[Hint: slopes of AB, BC and CA must be equal.]

6. Find k so that line passing through the points (k,9) and (2,7) is parallel to the line passing through the points (2,–2) and (6,4).

7. Using the concept of slope of a line, show that the points (–4,–1), (–2–4), (4,0) and (2,3) taken in the given order are the vertices of a rectangle.

8. The vertices of a triangle ABC are A(–3,3), B(–1,– 4) and C(5,–2). M and N are the midpoints of AB and AC. Show that MN is parallel to BC and MN = $\frac{1}{2}$ BC.

13.9 INTERCEPTS MADE BY A LINE ON AXES

If a line l (not passing through the Origin) meets x-axis at A and y-axis at B as shown in Fig. 13.17, then

(i) OA is called the x-intercept or the intercept made by the line on x-axis.

(ii) OB is called y-intercept or the intercept made by the line on y-axis.

(iii) OA and OB taken together in this order are called the intercepts made by the line l on the axes.

(iv) AB is called the portion of the line intercepted between the axes.

(v) The coordinates of the point A on x-axis are (a,0) and those of point B are (0,b)

To find the intercept of a line in a given plane on x-axis, we put y = 0 in the given equation of a line and the value of x so obtained is called the x intercept.

To find the intercept of a line on y-axis we put x = 0 and the value of y so obtained is called the y intercept.

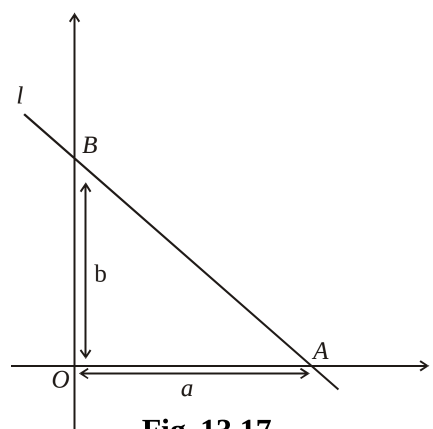


Fig. 13.17

Note: 1. A line which passes through origin makes no intercepts on axes.

2. A horizontal line has no x-intercept and vertical line has no y-intercept.

3. The intercepts on x- axis and y-axis are usually denoted by a and b respectively. But if only y-intercept is considered, then it is usually denoted by c.

Example 13.22 If a line is represented by $2x+3y=6$, find its x and y intercepts.

Solution : The given equation of the line is $2x+3y=6$ ... (i)

Putting x = 0 in (i), we get y = 2

Thus, y-intercept is 2.

Again putting $y = 0$ in (i), we get $2x = 6 \Rightarrow x = 3$

Thus, x-intercept is 3.

CHECK YOUR PROGRESS 13.8

1. Find x and y intercepts, if the equations of lines are :

(i) $x + 3y = 6$ (ii) $7x + 3y = 2$ (iii) $\frac{x}{2a} + \frac{y}{2b} = 1$ (iv) $ax + by = c$

(v) $\frac{y}{2} - 2x = 8$ (vi) $\frac{y}{3} - \frac{2x}{3} = 7$

13.10 ANGLE BETWEEN TWO LINES

Let l_1 and l_2 be two non vertical and non perpendicualr lines with slopes m_1 and m_2 respectively. Let α_1 and α_2 be the angles subtended by l_1 and l_2 respectively with the positive direction of x-axis. Then $m_1 = \tan \alpha_1$ and $m_2 = \tan \alpha_2$.

From figure 1, we have $\alpha_1 = \alpha_2 + \theta$

$\therefore \quad \theta = \alpha_1 - \alpha_2$

$\Rightarrow \quad \tan \theta = \tan (\alpha_1 - \alpha_2)$

i.e. $\quad \tan \theta = \dfrac{\tan \alpha_1 - \tan \alpha_2}{1 + \tan \alpha_1 . \tan \alpha_2}$

i.e. $\quad \tan \theta = \dfrac{m_1 - m_2}{1 + m_1 m_2} \quad ...(1)$

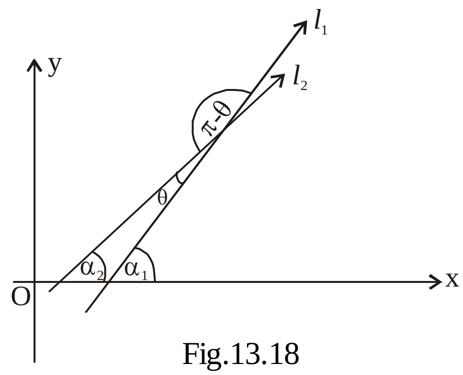


Fig.13.18

As it is clear from the figure that there are two angles θ and $\pi - \theta$ between the lines l_1 and l_2.

We know, $\quad \tan (\pi - \theta) = - \tan \theta$

$\therefore \quad \tan (\pi - \theta) = - \left(\dfrac{m_1 - m_2}{1 + m_1 m_2} \right)$

Let $\pi - \theta = \phi$

$$\therefore \quad \tan\phi = -\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) \quad ...(2)$$

- If $\frac{m_1 - m_2}{1 + m_1 m_2}$ is positive then tan θ is positive and tan φ is negative i.e. θ is acute and φ is obtuse.

- If $\frac{m_1 - m_2}{1 + m_1 m_2}$ is negative then tan θ is negative and tan φ is positive i.e. θ is obtuse and φ is acute.

Thus the acute angle (say θ) between lines l_1 and l_2 with slopes m_1 and m_2 respectively is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \text{where } 1 + m_1 m_2 \neq 0.$$

The obtuse angle (say φ) can be found by using the formula φ = 180° – θ.

Example 13.23 Find the acute and obtuse angles between the lines whose slopes are $\frac{3}{4}$ and $\frac{-1}{7}$.

Solution : Let θ and φ be the acute and obtuse angle between the lines respectively.

$$\therefore \quad \tan\theta = \left|\frac{\frac{3}{4} + \frac{1}{7}}{1 + \left(\frac{3}{4}\right)\left(\frac{-1}{7}\right)}\right| = \left|\frac{21 + 4}{28 - 3}\right| = |1| = 1$$

$$\Rightarrow \quad \theta = 45^\circ$$

$$\therefore \quad \phi = 180^\circ - 45^\circ = 135^\circ.$$

Example 13.24 Find the angle (acute or obtuse) between x-axis and the line joining the points (3, –1) and (4, –2),

Solution : Slope of x-axis (say m_1) = 0

Slope of given line (say m_2) = $\frac{-2 + 1}{4 - 3} = -1$

$$\therefore \quad \tan\theta = \left|\frac{0 + 1}{1 + (0)(-1)}\right| = 1$$

$$\Rightarrow \quad \theta = 45^\circ \text{ as acute angle.}$$

Example 13.25 If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Solution : Here, $\tan\frac{\pi}{4} = \left|\dfrac{\frac{1}{2}-m_2}{1+\left(\frac{1}{2}\right)(m_2)}\right|$

$$\Rightarrow \quad \left|\frac{1-2m_2}{2+m_2}\right| = 1$$

$$\Rightarrow \quad \frac{1-2m_2}{2+m_2} = 1 \text{ or } \frac{1-2m_2}{2+m_2} = -1.$$

$$\Rightarrow \quad m_2 = -\frac{1}{3} \text{ or } m_2 = 3.$$

$\therefore$ Slope of other line is 3 or $-\frac{1}{3}$.

CHECK YOUR PROGRESS 13.9

1. Find the acute angle between the lines with slopes 5 and $\frac{2}{3}$.
2. Find the obtuse angle between the lines with slopes 2 and –3.
3. Find the acute angle between the lines l_1 and l_2 where l_1 is formed by joining the points (0, 0) and (2, 3) and l_2 by joining the points (2, –2) and (3, 5)

13.11 SHIFTING OF ORIGIN :

We know that by drawing x-axis and y-axis, any plane is divided into four quadrants and we represent any point in the plane as an ordered pair of real numbers which are the lengths of perpendicular distances of the point from the axes drawn. We also know that these axes can be chosen arbitrarily and therefore the position of these axes in the plane is not fixed. Position of the axes can be changed. When we change the position of axes, the coordinates of a point also get changed correspondingly. Consequently equations of curves also get changed.

The axes can be changed or transformed in the following ways :

(i) Translation of axes (ii) Rotation of axes (iii) Translation and rotation of axes. In the present section we shall discuss only one transformation i.e. translation of axes.

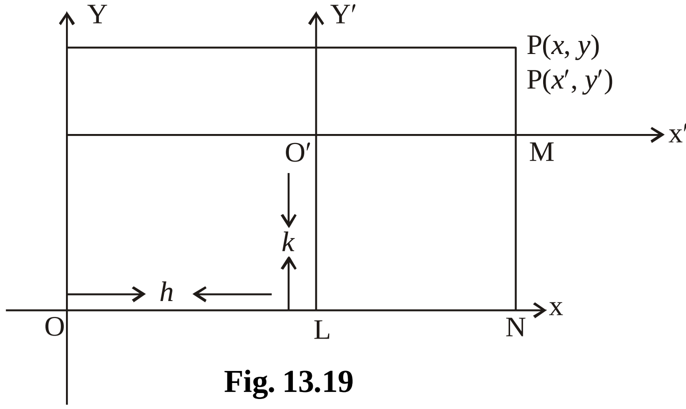


Fig. 13.19

The transformation obtained, by shifting the origin to a given point in the plane, without changing the directions of coordinate axes is called **translation of axes**.

Let us see how coordinates of a point in a plane change under a translation of axes. Let $\overrightarrow{OX}$ and $\overrightarrow{OY}$ be the given coordinate axes. Suppose the origin O is shifted to O′(*h*, *k*) by the translation of the axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$. Let $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$ be the new axes as shown in the above figure. Then with reference to $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$ the point O' has coordinates (0, 0).

Let P be a point with coordinates (x, y) in the system $\overrightarrow{OX}$ and $\overrightarrow{OY}$ and with coordinates (x', y') in the system $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$. Then $O'L = K$ and $OL = h$.

Now $\quad x = ON = OL + LN$

$$= OL + O'M$$

$$= h + x'.$$

and $y = PN = PM + MN = PM + O'L = y' + k$.

Hence $x = x' + h;\ y = y' + k$

or $x' = x - h,\ y' = y - k$

- If the origin is shifted to (h, k) by translation of axes then coordinates of the point P(x, y) are transformed to P($x - h$, $y - k$) and the equation F(x, y) = 0 of the curve is transformed to F($x' + h$, $y' + k$) = 0.

- Translation formula always hold, irrespective of the quadrant in which the origin of the new system happens to lie.

Example 13.26 When the origin is shifted to (–3, 2) by translation of axes find the coordinates of the point (1, 2) with respect to new axes.

Solution : Here $(h, k) = (-3, 2)$, $(x, y) = (1, 2)$, $(x', y') = ?$

$x' = x - h = 1 + 3 = 4$

$y' = y - k = 2 - 2 = 0$

Therefore $(x', y') = (4, 0)$

Example 13.27 When the origin is shifted to the point (3, 4) by the translation of axes, find the transformed equation of the line $3x + 2y - 5 = 0$.

Solution : Here $(h, k) = (3, 4)$

$\therefore \quad x = x' + 3$ and $y = y' + 4$.

Substituting the values of x and y in the equation of line

we get $3(x' + 3) + 2(y' + 4) - 5 = 0$

i.e. $3x' + 2y' + 12 = 0$.

CHECK YOUR PROGRESS 13.10

1. (i) Does the length of a line segment change due to the translation of axes? Say yes or no.

   (ii) Are there fixed points with respect to translation of axes? Say yes or no.

   (iii) When the origin is shifted to the point (4, –5) by the translation of axes, the coordinates of the point (0, 3) are ...

   (iv) When the origin is shifted to (2, 3), the coordinates of a point P changes to (4, 5), coordinates of point P in original system are ...

   (v) If due to translation of axes the point (3, 0) changes to (2, –3), then the origin is shifted to the point ...

LET US SUM UP

- Distance between any two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

- Coordinates of the point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

- Coordinates of the point dividing the line segment joining the the points (x_1, y_1) and (x_2, y_2) externally are in the ratio $m_1 : m_2$ are.

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$

- Coordinates of the mid point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- The area of a triangle with vertices (x_1, y_1) and (x_2, y_2) and (x_3, y_3) is given by

$$\frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right]$$

- Three points A, B, and C are collinear if the area of the triangle formed by them is zero.
- If θ is the angle which a line makes with the positive direction of x-axis, then the slope of the line is $m = \tan\theta$.
- Slope (m) of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- A line with the slope m_1 is parallel to the line with slope m_2 if $m_1 = m_2$.
- A line with the slope m_1 is perpendicular to the line with slope m_2 if $m_1 \times m_2 = -1$.
- If a line l (not passing through the origin) meets x- axis at A and y- axis at B then OA is called the x- intercept and OB is called the y- intercept.
- If θ be the angle between two lines with slopes m_1 and m_2, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

where $1 + m_1m_2 \neq 0$

- If $\tan\theta$ is +ve, the angle (θ) between the lines is acute and if $\tan\theta$ is -ve then it is obtuse.
- When origin is shifted to (h,k) then transformed coordinates (x',y') (say) of a point P (x,y) are (x-h, y-k)

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=VhNkWdLGpmA
http://www.youtube.com/watch?v=5ctsUsvIp8w
http://www.youtube.com/watch?v=1op92ojA6q0

TERMINAL EXERCISE

1. Find the distance between the pairs of points:

   (a) (2, 0) and (1, cot θ) (b) (–sin A, cosA) and (sin B, cos B)

2. Which of the following sets of points form a triangle?

   (a) (3, 2), (–3, 2) and (0, 3) (b) (3, 2), (3, –2) and (3, 0)

3. Find the midpoint of the line segment joining the points (3. –5) and (–6, 8).

4. Find the area of the triangle whose vertices are:

   (a) (1, 2), (–2, 3), (–3, –4) (b)(c, a), (c + a, a), (c – a, –a)

5. Show that the following sets of points are collinear (by showing that area formed is 0).

   (a) (–2, 5) (2, –3) and (0, 1) (b) (a, b + c), (b, c + a) and (c, a + b)

6. If (–3, 12), (7, 6) and (x, a) are collinear, find x.

7. Find the area of the quadrilateral whose vertices are (4,3) (–5,6) (0,7) and (3,–6).

8. Find the slope of the line through the points

   (a) (1,2), (4,2) (b) (4, –6), (–2, –5)

9. What is the value of y so that the line pasing through the points (3, y) and (2,7) is parallel to the line passing through the points (–1, 4) and (0, 6) ?

10. Without using Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right-angled triangle.

11. Using the concept of slope, determine which of the following sets of points are collnear:

    (i) (–2, 3) , (8, –5) and (5, 4), (ii) (5, 1) , (1, –1) and (11, 4),

12. If A (2, –3) and B (3, 5) are two vertices of a rectangle ABCD, find the slope of

    (i) BC (ii) CD (iii) DA.

13. A quadrilateral has vertices at the points (7, 3), (3, 0), (0, –4) and (4, –1). Using slopes, show that the mid-points of the sides of the quadrilatral form a parallelogram.

14. Find the x-intercepts of the following lines:

    (i) $2x-3y=8$ (ii) $3x-7y+9=0$ (iii) $x-\frac{y}{2}=3$

15. When the origin is shifted to the point (3, 4) by translation of axes, find the transformed equation of $2x^2+4xy+5y^2=0$.

16. If the origin is shifted to the point (3, –4), the transformed equation of a curve is $(x^1)^2+(y^1)^2=4$, find the original equation of the curve.

17. If A(–2, 3), B(3, 8) and C(4, 1) are the vertices of a ΔABC. Find $\angle$ABC of the triangle.

18. Find the acute angle between the diagonals of a quadrilateral ABCD formed by the points A(9, 2), B(17, 11), C(5, – 3) and D(–3, –2) taken in order.

19. Find the acute angle between the lines AB and BC given that A = (5, –3), B = (–3, –2) and C = (9, 12).

ANSWERS

CHECK YOUR PROGRESS 13.1

(a) $\sqrt{58}$ (b) $\sqrt{2(a^2+b^2)}$

CHECK YOUR PROGRESS 13.2

1. (a) $\left(\frac{1}{2},4\right)$ (b) (2,5) 2. (1,4) 3. (a) (2,6)

4. $\left(3,\frac{5}{3}\right),\left(4,\frac{1}{3}\right)$

CHECK YOUR PROGRESS 13.3

1. (a) $\frac{25}{2}$ sq. units (b) 12 sq. units (c) $\frac{a^2}{2}$ sq. units

2. $k=\frac{5}{3}$ 3. 80 sq. units 4. $\frac{41}{2}$ sq. units

CHECK YOUR PROGRESS 13.4

5. $k = 3$ 6. $k = \frac{1}{2}, -1$

CHECK YOUR PROGRESS 13.5

1. (*i*) $\sqrt{3}$ (*ii*) $-\frac{1}{\sqrt{3}}$ 2. $-\sqrt{3}$ 3. $-\sqrt{3}$

CHECK YOUR PROGRESS 13.6

1. -3 2. 5 3. $-\frac{7}{3}$ 4. $\frac{5}{3}$

CHECK YOUR PROGRESS 13.7

2. $\frac{1}{3}$ 3. $\frac{14}{3}$. 6. $k=\frac{10}{3}$

CECK YOUR PROGRESS 13.8

1. (*i*) x-intercept = 6, y-intercept = 2

(*ii*) x-intercept = $\frac{2}{7}$, y-intercept = $\frac{2}{3}$

(*iii*) x-intercept = 2a, y-intercept = $2b$

(*iv*) x-intercept $= \frac{c}{a}$, y-intercept $= \frac{c}{b}$

(*v*) x-intercept $= -4$, y-intercept $= 16$

(*vi*) x-intercept $= \frac{-21}{2}$, y-intercept = 21

CHECK YOUR PROGRESS 13.9

1. 45° 2. 135° 3. $\tan = \frac{11}{23}$

CHECK YOUR PROGRESS 13.10

1. (i) No (ii) No (iii) (-4,8) (iv) (6,8) (v) (1,3)

TERMINAL EXERCISE

1. (a) cosec θ (b) $2 \sin \frac{A+B}{2}$
2. None of the given sets forms a triangle.
3. $\left(-\frac{3}{2}, \frac{3}{2}\right)$
4. (a) 11 sq. unit (b) a^2 sq. unit.
6. $\frac{51-5a}{3}$
7. 29 sq. unit.
8. (*a*) 0 (b) $-\frac{1}{6}$
9. $y = 3$
11. Only (*ii*)
12. (*i*) $-\frac{1}{8}$ (*ii*) 8 (*iii*) $-\frac{1}{8}$
14. (*i*) 4 (*ii*) -3 (*iii*) 3
15. $x^2 + 4y^2 + 4xy + 116x + 2y + 259 = 0$
16. $x^2 + y^2 - 6x + 8y + 21 = 0$
17. $\tan^{-1}\left(\frac{4}{3}\right)$
18. $\tan^{-1}\left(\frac{48}{145}\right)$
19. $\tan^{-1}\left(\frac{62}{55}\right)$

14

STRAIGHT LINES

We have read about lines, angles and rectilinear figures in geometry. Recall that a line is the join of two points in a plane continuing endlessly in both directions. We have also seen that graphs of linear equations, which came out to be straight lines

Interestingly, the reverse problem of the above is finding the equations of straight lines, under different conditions, in a plane. The analytical geometry, more commonly called coordinate geomatry, comes to our help in this regard. In this lesson. We shall find equations of a straight line in different forms and try to solve problems based on those.

OBJECTIVES

After studying this lesson, you will be able to :

- derive equations of a line parallel to either of the coordinate axes;
- derive equations of a line in different forms (slope-intercept, point-slope, two point, intercept, and perpendicular)
- find the equation of a line in the above forms under given conditions;
- state that the general equation of first degree represents a line;
- express the general equation of a line into

  (i) slope-intercept form (ii) intercept form and (iii) perpendicular form;
- derive an expression for finding the distance of a given point from a given line;
- calculate the distance of a given point from a given line;
- derive the equation of a line passing through a given point and parallel/perpendicular to a given line;
- find equation of family of lines passing through the point of intersection of two lines.

EXPECTED BACKGROUND KNOWLEDGE

- Congruence and similarity of traingles

14.1 STRAINGHT LINE PARALLEL TO AN AXIS

If you stand in a room with your arms stretched, we can have a line drawn on the floor parallel to one side. Another line perpendicular to this line can be drawn intersecting the first line between your legs.

In this situation the part of the line in front of you and going behind you is the *y*-axis and the one being parallel to your arms is the *x*-axis.

The direction part of the *y*-axis in front of you is positive and behind you is negative.

The direction of the part *x*-axis to your right is positive and to that to your left is negative.

Now, let the side facing you be at *b* units away from you, then the equation of this edge will be $y = b$ (parallel to *x*-axis)

where *b* is equal in absolute value to the distance from the *x*-axis to the opposite side.

If $b > 0$, then the line lies in front of you, i.e., above the *x*-axis.

If $b < 0$, then the line lies behind you, i.e., below the *x*-axis.

If $b = 0$, then the line passes through you and is the *x*-axis itself.

Again, let the side of the right of you is at *c* units apart from you, then the equation of this line will be $x = c$ (parallel to *y* - axis)

where *c* is equal in absolute value, to the distance from the *y*-axis on your right.

If $c > 0$, then the line lies on the right of you, i.e., to the right of *y*-axis.

If $c < 0$, then the line lies on the left of you, i.e., to the left of *y*-axis

If $c = 0$, then the line passes through you and is the *y*-axis.

Example 14.1 Find the equation of the line passing through (–2, –3) and

(i) parallel to *x*-axis (ii) parallel to *y*-axis

Solution :

(i) The equation of any line parallel to *x*-axis is $y = b$

Since it passes through (–2, –3), hence $-3 = b$

∴ The required equation of the line is $y = -3$

(i) The equation of any line parallel to *y*-axis is $x = c$

Since it passes through (–2, –3), hence –2 = c

∴ The required cquation of the line is $x = -2$

CHECK YOUR PROGRESS 14.1

1. Find the equation of the line passing through (–3, –4) and

   (a) parallel to x-axis. (b) parallel to y-axis.

2. Find the equation of a line passing through (5, –3) and perpendicular to x-axis.

3. Find the equation of the line passing through (–3, –7) and perpendicular to y-axis.

14.2 DERIVATION OF THE EQUATION OF STRAIGHT LINE IN VARIOUS STANDARD FORMS

So far we have studied about the inclination, slope of a line and the lines parallel to the axes. Now the questions is, can we find a relationship between x and y, where (x, y) is any arbitrary point on the line?

The relationship between x and y which is satisfied by the co-ordinates of arbitrary point on the line is called the equation of a straight line. The equation of the line can be found in various forms under the given conditions, such as

(a) When we are given the slope of the line and its intercept on y-axis.

(b) When we are given the slope of the line and it passes through a given point.

(c) When the line passes through two given points.

(d) When we are given the intercepts on the axes by the line.

(e) When we are given the length of perpendicular from origin on the line and the angle which the perpendicualr makes with the positive direction of x-axis.

We will discuss all the above cases one by one and try to find the equation of line in its standard forms.

(A) SLOPE-INTECEPT FORM

Let AB be a straight line making an angle θ with x-axis and cutting off an intercept $OD = c$ from OY.

As the line makes intercept OD = c on y-axis, it is called y-intercept.

Let AB intersect OX' at T.

Take any point $P(x, y)$ on AB. Draw $PM \perp OX$.

The $OM = x$, $MP = y$.

Draw $DN \perp MP$.

From the right-angled triangle DNP, we have

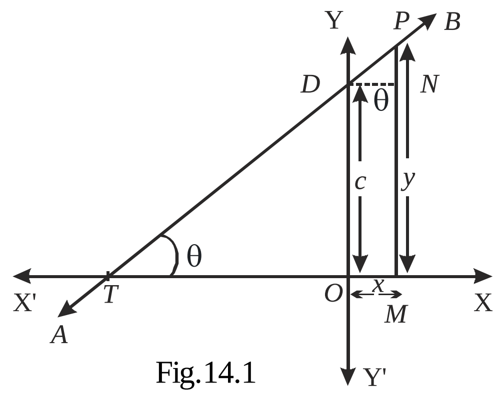


Fig.14.1

$$\tan\theta = \frac{NP}{DN} = \frac{MP - MN}{OM}$$

$$= \frac{y - OD}{OM}$$

$$= \frac{y - c}{x}$$

$\therefore \quad y = x\tan\theta + c$

$\tan\theta = m$ (slope)

$\therefore \quad y = mx + c$

Since, this equation is true for every point on AB, and clearly for no other point in the plane, hence it represents the equation of the line AB.

***Note :** (1) When $c = 0$ and $m \neq 0 \Rightarrow$ the line passes through the origin and its equation is $y = mx$*

(2) When $c = 0$ and $m = 0 \Rightarrow$ the line coincides with x – axis and its equation is of the form $y = o$

(3) When $c \neq 0$ and $m = 0 \Rightarrow$ the line is parallel to x-axis and its equation is of the form $y = c$

Example 14.2 Find the equation of a line with slope 4 and y-intercept 0.

Solution : Putting $m = 4$ and $c = 0$ in the slope intercept form of the equation, we get $y = 4x$

This is the desired equation of the line.

Example 14.3 Determine the slope and the y-intercept of the line whose equation is

$8x + 3y = 5.$

Solution : The given equation of the line is

$$8x + 3y = 5 \quad \text{or,} \quad y = -\frac{8}{3}x + \frac{5}{3}$$

Comparing this equation with the equation $y = mx + c$ (Slope intercept form) we get

$$m = -\frac{8}{3} \text{ and } c = \frac{5}{3}$$

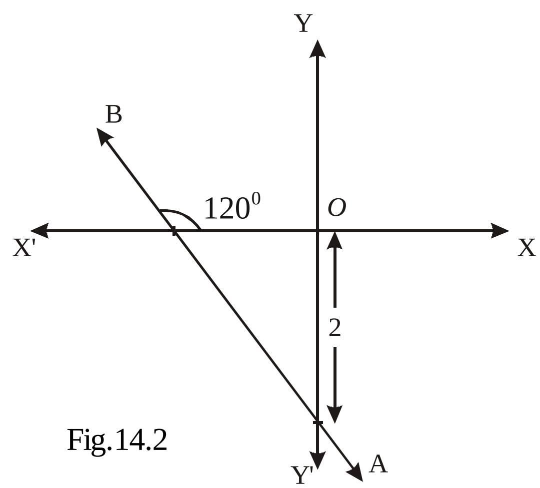


Fig.14.2

Therefore, slope of the line is $-\frac{8}{3}$ and its y-intercept is $\frac{5}{3}$.

Example 14.4 Find the equation of the line cutting off an intercept of length 2 from the negative direction of the axis of y and making an angle of 120^0 with the positive direction x-axis

Solution : From the slope intercept form of the line $\therefore \quad y = x \tan 120° + (-2)$

$$= -\sqrt{3}\,x - 2 \text{ or, } y + \sqrt{3}\,x + 2 = 0$$

Here $m = \tan 120°$, and $c = -2$, because the intercept is cut on the negative side of y-axis.

(b) POINT-SLOPE FORM

Here we will find the equation of a line passing through a given point $A(x_1, y_1)$ and having the slope m.

Let $P(x, y)$ be any point other than A on given the line. Slope $(\tan\theta)$ of the line joining $A(x_1, y_1)$ and $P(x, y)$ is given by

$$\text{m} = \tan\theta = \frac{y - y_1}{x - x_1}$$

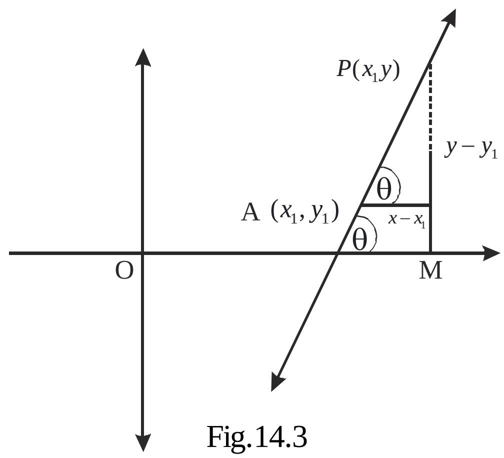


Fig.14.3

The slope of the line AP is given to be m.

$$\therefore \quad m = \frac{y - y_1}{x - x_1}$$

$\therefore$ The equation of the required line is, $y - y_1 = m(x - x_1)$

__Note :__ Since, the slope m is undefined for lines parallel to y-axis, the point-slope form of the equation will not give the equation of a line though A (x_1, y_1) parallel to y-axis. However, this presents no difficulty, since for any such line the abscissa of any point on the line is x_1. Therefore, the equation of such a line is $x = x_1$.

Example 14.5 Determine the equation of the line passing through the point (2,– 1) and having slope $\frac{2}{3}$.

Solution : Putting $x_1 = 2, y_1 = -1$ and $m = \frac{2}{3}$ in the equation of the point-slope form of the line we get, $y - (-1) = \frac{2}{3}(x - 2)$

$$\Rightarrow \quad y + 1 = \frac{2}{3}(x - 2) \quad \Rightarrow \quad y = \frac{2}{3}x - \frac{7}{3}$$

which is the required equation of the line.

(c) TWO POINT FORM

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given distinct points.

Slope of the line passing through these points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad (x_2 \neq x_1)$$

From the equation of line in point slope form, we get

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

which is the required equation of the line in two-point form.

Example 14.6 Find the equation of the line passing through (3, – 7) and (– 2,– 5).

Solution : The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \cdots \text{(i)}$$

Since $x_1 = 3$, $y_1 = -7$ and $x_2 = -2$, and $y_2 = -5$, equation (i) becomes,

$$y + 7 = \frac{-5 + 7}{-2 - 3}(x - 3)$$

or, $y + 7 = \dfrac{2}{-5}(x - 3)$ or, $2x + 5y + 29 = 0$

(d) INTERCEPT FORM

We want to find the equation of a line which cuts off given intercepts on both the co-ordinate axes.

Let PQ be a line meeting x-axis in A and y-axis in B. Let $OA = a$, OB = b.

Then the co-ordinates of A and B are $(a,0)$ and (0, b,) respectively.

The equation of the line joining A and B is

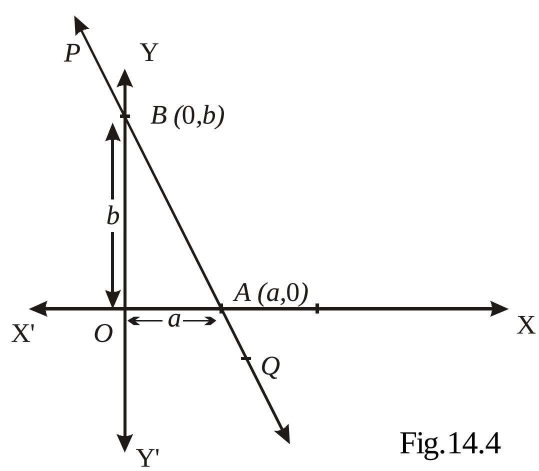


Fig.14.4

$$y - 0 = \frac{b-0}{0-a}(x-a) \text{ or, } \quad y = -\frac{b}{a}(x-a)$$

$$\text{or, } \frac{y}{b} = -\frac{x}{a} + 1 \text{ or, } \frac{x}{a} + \frac{y}{b} = 1$$

This is the required equation of the line having intercepts a and b on the axes.

Example 14.7 Find the equation of a line which cuts off intercepts 5 and –3 on x and y axes respectively.

Solution : The intercepts are 5 and –3 on x and y axes respectively. i.e., $a = 5$, $b = -3$

The required equation of the line is

$$\frac{x}{5} + \frac{y}{-3} = 1, \; 3x - 5y - 15 = 0$$

Example 14.8 Find the equation of a line which passes through the point (3, 4) and makes intercepts on the axes equl in magnitude but opposite in sign.

Solution : Let the x-intercept and y-intercept be a and –a respectively

$\therefore$ The equation of the line is, $\frac{x}{a} + \frac{y}{-a} = 1$, $x - y = a$ $\cdots$ (i)

Since (i) passes through (3, 4)

$\therefore$ $3 - 4 = a$ or $a = -1$

Thus, the required equation of the line is

$x - y = -1$ or $x - y + 1 = 0$

Example 14.9 Determine the equation of the line through the point (– 1,1) and parallel to x - axis.

Solution : Since the line is parallel to x-axis, so its slope ia zero. Therefore from the point slope form of the equation, we get, $y - 1 = 0\,[\,x - (-1)]$, $y - 1 = 0$

which is the required equation of the given line

Example 14.10 Find the intercepts made by the line

$3x - 2y + 12 = 0$ on the coordinate axes

Solution : Equation of the given line is, $3x - 2y = -12$.

Dividing by – 12, we get, $\frac{x}{-4} + \frac{y}{6} = 1$

Comparing it with the standard equation of the line in intercept form, we find $a = -4$ and $b =$

6. Hence the intercepts on the x-axis and y-axis repectively are –4. and 6.

Example 14.11 The segment of a line, intercepted between the coordinate axes is bisected at the point (x_1, y_1). Find the equation of the line

Solution : Let $P(x_1, y_1)$ be the middle point of the segment CD of the line AB intercepted between the axes. Draw $PM \perp OX$

$\therefore \quad OM = x_1$ and $MP = y_1$

$\therefore \quad OC = 2x_1$ and $OD = 2y_1$

Now, from the intercept form of the line

$$\frac{x}{2x_1} + \frac{y}{2y_1} = 1 \text{ or, } \quad \frac{x}{x_1} + \frac{y}{y_1} = 2$$

which is the required equation of the line.

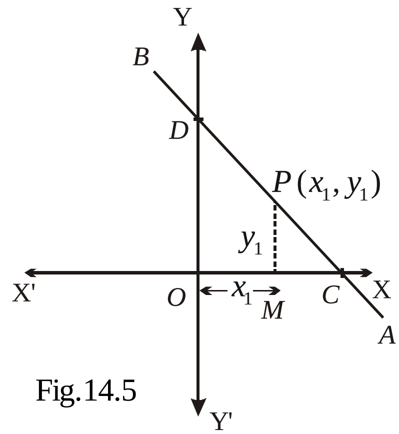


Fig.14.5

(e) PERPENDICULAR FORM (NORMAL FORM)

We now derive the equation of a line when p be the length of perpendicular from the origin on the line and α, the angle which this perpendicular makes with the positive direction of x-a i x is are given.

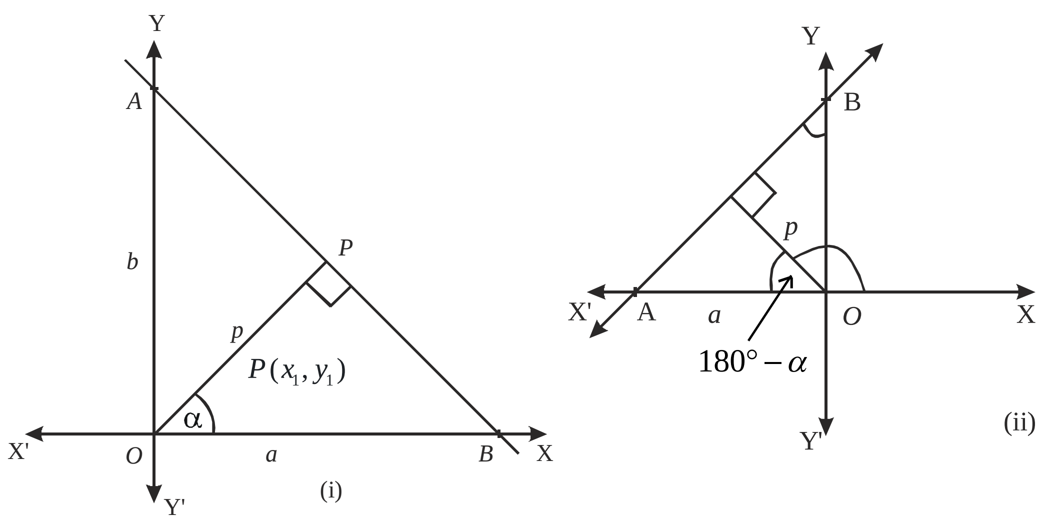


Fig. 14.6

(i) Let AB be the given line cutting off intercepts a and b on x–*axis* and y –*axis* respectively. Let OP be perpendicular from origin O on AB and $\angle POB = \alpha$ (See Fig. 14.6 (i))

$\therefore \quad \dfrac{p}{a} = \cos\alpha \Rightarrow a = p\sec\alpha$ and, $\dfrac{p}{b} = \sin\alpha \Rightarrow b = p\,\text{cosec}\,\alpha$

$\therefore$ The equation of line AB is

$$\frac{x}{p\sec\alpha}+\frac{y}{p\operatorname{cosec}\alpha}=1$$

or, $x\cos\alpha + y\sin\alpha = p$

(ii) $\frac{p}{a} = \cos(180° - \alpha) = -\cos\alpha$ [From Fig. 14.6 (ii)]

$\Rightarrow$ $a = -p\sec\alpha$

similary, b = $p\operatorname{cosec}\alpha$

$\therefore$ The equation of the line AB is $\frac{x}{-a}+\frac{y}{b}=1$ *or* $x\cos\alpha + y\sin\alpha = p$

> ***Note :*** *1. p is the length of perpendicular from the origin on the line and is always taken to be positive.*
>
> *2. α is the angle between positive direction of x-axis and the line perpendicular from the origin to the given line.*

Example 14.12 Determine the equation of the line with $\alpha = 135°$ and perpendicular distance $p = \sqrt{2}$ from the origin.

Solution : From the standard equation of the line in normal form have

$$x\cos 135° + y\sin 135° = \sqrt{2}$$

or, $-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}=\sqrt{2}$ or, $-x + y - 2 = 0$

or, $x - y + 2 = 0$, which is the required equation of the straight line.

Example 14.13 Find the equation of the line whose perpendicular distance from the origin is 6 units and the perpendicular from the origin to line makes an angle of $30°$ with the positive direction of x-axis.

Solution : Here $\alpha = 30°$, $p = 6$ $\therefore$ The equation of the line is, $x\cos 30° + y\sin 30° = 6$

or, $x\left(\frac{\sqrt{3}}{2}\right)+y\left(\frac{1}{2}\right)=6$ or, $\sqrt{3}\,x + y = 12$

CHECK YOUR PROGRESS 14.2

1. (a) Find the equation of a line with slope 2 and y – intercept equal to –2.

   (b) Determine the slope and the intercepts made by the line on the axes whose equation is $4x + 3y = 6$.

2. Find the equation of the line cutting off an interecept $\frac{1}{\sqrt{3}}$ on negative direction of axis of y and inclined at 120° to the positive direction of x-axis.

3. Find the slope and y-intercept of the line whose equation is $3x - 6y = 12$.

4. Determine the equation of the line passing through the point (–7, 4) and having the slope $-\frac{3}{7}$.

5. Determine the equation of the line passing through the point (1, 2) which makes equal angles with the two axes.

6. Find the equation of the line passing through (2, 3) and parallel to the line joining the points (2, –2) and (6, 4).

7. (a) Determine the equation of the line through (3, –4) and (–4, 3).

   (b) Find the equation of the diagonals of the rectangle ABCD whose vertices are A (3, 2), B (11, 8), C (8, 12) and D (0, 6).

8. Find the equation of the medians of a triangle whose vertices are (2, 0), (0, 2) and (4, 6).

9. Find the equation of the line which cuts off intercepts of length 3 units and 2 units on x-$ax$is and y-axis respectively.

10. Find the equation of a line such that the segment between the coordinate axes has its mid point at the point (1, 3)

11. Find the equation of a line which passes through the point (3, –2) and cuts off positive intercepts on x and y axes in the ratio of 4 : 3.

12. Determine the equation of the line whose perpendicular from the origin is of length 2 units and makes an angle of 45° with the positive direction of x-axis.

13. If p is the length of the perpendicular segment from the origin, on the line whose intercept on the axes are a and b, then show that, $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

14.3 GENERAL EQUATION OF FIRST DEGREE

You know that a linear equation in two variables x and y is given by $Ax + By + C = 0$ ...(1)

In order to understand its graphical representation, we need to take the following thres cases.

Case-1: (When both A and B are equal to zero)

In this case C is automaticaly zero and the equation does not exist.

Case-2: (When $A = 0$ and $B \neq 0$)

In this case the equation (1) becomes $By + C = 0$.

or $y = -\frac{C}{B}$ and is satisfied by all points lying on a line which is parallel to x-axis and the y-coordinate of every point on the line is $-\frac{C}{B}$. Hence this is the equation of a straight line. The case where B = 0 and A $\neq$ 0 can be treated similarly.

Case-3: (When A $\neq$ 0 and B $\neq$ 0)

We can solve the equation (1) for y and obtain., $y = -\frac{A}{B}x - \frac{C}{B}$

Clearly, this represents a straight line with slope $-\frac{A}{B}$ and y – intercept equal to $-\frac{C}{B}$.

14.3.1 CONVERSION OF GENERAL EQUATION OF A LINE INTO VARIOUS FORMS

If we are given the general equation of a line, in the form Ax + By + C = 0, we will see how this can be converted into various forms studied before.

14.3.2 CONVERSION INTO SLOPE-INTERCEPT FORM

We are given a first degree equation in x and y as $Ax + By + C = O$

Are you able to find slope and y-intercept ?

Yes, indeed, if we are able to put the general equation in slope-intercept form. For this purpose, let us re-arrange the given equation as.

$$Ax + By + C = 0 \text{ as, } By = -Ax - C$$

or $$y = -\frac{A}{B}x - \frac{C}{B} \text{ (Provided } B \neq 0)$$

which is the required form. Hence, the slope = $-\frac{A}{B}$, y – intercept = $-\frac{C}{B}$.

Example 14.14 Reduce the equation $x + 7y - 4 = 0$ to the slope – intercept form.

Here find its slope and y intercept.

Solution : The given equation is, $x + 7y - 4 = 0$

or $$7y = -x + 4, \text{ or } y = -\frac{1}{7}x + \frac{4}{7}$$

Here slope $= -1/7$ and y interecept $= 4/7$

14.3.3 CONVERSION INTO INTERCEPT FORM

Suppose the given first degree equation in x and y is $Ax + By + C = 0.$ ...(i)

In order to convert (i) in intercept form, we re arrange it as $Ax + By = -C$ or $\frac{Ax}{-C} + \frac{By}{-C} = 1$

or $\frac{x}{(-\frac{C}{A})} + \frac{y}{(-\frac{C}{B})} = 1$ (Provided $A \neq 0$ and $B \neq 0$)

which is the requied converted form. It may be noted that intercept on x – axis = $\frac{-C}{A}$ and

intercept on y – axis = $\frac{-C}{B}$

Example 14.15 Reduce $3x + 5y = 7$ into the intercept form and find its intercepts on the axes.

Solution : The given equation is, $3x + 5y = 7$

or, $\frac{3}{7}x + \frac{5}{7}y = 1$ or, $\frac{x}{\frac{7}{3}} + \frac{y}{\frac{7}{5}} = 1$

$\therefore$ The x– intercept = $\frac{7}{3}$ and, y – intercept = $\frac{7}{5}$

14.3.4 CONVERSION INTO PERPENDICULAR FORM

Let the general first degree equation in x and y be, $Ax + By + C = 0$ ... (i)

We will convert this general equation in perpendicular form. For this purpose let us re-write the given equation (i) as $Ax + By = -C$

Multiplying both sides of the above equation by λ, we have

$$\lambda Ax + \lambda By = -\lambda C \qquad \text{... (ii)}$$

Let us choose λ such that $(\lambda A)^2 + (\lambda B)^2 = 1$

or $\lambda = \frac{1}{\sqrt{(A^2 + B^2)}}$ (Taking positive sign)

Substituting this value of λ in (ii), we have

$$\frac{Ax}{\sqrt{(A^2 + B^2)}} + \frac{By}{\sqrt{(A^2 + B^2)}} = -\frac{C}{\sqrt{(A^2 + B^2)}} \qquad \text{... (iii)}$$

This is required conversion of (i) in perpendicular form. Two cases arise according as C is negative or positive.

(i) If $C < 0$, the equation (ii) is the required form.

(ii) If $C > 0$, the R. H. S. of the equation of (iii) is negative.

$\therefore$ We shall multiply both sides of the equation of (iii) by –1.

$\therefore$ The required form will be $-\dfrac{Ax}{\sqrt{(A^2+B^2)}} - \dfrac{By}{\sqrt{(A^2+B^2)}} = \dfrac{C}{\sqrt{(A^2+B^2)}}$

Thus, length of perpendicular from the origin = $\dfrac{|C|}{\sqrt{(A^2+B^2)}}$

Inclination of the perpendicular with the positwe direction of x-ax is

is given by $\cos\theta = \mp\dfrac{A}{\sqrt{A^2+B^2}}$

or $\quad \sin\theta = \left(\mp\dfrac{B}{\sqrt{\left(A^2B^2\right)}}\right)$

where the upper sign is taken for $C > 0$ and the lower sign for $C < 0$. If $C = 0$, the line passes through the origin and there is no perpendicular from the origin on the line.

With the help of the above three cases, we are able to say that

> ***"The general equation of first degree in x and y always represents a straight line provided A and B are not both zero simultaneously."***

Is the converse of the above statement true? ***The converse of the above statement is that every straight line can be expressed as a general equation of first degree in x and y.***

In this lesson we have studied about the various forms of equation of straight line. For example, let us take some of them as $y = mx + c$, $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $x\cos\alpha + y\sin\alpha = p$. Obviously, all are linear equations in x and y. We can re-arrange them as $y - mx - c = 0$, $bx + ay - ab = 0$ and $x\cos\alpha + y\sin\alpha - p = 0$ respectively. Clearly, these equations are nothing but a different arrangement of general equation of first degree in x and y. Thus, we have established that

> ***"Every straight line can be expressed as a general equation of first degree in x and y".***

Example 14.16 Reduce the equation $x + \sqrt{3}\,y + 7 = 0$ into perpendicular form.

Solution : The equation of given line is $x + \sqrt{3}y + 7 = 0$ ... (i)

Comparing (i) with general equation of straight line, we have, $A = 1$ and $B = \sqrt{3}$

$$\therefore \quad \sqrt{A^2 + B^2} = 2$$

Dividing equation (i) by 2, we have, $\frac{x}{2} + \frac{\sqrt{3}}{2} y + \frac{7}{2} = 0$

or $\quad \left(-\frac{1}{2}\right)x + \left(-\frac{\sqrt{3}}{2}\right)y - \frac{7}{2} = 0$ or $\quad x \cos \frac{4\pi}{3} + y \sin \frac{4\pi}{3} = \frac{7}{2}$

($\cos\theta$ and $\sin\theta$ being both negative in the third quadrant, value of θ will lie in the third quadrant).

This is the representation of the given line in perpendicular form.

Example 14.17 Find the perpendicular distance from the origin on the line $\sqrt{3}\, x - y + 2 = 0$. Also, find the inclination of the perpendicular from the origin.

Solution : The given equation is $\sqrt{3}\, x - y + 2 = 0$

Dividing both sides by $\sqrt{(\sqrt{3})^2 + (-1)^2}$ or 2, we have

$$\frac{\sqrt{3}}{2} x - \frac{1}{2} y + 1 = 0 \text{ or, } \quad \frac{\sqrt{3}}{2} x - \frac{1}{2} y = -1$$

Multiplying both sides by –1, we have, $-\frac{\sqrt{3}}{2} x + \frac{1}{2} y = 1$

or, $\quad x \cos \frac{5\pi}{6} + y \sin \frac{5\pi}{6} = 1$ ($\cos\theta$ is –ve in second quadrant and $\sin\theta$ is +ve in second quadrant, so value of θ lies in the second quadrant).

Thus, inclination of the perpendicular from the origin is 150° and its length is equal to 1.

Example 14.18 Find the equation of a line which passess through the point (3,1) and bisects the portion of the line $3x + 4y = 12$ intercepted between coordinate axes.

Solution : First we find the intercepts on coordinate axes cut off by the line whose equation is

$$3x + 4y = 12 \text{ or } \frac{3x}{12} + \frac{4y}{12} = 1 \text{ or } \qquad \frac{x}{4} + \frac{y}{3} = 1$$

Hence, intercepts on x-axis and y-axis are 4 and 3 respectively.

Thus, the coordinates of the points where the line meets the coordinate axes are A (4, 0) and B (0, 3).

$\therefore$ Mid – point of AB is $\left(2, \frac{3}{2}\right)$ is

Hence the equation of the line through (3,1) and

is and $\left(2, \frac{3}{2}\right)$ *is*, $y - 1 = \dfrac{\frac{3}{2} - 1}{2 - 3}(x - 3)$

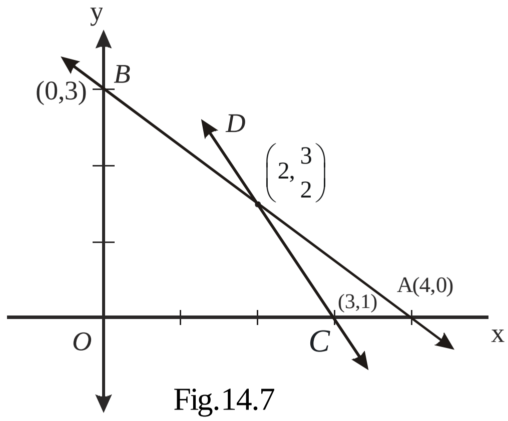


Fig.14.7

or $y - 1 = -\frac{1}{2}(x - 3)$

or $2(y - 1) + (x - 3) = 0$

or $2y - 2 + x - 3 = 0$, or $x + 2y - 5 = 0$

Example 14.19 Prove that the line through (8, 7) and (6, 9) cuts off equal intercepts on coordinate axes.

Solution : The equation of the line passing through (8, 7) and (6, 9) is, $y - 7 = \dfrac{9 - 7}{6 - 8}(x - 8)$

or $y - 7 = -(x - 8)$, or $x + y = 15$

or $\dfrac{x}{15} + \dfrac{y}{15} = 1$

Hence, intercepts on both axes are 15 each.

Example 14.20 Find the ratio in which the line joining (–5, 1) and (1, –3) divides the join of (3, 4) and (7, 8).

Solution : The equation of the line joining C (–5, 1) and D (1, –3) is

$$y - 1 = \frac{-3 - 1}{1 + 5}(x + 5), \text{ or}$$

$$y - 1 = -\frac{4}{6}(x + 5)$$

or $3y - 3 = -2x - 10$, or $2x + 3y + 7 = 0$ ... (i)

Let line (i) divide the join of A (3, 4) and B (7, 8) at the point P.

If the required ratio is $\lambda : 1$ in which line (i) divides the join of A (3, 4) and B (7, 8), then the coordinates of P are

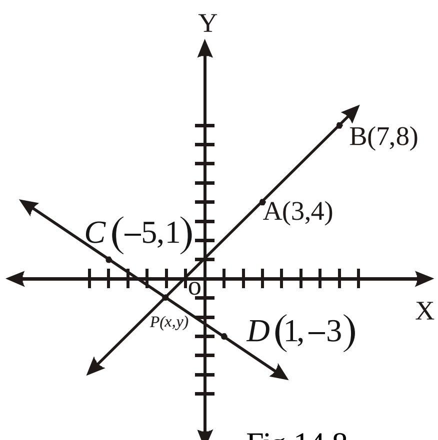


Fig.14.8

$$\left(\frac{7\lambda+3}{\lambda+1},\frac{8\lambda+4}{\lambda+1}\right)$$

Since P lies on the line (i), we have

$$2\left(\frac{7\lambda+3}{\lambda+1}\right)+3\left(\frac{8\lambda+4}{\lambda+1}\right)+7=0$$

$$\Rightarrow \quad 14\lambda + 6 + 24\lambda + 12 + 7\lambda + 7 = 0, \Rightarrow \quad 45\lambda + 25 = 0 \Rightarrow \lambda = -\frac{5}{9}$$

Hence, the line joining (–5, 1) and (1, –3) divides the join of (3, 4) and (7, 8) externally in the ratio 5 : 9.

CHECK YOUR PROGRESS 14.3

1. Under what condition, the general equation $Ax + By + C = 0$ of first degree in x and y represents a line?
2. Reduce the equation $2x + 5y + 3 = 0$ to the slope intercept form.
3. Find the x and y intercepts for the following lines :

   (a) $y = mx + c$ (b) $3y = 3x + 8$ (c) $3x - 2y + 12 = 0$
4. Find the length of the line segment AB intercepted by the straight line $3x - 2y + 12 = 0$ between the two axes.
5. Reduce the equation $x \cos \alpha + y \sin \alpha = p$ to the intercept form of the equation and also find the intercepts on the axes.
6. Reduce the following equations into normal form.

   (a) $3x - 4y + 10 = 0$ (b) $3x - 4y = 0$
7. Which of the lines $2x - y + 3 = 0$ and $x - 4y - 7 = 0$ is nearer from the origin?

14.4 DISTANCE OF A GIVEN POINT FROM A GIVEN LINE

In this section, we shall discuss the concept of finding the distance of a given point from a given line or lines.

Let $P(x_1, y_1)$ be the given point and l be the line $Ax + By + C = 0$.

Let the line l intersect x axis and y axis R and Q respectively.

Draw $PM \perp l$ and let $PM = d$.

Let the coordinates of M be (x_2, y_2)

$$d = \sqrt{\left\{(x_1 - x_2)^2 + (y_1 - y_2)^2\right\}} \qquad ...(i)$$

$\because$ M lies on l, $\therefore$ $Ax_2 + By_2 + C = 0$ or $C = -(Ax_2 + By_2)$ ...(ii)

The coordinates of R and Q are $\left(-\frac{C}{A}, 0\right)$ and $\left(0, -\frac{C}{B}\right)$ respectively.

$$\text{The slope of } QR = \frac{0 + \frac{C}{B}}{-\frac{C}{A} - 0} = -\frac{A}{B} \text{ and,}$$

$$\text{the slope of } PM = \frac{y_2 - y_1}{x_2 - x_1}$$

As $PM \perp QR \Rightarrow \frac{y_2 - y_1}{x_2 - x_1} \times \left(-\frac{A}{B}\right) = -1.$ or $\frac{y_1 - y_2}{x_1 - x_2} = \frac{B}{A}$ ...(iii)

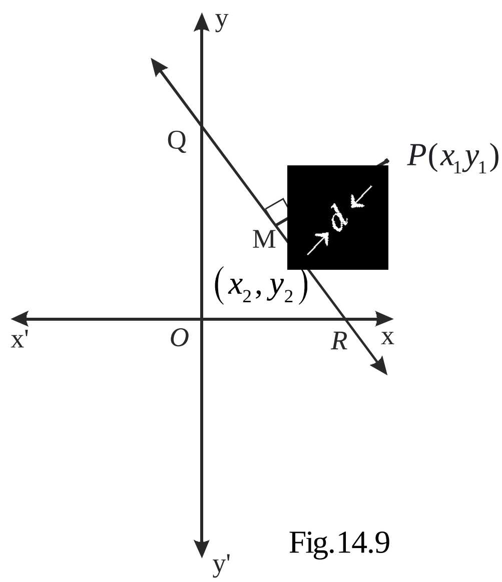


Fig.14.9

From (iii) $\frac{x_1 - x_2}{A} = \frac{y_1 - y_2}{B} = \frac{\sqrt{\left\{(x_1 - x_2) + (y_1 - y_2)^2\right\}}}{\sqrt{(A^2 + B^2)}}$ ...(iv)

(Using properties of Ratio and Proportion)

Also $\frac{x_1 - x_2}{A} = \frac{y_1 - y_2}{B} = \frac{A(x_1 - x_2) + B(y_1 - y_2)}{A^2 + B^2}$ ...(v)

From (iv) and (v), we get

$$\frac{\sqrt{\left\{\left(x_1-x_2\right)^2\left(y_1-y_2\right)^2\right\}}}{\sqrt{\left(A^2+B^2\right)}}=\frac{A\left(x_1-x_2\right)+B\left(y_1-y_2\right)}{A^2+B^2}$$

or $$\frac{d}{\sqrt{A^2+B^2}}+\frac{Ax_1+By_1-(Ax_2+By_2}{A^2+B^2}$$ [Using (i)]

or $$\frac{Ax_1+By_1+C}{\sqrt{\left(A^2+B^2\right)}}$$ [Using (ii)]

Since the distance is always positive, we can write

$$d=\left|\frac{Ax_1+By_1+C}{\sqrt{\left(A^2+B^2\right)}}\right|$$

***Note :** The perpendicular distance of the origin (0, 0) from Ax + By + C = 0 is*

$$\frac{A(0)+B(0)+C}{\sqrt{\left(A^2+b^2\right)}}=\frac{C}{\sqrt{\left(A^2+B^2\right)}}$$

Example 14.21 Find the points on the x-axis whose perpendicular distance from the straight line $\frac{x}{a}+\frac{y}{b}=1$ is a.

Solution : Let $(x_1, 0)$ be any point on x-axis.

Equation of the given line is $bx + ay - ab = 0$. The perpendicular distance of the point $(x_1, 0)$ from the given line is, $a=\pm\frac{bx_1+a.0-ab}{\sqrt{\left(a^2+b^2\right)}}$ $\therefore$ $x_1=\frac{a}{b}\left\{b\pm\sqrt{\left(a^2+b^2\right)}\right\}$

Thus, the point on x-axis is $x_1=\left(\frac{a}{b}b\pm\sqrt{\left(a^2+b^2\right)},0\right)$

CHECK YOUR PROGRESS 14.4

1. Find the perpendicular distance of the point (2, 3) from $3x + 2y + 4 = 0$.
2. Find the points on the axis of y whose perpendicular distance from the straight line

$\frac{x}{a} + \frac{y}{b} = 1$ is b.

3. Find the points on the axis of y whose perpendicular distance from the straight line 4x + 3y = 12 is 4.

4. Find the perpendicular distance of the origin from 3x + 7y + 14 = 0

14.6 EQUATION OF PARALLEL (OR PERPENDICULAR) LINES

Till now, we have developed methods to find out whether the given lines are prallel or perpendicular. In this section, we shall try to find, the equation of a line which is parallel or perpendicular to a given line.

14.6.1 EQUATION OF A STRAIGHT LINE PARALLEL TO THE GIVEN LINE

$$\boldsymbol{Ax + By + c = 0}$$

Let $A_1x + B_1y + C_1 = 0$ ...(i)

be any line parallel to the given line, $Ax + By + C = 0$

The condition for parallelism of (i) and (ii) is ...(ii)

$$\frac{A_1}{A} = \frac{B_1}{B} = K_1 \quad \text{(say)} \Rightarrow \qquad A_1 = AK_1,\ B_1 = \text{B}K_1$$

with these values of A_1 and B_1, (i) gives

$$AK_1x + BK_1\,y + C_1 = 0 \quad \text{or} \qquad Ax + By + \frac{C_1}{K_1} = 0$$

or $\quad Ax + By + K = 0,\qquad$ where $K = \frac{C_1}{K_1}$ ... (iii)

This is a line parallel to the given line. From equations (ii) and (iii) we observe that

(i) coefficients of *x* and *y* are same

(ii) constants are different, and are to evaluated from given conditions.

Example 14.22 Find equation of the straight line, which passes through the point (1, 2) and which is parallel to the straight line $2x + 3y + 6 = 0$.

Solution : Equation of any straight line parallel to the given equation can be written if we put

(i) the coefficients of *x* and *y* as same as in the given equation.

(ii) constant to be different from the given equation, which is to be evaluated under given condition.

Thus, the required equation of the line will be, $2x + 3y + K = 0$ for some constant K

Since it passes through the point (1, 2) hence, $2 \times 1 + 3 \times 2 + K = 0$

or $\quad K = -8$

$\therefore$ $\quad$ Required equation of the line is $2x + 3y = 8$.

14.7 STRAIGHT LINE PERPENDICULAR TO THE GIVEN LINE

$\boldsymbol{Ax + By + C = 0}$

Let $\quad A_1 x + B_1 y + C_1 = 0$ $\quad$ ... (i), be any line perpendicular to the given line

$Ax + By + C = 0$

Condition for perpendicularity of lines (i) and (ii) is $\quad$ ... (ii)

$$AA_1 + BB_1 = 0 \;\Rightarrow\; \frac{A_1}{B} = -\frac{B_1}{A} = K_1 \quad \text{(say)}$$

$\Rightarrow$ $\quad A_1 = BK_1$ and $B_1 = -AK_1$

With these values of A_1 and $B_{1,}$ (i) gives, $Bx - Ay + \frac{C_1}{K_1} = 0 = 0$

or $\quad Bx - Ay + K = 0$ where $K = \frac{C_1}{K_1}$ $\quad$ ... (iii)

Hence, the line (iii) is perpendicular to the given line (ii)

We observe that in order to get a line perpendicular to the given line we have to follow the following procedure : (i) Interchange the coefficients of x and y

(ii) Change the sign of one of them.

(iii) Change the Constant term to a new constant K (say), and evaluate it from given condition.

Example 14.23 Find the equation of the line which passes through the point (1, 2) and is perpendicular to the line $2x + 3y + 6 = 0$.

Solution : Following the procedure given above, we get the equation of line perpendicular to the given equation as $3x - 2y + K = 0$ $\quad$ ...(i)

(i) passes through the point (1, 2), hence

$3 \times 1 - 2 \times 2 + K = 0$ or $K = 1$

$\therefore$ $\quad$ Required equation of the straight line is $3x - 2y + 1 = 0$.

Example 14.24 Find the equation of the line which passes through the point (x_2, y_2) and is perpendicular to the straight line $y y_1 = 2a(x + x_1)$.

Solution : The given straight line is $yy_1 - 2ax - 2ax_1 = 0$ ...(i)

Any straight line perpendicular to (i) is $2ay + xy_1 + C = 0$

This passes through the point (x_2, y_2) $\therefore$ $2ay_2 + x_2 y_1 + C = 0$

$\Rightarrow$ $C = -2ay_2 - x_2y_1$

$\therefore$ Required equation of the spraight line is, $2a(y - y_2) + y_1(x - x_2) = 0$

CHECK YOUR PROGRESS 14.5

1. Find the equation of the straight line which passes through the point (0, –2) and is parallel to the straight line 3x + y = 2.
2. Find the equation of the straight line which passes through the point (–1, 0) and is parallel to the straight line y = 2x + 3.
3. Find the equation of the straight line which passes through the point (0, –3) and is perpendicular to the straight line x + y + 1 = 0.
4. Find the equation of the line which passes through the point (0, 0) and is perpendicular to the straight line x + y = 3.
5. Find the equation of the straight line which passes through the point (2, –3) and is perpendicular to the given straight line 2a (x + 2) + 3y = 0.
6. Find the equation of the line which has x - intercept –8 and is perpendicular to the line 3x + 4y – 17 = 0.
7. Find the equation of the line whose y-intercept is 2 and is parallel to the line 2x – 3y + 7 = 0.
8. Prove that the equation of a straight line passing throngh $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the sine $x\sec\theta + y\,\text{cosec}\,\theta = a$ is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

14.8 EQUATION OF FAMILY OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO LINES :

Let $l_1 : a_1x + b_1y + c_1 = 0$ ...(i)

and $l_2 : a_2x + b_2y + c_2 = 0$, be two intersecting lines.

Let P(*h*, *k*) be the point of intersection of l_1 and l_2, then

$$a_1h + b_1k + c_1 = 0 \qquad ...(iii)$$

and
$$a_2h + b_2k + c_2 = 0 \qquad ...(iv)$$

Now consider the equation

$$(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2) = 0 \qquad ...(v)$$

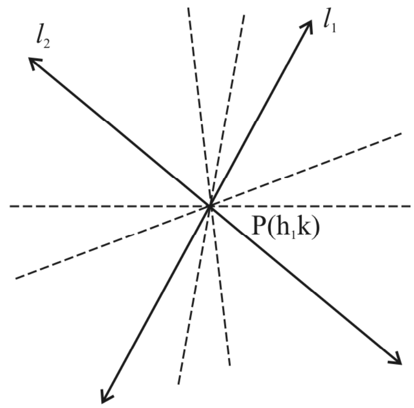


Fig.14.10

It is a first degree equation in x and y. So it will represent different lines for different values of λ. If we replace x by h and y by k we get

$$(a_1h+b_1k+c_1)+\lambda(a_2h+b_2k+c_2) = 0 \qquad ...(vi)$$

using (iii) and (iv) in (vi) we get

$$0 + \lambda 0 = 0 \text{ i.e. } 0 = 0 \text{ which is true.}$$

So equation (v) represents a family of lines passing throught the point (h, k) i.e. the point of intersection of the given lines l_1 and l_2.

- A particular member of the family is obtained by giving a particular value to λ. This value of λ can be obtained from other given conditions.

Example 14.25 Find the equation of the line passing through the point of intersection of the lines $x + y + 1 = 0$ and $2x - y + 7 = 0$ and containing the point (1, 2).

Solution : Equation of family of lines passing through the intersection of given lines is $(x + y + 1) + \lambda\ (2x - y + 7) = 0$

This line will contain the point (1, 2) if

$$(1 + 2 + 1) + \lambda\ (2 \times 1 - 1 \times 2 + 7) = 0$$

i.e. $\quad 4 + 7\lambda = 0 \Rightarrow \lambda = -\frac{4}{7}.$

Therefore the equation of required line is, $(x+y+1)-\frac{4}{7}(2x-y+7) = 0$

i.e. $\quad 7(x+y+1)-4(2x-y+7) = 0$ i.e. $-x + 11y - 21 = 0$

or $\quad x - 11y + 21 = 0$

Example 14.26 Find the equation of the line passing through the intersection of lines $3x + y - 9 = 0$ and $4x + 3y - 7 = 0$ and parallel to y-axis.

Solution : Equation of family of lines passing through the intersection of given lines is

$$(3x + y - 9) + \lambda(4x + 3y - 7) = 0, \text{ i.e. } (3 + 4\lambda)x + (1 + 3\lambda)y - (9 + 7\lambda) = 0 \quad \text{...(i)}$$

We know that if a line is parallel to y-axis then co-efficient of y in its equation must be zero.

$\therefore\ 1 + 3\lambda = 0 \Rightarrow \lambda = -1/3.$

Hence, equation of the required line is, $\left\{3 + 4\left(-\frac{1}{3}\right)\right\}x + 0y - \left\{9 + 7\left(\frac{-1}{3}\right)\right\} = 0$

i.e. $x = 4$

CHECK YOUR PROGRESS 14.6

1. Find the equation of the line passing through the intersection of the lines $x + y = 5$ and $2x - y - 7 = 0$ and parallel to x-axis.

2. Find the equation of the line passing though the intersection of the lines $x + y + 1 = 0$ and $x - y - 1 = 0$ and containing the point $(-3, 1)$

LET US SUM UP

- The equation of a line parallel to y-axis is $x = a$ and parallel to x-axis is $y = b$.
- The equation of the line which cuts off intercept c on y-axis and having slope m is $y = mx + c$
- The equation of the line passing through $A(x_1, y_1)$ and having the slope m is $y - y_1 = m(x - x_1)$
- The equation of the line passing through two points $A(x, y_1)$ and $B(x_2, y_2)$ is

  $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

- The equation of the line which cuts off intercepts a and b on x – axis and y – axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$
- The equation of the line in normal or perpendicular form is $x \cos\alpha + y \sin\alpha = p$

  where p is the length of perpendicular from the origin to the line and α is the angle which this perpendicular makes with the positive direction of the x-axis.

- The general equation of first degree in x and y always represents a straight line provided A and B are not both zero simultaneously.
- From general equation $Ax + By + C = 0$ we can evaluate the following :

  (i) Slope of the line $= -\dfrac{A}{B}$ (ii) x-intercept $= -\dfrac{C}{A}$ (iii) y-intercept $= -\dfrac{C}{B}$

  (iv) Length of perpendicular from the origin to the line $= \dfrac{|C|}{\sqrt{(A^2 + B^2)}}$

  (v) Inclination of the perpendicular from the origin is given by

  $$\cos\alpha = \frac{\mp A}{\sqrt{(A^2+B^2)}}\ ;\ \sin\alpha = \frac{\mp B}{\sqrt{(A^2+B^2)}}$$

  where the upper sign is taken for $C > 0$ and the lower sign for $C < 0$; but if $C = 0$ then either only the upper sign or only the lower sign are taken.

- Distance of a given point (x_1, y_1) from a given line $Ax + By + C = 0$ is $d = \left|\dfrac{Ax_1 + By_1 + C}{\sqrt{(A^2 + B^2)}}\right|$
- Equation of a line parallel to the line $Ax + By + C = 0$ is $Ax + By + k = 0$
- Equation of a line perpendicular to the line $Ax + By + C = 0$ is $Bx - Ay + k = 0$
- Equation of a line passing through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1) + k\,(a_2x + b_2y + c_2) = 0$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Straight_lines
http://mathworld.wolfram.com/Straight_lines

TERMINAL EXERCISE

1. Find the equation of the straight line whose y-intercept is –3 and which is:

   (a) parallel to the line joining the points (–2,3) and (4, –5).

   (b) perpendicular to the line joining the points (0, –5) and (–1, 3).

2. Find the equation of the line passing through the point (4, –5) and

   (a) parallel to the line joining the points (3, 7) and (–2, 4).

   (b) perpendicular to the line joining the points (–1, 2) and (4, 6).

3. Show that the points $(a, 0)$, $(0, b)$ and $(3a, -2b)$ are collinear. Also find the equation of the line containing them.

4. A (1, 4), B (2, –3) and C (–1, –2) are the vertices of triangle ABC. Find

   (a) the equation of the median through A.

   (b) the equation of the altitude througle A.

   (c) the right bisector of the side BC.

5. A straight line is drawn through point A (2, 1) making an angle of $\frac{\pi}{6}$ with the positive direction of x-axis. Find the equation of the line.

6. A straight line passes through the point (2, 3) and is parallel to the line $2x + 3y + 7 = 0$. Find its equation.

7. Find the equation of the line having a and b as x-intercept and y-intercepts respectively.

8. Find the angle between the lines $y = (2 - \sqrt{3})\,x + 5$ and $y = (2 + \sqrt{3})\,x - d$.

9. Find the angle between the lines $2x + 3y = 4$ and $3x - 2y = 7$

10. Find the length of the per pendicular drawn from the point (3, 4) on the straight line $12\,(x + 6) = 5\,(y - 2)$.

11. Find the length of the perpendicula from (0, 1) on $3x + 4\,y + 5 = 0$.

12. Find the distance between the lines $2x + 3y = 4$ and $4x + 6y = 20$

13. Find the length of the perpendicular drawn from the point (–3, – 4) on the line $4x - 3y = 7$.

14. Show that the product of the perpendiculars drawn from the points on the straight line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

15. Prove that the equation of the straight line which passes through the point $(a\cos^3\theta, b\sin^3\theta)$ and is perpendicular to $x\sec\theta + y\,\text{cosec}\,\theta = a$ is $x\cos\theta - y\,\text{cosec}\,\theta = a\cos 2\theta$

16. Find the equation of a straight line passing through the point of intersection of the lines $3x + y - 9 = 0$ and $4x + 4y - 7 = 0$ and perpendicular to the line $5x - 4y + 1 = 0$.

17. Find the equation of a straight line passing thought the point of intersection of the lines $2x + 3y - 2 = 0$ and $x - 2y + 1 = 0$ and having x-intercept equal to 3.

18. Find the equation of a line through the point of intersection of the lines $3x + 4y - 7 = 0$ and $x - y + 2 = 0$ and with slope 5.

19. Find the equation of a line through the point of intersection of the lines $5x - 3y = 1$ and $2x + 3y = 23$ and perpendicular to the line $5x - 3y - 1 = 0$.

20. Find the equation of a line through the intersection of the lines $3x - 4y + 1 = 0$ and $5x + y - 1 = 0$ which cut off equal intercepts on the axes.

ANSWERS

CHECK YOUR PROGRESS 14.1

1. (a) $y = -4$ (b) $x = -3$
2. $x = 5$
3. $y + 7 = 0$

CHECK YOUR PROGRESS 14.2

1. (a) $y = 2x - 2$ (b) Slope $= \frac{-4}{3}$, y – intercept = 2
2. $\sqrt{3}y = -3x - 1$
3. Slope $= \frac{1}{2}$, y-intercept $= -2$
4. $3x + 7y = 7$
5. $y = x + 1$; $x + y - 3 = 0$
6. $3x - 2y = 0$
7. (a) $x + y = -1$ (b) Equation of the diagonal AC $= 2x - y - 4 = 0$
   Equation of the diagonal BD $= 2x - 11y + 66 = 0$
8. $x - 2 = 0$, $x - 3y + 6 = 9$ and $5x - 3y - 2 = 0$
9. $2x + 3y = 6$
10. $3x + y = 6$
11. $3x + 4y = 1$
12. $x + y = 2\sqrt{2}$

CHECK YOUR PROGRESS 14.3

1. A and B are not both simltaneously zero
2. $y = \frac{-2}{5}x - \frac{3}{5}$
3. (a) x–intercept $= \frac{-c}{m}$; y-intercept = c

(b) x-intercept $= \frac{-8}{3}$; y-intercept $= \frac{8}{3}$

(c) x-intercept $= -4$; y-intercept $= 6$

4. $2\sqrt{13}$ units

5. $$\frac{x}{p\sec\alpha} + \frac{x}{p\,\mathrm{cosec}\,\alpha} = 1$$

6. (a) $\frac{-3}{5}x + \frac{4}{5}y - 2 = 0$ (b) $\frac{-3}{5}x + \frac{4}{5}y = 0$

7. The first line is nearer from the origin.

CHECK YOUR PROGRESS 14.4

1. $d = \frac{16}{\sqrt{13}}$

2. $\left(0, \frac{b}{a}(a \pm \sqrt{a^2 + b^2})\right)$

3. $\left(0, \frac{32}{3}\right)$

4. $\frac{14}{\sqrt{58}}$

CHECK YOUR PROGRESS 14.5

1. $3x + y + 2 = 0$
2. $y = 2x + 2$
3. $x - y = 3$
4. $y = x$
5. $3x - 2ay = 6(a - 1)$
6. $4x - 3y + 32 = 0$
7. $2x - 3y + 6 = 0$

CHECK YOUR PROGRESS 14.6

1. $y=1$
2. $2x+3y+3=0$

TERMINAL EXERCISE

1. (a) $4x + 3y + 9 = 0$ (b) $x - 8y - 24 = 0$

2. (a) $3x - 5y - 37 = 0$ (b) $5x - 8y - 60 = 0$

4. (a) $13x - y - 9 = 0$ (b) $3x - y + 1 = 0$

   (c) $3x - y - 4 = 0$

5. $x - \sqrt{3y} = 2 - \sqrt{3}$

6. $2x + 3y + 13 = 0$

7. $bx + ay = ab$

8. $\frac{\pi}{2}$

9. $\frac{\pi}{2}$

10. $\frac{98}{13}$

11. $\frac{9}{5}$

12. $\frac{6}{\sqrt{13}}$

13. $\frac{7}{5}$

16. $32x + 40y - 41 = 0$

17. $x + 5y - 3 = 0$

18. $35x - 7y + 18 = 0$

19. $63x + 105y - 781 = 0$

20. $23x + 23y - 11 = 0$

15

CIRCLES

Notice the path in which the tip of the hand of a watch moves. (see Fig. 15.1)

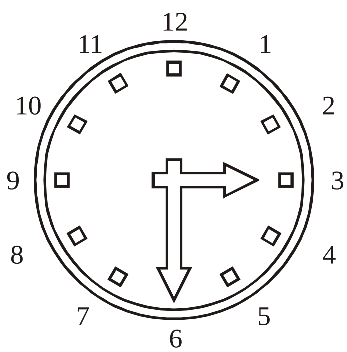


Fig.15.1

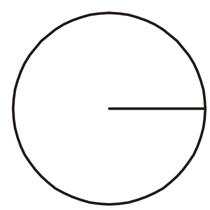


Fig.15.2

Again, notice the curve traced out when a nail is fixed at a point and a thread of certain length is tied to it in such a way that it can rotate about it, and on the other end of the thread a pencil is tied. Then move the pencil around the fixed nail keeping the thread in a stretched position (See Fig 15.2)

Certainly, the curves traced out in the above examples are of the same shape and this type of curve is known as a ***circle***.

The distance between the tip of the pencil and the point, where the nail is fixed is known as the ***radius*** of the circle.

We shall discuss about the curve traced out in the above examples in more details.

OBJECTIVES

After studying this lesson, you will be able to :

- derive and find the equation of a circle with a given centre and radius;
- state the conditions under which the general equation of second degree in two variables represents a circle;
- derive and find the centre and radius of a circle whose equation is given in general form;
- find the equation of a circle passing through :

  (*i*) three non-collinear points (*ii*) two given points and touching any of the axes;

MODULE-IV
Co-ordinate Geometry

Notes

EXPECTED BACKGROUND KNOWLEDGE

- Terms and concepts connected with circle.
- Distance between two points with given coordinates.
- Equation of a straight line in different forms.

15.1 DEFINITION OF THE CIRCLE

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point in the same plane remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

15.2 EQUATION OF A CIRCLE

Can we find a mathematical expression for a given circle?

Let us try to find the equation of a circle under various given conditions.

15.2.1 WHEN COORDINATES OF THE CENTRE AND RADIUS ARE GIVEN

Let C be the centre and a be they radius of the circle. Coordinates of the centre are given to be (h, k), say.

Take any point $P(x, y)$ on the circle and draw perpendiculars CM and PN on OX. Again, draw CL perpendicular to PN.

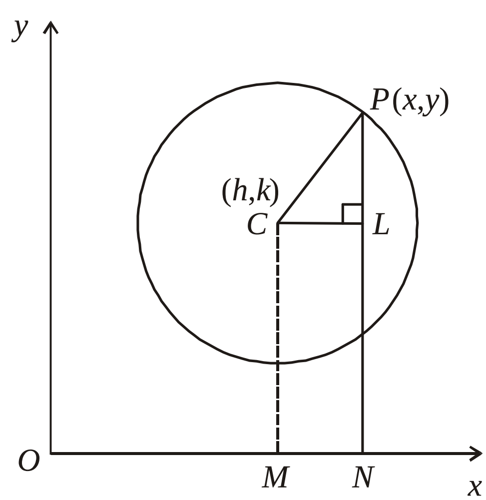


Fig.15.3

We have

$$CL = MN = ON - OM = x - h$$

and $\quad PL = PN - LN = PN - CM = y - k$

In the right angled triangle CLP, $CL^2 + PL^2 = CP^2$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = a^2 \qquad \ldots(1)$$

This is the required equation of the circle under given conditions. This form of the circle is known as ***standard form*** of the circle.

Conversely, if (x, y) is any point in the plane satisfying (1), then it is at a distance 'a' from (h, k). So it is on the circle.

What happens when the

(*i*) circle passes through the origin?

(*ii*) circle does not pass through origin and the centre lies on the *x*-axis?

(*iii*) circle passes through origin and the *x*-axis is a diameter?

(*iv*) centre of the circle is origin?

(*v*) circle touches the *x*-axis?

(*vi*) circle touches the *y*-axis?

(*vii*) circle touches both the axes?

We shall try to find the answer of the above questions one by one.

(*i*) In this case, since (0, 0) satisfies (1), we get

$$h^2 + k^2 = a^2$$

Hence the equation (1) reduces to $x^2 + y^2 - 2hx - 2ky = 0$ ...(2)

(*ii*) In this case $k = 0$

Hence the equation (1) reduces to $(x - h)^2 + y^2 = a^2$ ...(3)

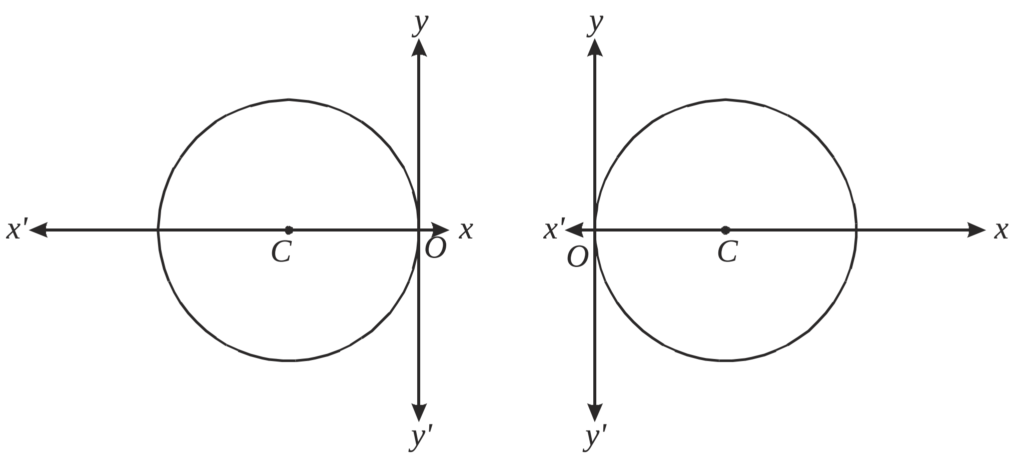


Fig.15.4

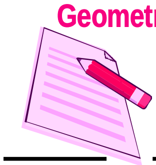
(*iii*) In this case $k = 0$ and $h = \pm a$ (see Fig. 15.4)

Hence the equation (1) reduces to $x^2 + y^2 \pm 2ax = 0$ ...(4)

(*iv*) In this case $h = 0 = k$, Hence the equation (1) reduces to $x^2 + y^2 = a^2$ ...(5)

(*v*) In this case $k = a$ (see Fig. 15.5)

Hence the equation (1) reduces to $x^2 + y^2 - 2hx - 2ay + h^2 = 0$ ...(6)



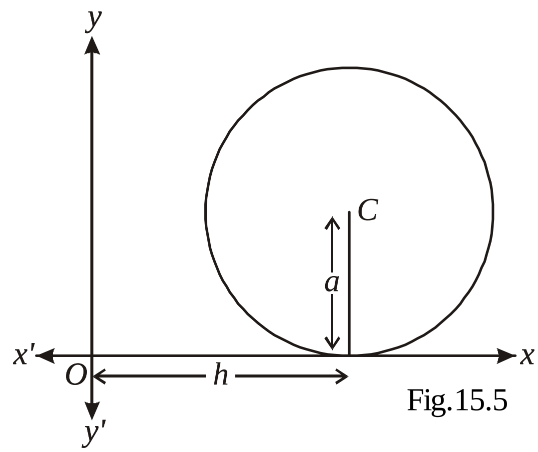


Fig.15.5

(*vi*) In this case $h = a$

Hence the equation (1) reduces to $x^2 + y^2 - 2ax - 2ky + k^2 = 0$ ...(7)

(*vii*) In this case $h = k = a$. (See Fig. 15.6)

Hence the equation (1) reduces to $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ ...(8)

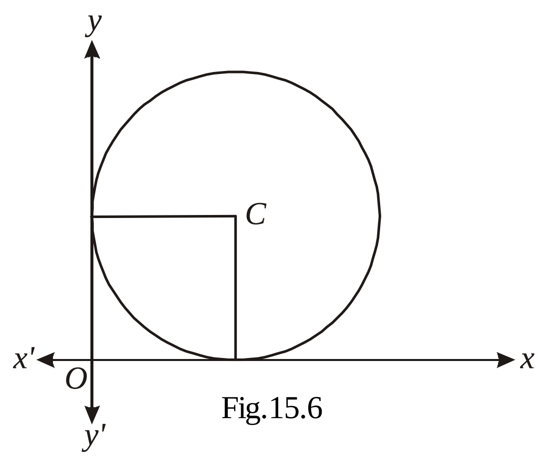


Fig.15.6

Example 15.1 Find the equation of the circle whose centre is (3, – 4) and radius is 6.

Solution : Comparing the terms given in equation (1), we have

$h = 3$, $k = -4$ and $a = 6$.

$\therefore \quad (x-3)^2 + (y+4)^2 = 6^2$ or $\quad x^2 + y^2 - 6x + 8y - 11 = 0$

Example 15.2 Find the centre and radius of the circle given by $(x+1)^2 + (y-1)^2 = 4$.

Solution: Comparing the given equation with $(x-h)^2 + (y-k)^2 = a^2$ we find that

$-h = 1, -k = -1, a^2 = 4$

$\therefore \quad h = -1, k = 1, a = 2.$

So the given circle has its centre (–1,1) and radius 2.

15.3GENERAL EQUATION OF THE CIRCLE IN SECOND DEGREE IN TWO VARIABLES

The standard equation of a circle with centre (*h*, *k*) and radius *r* is given by

$$(x-h)^2+(y-k)^2=r^2 \quad \text{... (1)}$$

or $$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0 \quad \text{... (2)}$$

This is of the form $x^2+y^2+2gx+2fy+c=0.$

$$x^2+y^2+2gx+2fy+c=0 \quad \text{... (3)}$$

$$\Rightarrow \quad (x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c$$

$$\Rightarrow \quad (x+g)^2+(y+f)^2=\left(\sqrt{g^2+f^2-c}\right)^2$$

$$\Rightarrow \quad [x-(-g)]^2+[y-(-f)]^2=\left(\sqrt{g^2+f^2-c}\right)^2 \quad \text{... (4)}$$

$$\Rightarrow \quad (x-h)^2+(y-k)^2=r^2$$

where $h=-g, \quad k=-f, \quad r=\sqrt{g^2+f^2-c}$

This shows that the given equation represents a circle with centre (–*g*, –*f*) and radius $= \sqrt{g^2+f^2-c}$

15.3.1 CONDITONS UNDER WHICH THE GENERAL EQUATION OF SECOND DEGREE IN TWO VARIABLES REPRESENTS A CIRCLE

Let the equation be $x^2+y^2+2gx+2fy+c=0$

(i) It is a second degree equation in *x*, *y* in which coefficients of the terms involving x^2 and y^2 are equal.

(ii) It contains no term involving *xy*

Note : *In solving problems, we keep the coefficients of x^2 and y^2 unity.*

Example 15.3 Find the centre and radius of the circle

$$45x^2+45y^2-60x+36y+19=0$$

Solution : Given equation can be written on dividing by 45 as

$$x^2+y^2-\frac{4}{3}x+\frac{4}{5}y+\frac{19}{45}=0$$

Comparing it with the equation, $x^2+y^2+2gx+2fy+c=0$ we get

$$g=-\frac{2}{3},\ f=\frac{2}{5}\text{ and } c=\frac{19}{45}$$

Thus, the centre is $\left(\frac{2}{3},-\frac{2}{5}\right)$ and radius is $\sqrt{g^2+f^2-c}=\frac{\sqrt{41}}{15}$

Example 15.4 Find the equation of the circle which passes through the points (1, 0), (0, – 6) and (3, 4).

Solution: Let the equation of the circle be, $x^2+y^2+2gx+2fy+c=0$ ...(1)

Since the circle passes through three given points so they will satisfy the equation (1). Hence

$$1+2g+c=0 \quad \cdots(2)$$

and

$$36-12f+c=0 \quad \cdots(3)$$

$$25+6g+8f+c=0 \quad \cdots(4)$$

Subtracting (2) from (3) and (3) from (4), we have $2g+12f=35$ and $6g+20f=11$

Solving these equations for *g* and *f*, we get $g=-\frac{71}{4},\quad f=\frac{47}{8}$

Substituting *g* in (2), we get $c=\frac{69}{2}$

and substituting *g*, *f* and *c* in (1), the required equation of the circle is

$$4x^2+4y^2-142x+47y+138=0$$

Exmaple 15. 5 Find the equation of the circles which touches the axis of *x* and passes through the points (1,–2) and (3, –4).

Solution : Since the circle touches the *x*-axis, put *k* = *a* in the standard form (See result 6) of the equation of the circle, we have, $x^2+y^2-2hx-2ay+h^2=0$ ... (1)

This circle passes through the point (1, –2) $\therefore$ $h^2-2h+4a+5=0$ ... (2)

Also, the circle passes through the point (3, – 4) $\therefore$ $h^2-6h+8a+25=0$ ...(3)

Eliminatong '*a*' from (2) and (3), we get $\Rightarrow$ $h^2+2h-15=0$

$$h=3 \text{ or } h=-5.$$

From (3) the corresponding values of a are – 2 and –10 respectively. On substituting the values of h and a in (1) we get , $x^2 + y^2 - 6x + 4y + 9 = 0$ ... (4)

and $x^2 + y^2 + 10x + 20y + 25 = 0$ ... (5)

(4) and (5) represent the required equations.

CHECK YOUR PROGRESS 15.1

1. Find the equation of the circle whose

   (a) centre is (0, 0) and radius is 3. (b) centre is (– 2,3) and radius is 4.

2. Find the centre and radius of the circle

   (a) $x^2 + y^2 + 3x - y = 6$ (b) $4x^2 + 4y^2 - 2x + 3y - 6 = 0$

3. Find the equation of the circle which passes through the points (0, 2) (2, 0) and (0, 0).

4. Find the equation of the circle which touches the y-axis and passes through the points (–1,2) and (–2,1)

LET US SUM UP

- **Standard form of the circle**

  $(x-h)^2 + (y-k)^2 = a^2$, Centre is (h, k) and radius is a

- The general form of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

  Its centre: $(-g, -f)$ and radius = $\sqrt{g^2 + f^2 - c}$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=6r1GQCxyMKI

www.purplemath.com/modules/**circle**2.htm

www.purplemath.com/modules/**circle**.htm

TERMINAL EXERCISE

1. Find the equation of a circle with centre (4, – 6) and radius 7.

2. Find the centre and radius of the circle $x^2 + y^2 + 4x - 6y = 0$

3. Find the equation of the circle passes through the point (1,0), (–1,0) and (0,1)

ANSWERS

CHECK YOUR PROGRESS 15.1

1. (a) $x^2 + y^2 = 9$

   (b) $x^2 + y^2 + 4x - 6y - 3 = 0$

2. (a) $\left(-\frac{3}{2}, 1\right); \frac{\sqrt{37}}{2}$

   (b) $\left(\frac{1}{4}, -\frac{3}{8}\right); \frac{\sqrt{109}}{8}$

3. $x^2 + y^2 - 2x - 2y = 0$

4. $x^2 + y^2 + 2x - 2y + 1 = 0$

TERMINAL EXERCISE

1. $x^2 + y^2 - 8x + 12y + 3 = 0$

2. Centre $(-2, 3)$; Radius = $\sqrt{13}$

3. $x^2 + y^2 = 1.$

16

CONIC SECTIONS

While cutting a carrot you might have noticed different shapes shown by the edges of the cut. Analytically you may cut it in three different ways, namely

(*i*) Cut is parallel to the base (see Fig.16.1)

(*ii*) Cut is slanting but does not pass through the base (see Fig.16.2)

(*iii*) Cut is slanting and passes through the base (see Fig.16.3)

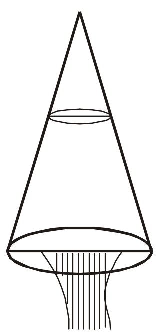


Fig.16.1

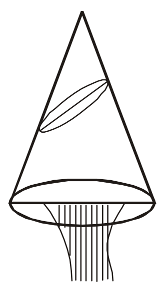


Fig.16.2

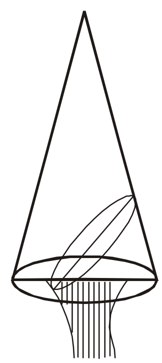


Fig.16.3

The different ways of cutting, give us slices of different shapes.

In the first case, the slice cut represent a circle which we have studied in previous lesson.

In the second and third cases the slices cut represent different geometrical curves, which we shall study in this lesson.

OBJECTIVES

After studying this leson, you will be able to :

- recognise a circle, parabola, ellipse and hyperbola as sections of a cone;
- recognise the parabola, ellipse and hyperbola as certain loci;
- identify the concept of eccentricity, directrix, focus and vertex of a conic section;
- identify the standard equations of parabola, ellipse and hyperbola;
- find the equation of a parabola, ellipse and hyperbola given its directrix and focus.

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of coordinate Geometry
- Various forms of equation of a straight line
- Equation of a circle in various forms

16.1 CONIC SECTION

In the introduction we have noticed the various shapes of the slice of the carrot. Since the carrot is conical in shape so the section formed are sections of a cone. They are therefore called conic sections.

Mathematically, a conic section is the locus of a point P which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed line.

The fixed point is called the ***focus*** and is usually denoted by *S*.

The fixed straight line is called the ***Directrix.***

The straight line passing through the focus and perpendicular to the directrix is called the ***axis.***

The constant ratio is called the ***eccentricity*** and is denoted by *e*.

What happens when

(*i*) $e<1$ (*ii*) $e=1$ (*iii*) $e>1$

In these cases the conic section obtained are known as ellipse, parabola and hyperbola respectively.

In this lesson we shall study about ellipse, parabola, and hyperbola.

16.2 ELLIPSE

Recall the cutting of slices of a carrot. When we cut it obliquely, slanting without letting the knife pass through the base, what do we observe?

You might have come across such shapes when you cut a boiled egg vertically.

The slice thus obtained represents an ellipse. Let us define the ellipse mathematically as follows:

> ***"An ellipse is the locus of a point which moves in a plane such that its distance from a fixed point bears a constant ratio to its distance from a fixed line and this ratio is less than unity".***

16.2.1 STANDARD EQUATION OF AN ELLIPSE

Let *S* be the focus, *ZK* be the directrix and *P* be a moving point. Draw *SK* perpendicular from *S* on the directrix. Let *e* be the eccentricity.

Divide *SK* internally and externally at *A* and *A'* (on *KS* produced) repectively in the ratio $e:1$, as $e<1$.

$SA = e.AK$ $\quad\cdots$ (1)

and $SA' = e.A'K$ $\quad\cdots$ (2)

Since A and A' are points such that their distances from the focus bears a constant ratio e ($e < 1$) to their respective distances from the directrix and so they lie on the ellipse. These points are called vertices of the ellipse.

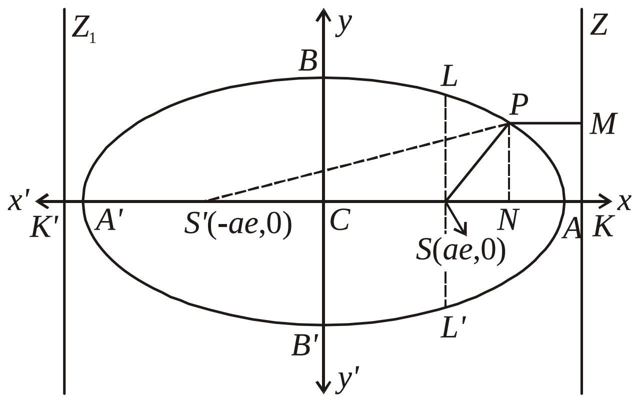


Fig.16.4

Let AA' be equal to $2a$ and C be its mid point, i.e., $CA = CA' = a$

The point C is called the centre of the ellipse.

Adding (1) and (2), we have

$SA + SA' = e.AK + e.A'K$

A

or $\quad AA' = e(CK - CA + A'C + CK)$ or $\quad 2a = e.2CK$ or $\quad CK = \dfrac{a}{e}$ $\quad\cdots$ (3)

Subtracting (1) from (2), we have

$SA' - SA = e(A'K - AK)$

or $\quad (SC + CA') - (CA - CS) = e.A'A$

or $\quad 2\,CS = e.2a$ or $CS = ae$ $\quad\cdots$ (4)

Let us choose C as origin, CAX as x-axis and CY, a line perpendicular to CX as y-axis.

$\therefore$ Coordinates of S are then $(ae, 0)$ and equation of the directrix is $x = \dfrac{a}{e}$

Let the coordinates of the moving point P be (x, y). Join SP, draw $PM \perp ZK$.

By definition $SP = e.PM$ or $\quad SP^2 = e^2 \,.\, PM^2$

or $\quad SN^2 + NP^2 = e^2.(NK)^2$ or $\quad (CN - CS)^2 + NP^2 = e^2.(CK - CN)^2$

or $\quad (x-ae)^2+y^2=e^2\left(\dfrac{a}{e}-x\right)^2$ or $x^2(1-e^2)+y^2=a^2(1-e^2)$

or $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{a^2(1-e^2)}=1$ [On dividing by $a^2(1-e^2)$]

Putting $a^2(1-e^2)=b^2$, we have the standard form of the ellipse as, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Major axis : The line joining the two vertices *A*' and *A*, i.e., *A*'*A* is called the major axis and its length is 2*a*.

Minor axis : The line passing through the centre perpendicular to the major axis, i.e., *BB*' is called the minor axis and its length is 2*b*.

Principal axis : The two axes together (major and minor) are called the principal axes of the ellipse.

Latus rectum : The length of the line segment *LL*' is called the latus rectum and it is given by

$$\frac{2b^2}{a}$$

Equation of the directrix : $x=\pm\dfrac{a}{e}$

Eccentricity : *e* is given by $e^2=1-\dfrac{b^2}{a^2}$

Example 16.1 Find the equation of the ellipse whose focus is (1,–1), eccentricity $e=\dfrac{1}{2}$ and the directrix is $x-y=3$.

Solution : Let *P* (*h*,*k*) be any point on the ellipse then by the definition, its distance from the focus = *e*. Its distance from directrix or $SP^2=e^2.PM^2$

(*M* is the foot of the perpendicular drawn from *P* to the directrix).

or $\quad (h-1)^2+(k+1)^2=\dfrac{1}{4}\left(\dfrac{h-k-3}{\sqrt{1+1}}\right)^2$

or $\quad 7(h^2+k^2)+2hk-10h+10k+7=0$

$\therefore$ The locus of P is, $7(x^2+y^2)+2xy-10x+10y+7=0$

which is the required equation of the ellipse.

Example 16.2 Find the eccentricity, coordinates of the foci and the length of the axes of the ellipse $3x^2 + 4y^2 = 12$

Solution : The equation of the ellipse can be written in the following form, $\frac{x^2}{4} + \frac{y^2}{3} = 1$

On comparing this equation with that of the standard equation of the ellipse, we have $a^2 = 4$ and $b^2 = 3$, then

(i) $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$

(ii) coordinates of the foci are (1,0) and $(-1,0)$

[$\because$ The coordinate are ($\pm$ ae, 0)]

(*iii*) Length of the major axes $2a = 2 \times 2 = 4$ and

length of the minor axis = $2b = 2 \times \sqrt{3} = 2\sqrt{3}.$

CHECK YOUR PROGRESS 16.1

1. Find the equation of the ellipse referred to its centre

   (a) whose latus rectum is 5 and whose eccentricity is $\frac{2}{3}$

   (b) whose minor axis is equal to the distance between the foci and whose latus rectum is 10.

   (c) whose foci are the points (4,0) and (– 4,0) and whose eccentricity is $\frac{1}{3}$.

2. Find the eccentricity of the ellipse, if its latus rectum be equal to one half its minor axis.

16.3 PARABOLA

Recall the cutting of slice of a carrot. When we cut obliquely and letting the knife pass through the base, what do we observe?

Also when a batsman hits the ball in air, have you ever noticed the path of the ball?

Is there any property common to the edge of the slice of the carrot and the path traced out by the ball in the example cited above?

Yes, the edge of such a slice and path of the ball have the same shape which is known as a parabola. Let us define parabola mathematically.

"A parabola is the locus of a point which moves in a plane so that its distance

MODULE-IV
Co-ordinate Geometry

Notes

from a fixed point in the plane is equal to its distance from a fixed line in the plane."

16.3.1 STANDARD EQUATION OF A PARABOLA

Let S be the fixed point and ZZ' be the directrix of the parabola. Draw SK perpendicular to ZZ'. Bisect SK at A.

Since $SA = AK$, by the definition of the parabola A lies on the parabola. A is called the vertex of the parabola.

Take A as origin, AX as the x-axis and AY perpendicular to AX through A as the y-axis.

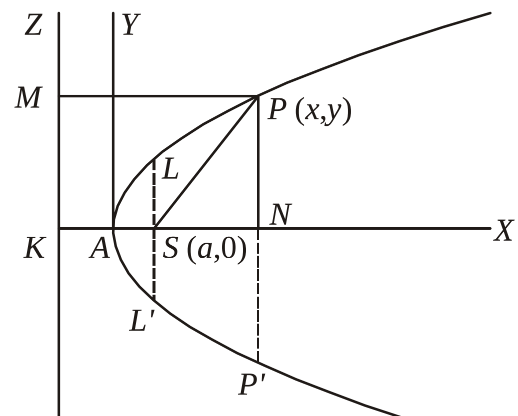


Fig.16.5

Let $KS = 2a$ $\quad \therefore AS = AK = a$

$\therefore$ The coordinates of A and S are (0,0) and $(a,0)$ respectively.

Let $P(x,y)$ be any point on the parabola. Draw $PN \perp AS$ produced

$\therefore$ $AN = x$ and $NP = y$

Join SP and draw $PM \perp ZZ'$

$\therefore$ By definition of the parabola

$SP = PM$ or $\quad SP^2 = PM^2$

or $\quad (x-a)^2 + (y-0)^2 = (x+a)^2 \quad [\because PM = NK = NA + AK = x + a]$

or $\quad (x-a)^2 - (x+a)^2 = -y^2$ or $\quad y^2 = 4ax$

which is the ***standard equation*** of the parabola.

Note : *In this equation of the parabola*

(*i*) Vertex is (0,0)

(*ii*) Focus is $(a,0)$

(*iii*) Equation of the axis is $y = 0$

(*iv*) Equation of the directrix is $x + a = 0$

(*v*) Latus rectum = $4a$

16.3.2 OTHER FORMS OF THE PARABOLA

What will be the equation of the parabola when

(i) focus is $(-a, 0)$ and directrix is $x - a = 0$

(ii) focus is $(0, a)$ and directrix is $y + a = 0$,

(iii) focus is $(0, -a)$ and directrix is $y - a = 0$?

It can easily be shown that the equation of the parabola with above conditions takes the following forms:

(i) $y^2 = -4ax$ (ii) $x^2 = 4ay$ (iii) $x^2 = -4ay$

The figures are given below for the above equations of the parabolas.

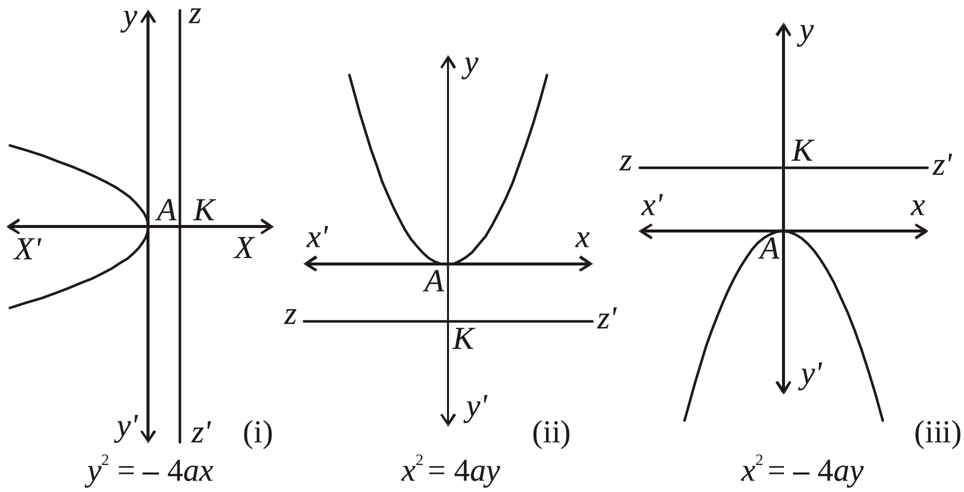


Fig. 16.6

Corresponding results of above forms of parabolas are as follows:

Forms	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	(0,0)	(0,0)	(0,0)	(0,0)
Coordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Coordinates of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Coordinates of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
length of Latus rectum	$4a$	$4a$	$4a$	$4a$

Example 16.3 Find the equation of the parabola whose focus is the origin and whose directrix is the line $2x + y - 1 = 0$.

Solution : Let S (0,0) be the focus and ZZ' be the directrix whose equation is $2x+y-1=0$

Let $P(x, y)$ be any point on the parabola.

Let PM be perpendicular to the directrix (See Fig. 16.5)

$\therefore$ By definition $SP = PM$ or $SP^2 = PM^2$

or $$x^2+y^2=\frac{(2x+y-1)^2}{\left(\sqrt{2^2+1}\right)^2}$$

or $5x^2+5y^2=4x^2+y^2+1+4xy-2y-4x$ or $x^2+4y^2-4xy+2y+4x-1=0.$

Example 16.4 Find the equation of the parabola, whose focus is the point (2, 3) and whose directrix is the line $x - 4y + 3 = 0$.

Solution : Given focus is $S(2,3)$; and the equation of the directrix is $x-4y+3=0$.

$\therefore$ As in the above example, $$(x-2)^2+(y-3)^2=\left\{\frac{x-4y+3}{\sqrt{1^2+4^2}}\right\}^2$$

$\Rightarrow$ $16x^2+y^2+8xy-74x-78y+212=0$

CHECK YOUR PROGRESS 16.2

1. Find the equation of the parabola whose focus is (a, b) and whose directrix is $\frac{x}{a}+\frac{y}{b}=1$.
2. Find the equation of the parabola whose focus is (2,3) and whose directrix is $3x+4y=1$.

16.4 HYPERBOLA

Hyperbola is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its distance from a fixed straight line in the same plane is greater than one. In other words hyperbola is the conic in which eccentricity is greater than unity. The fixed point is called focus and the fixed straight line is called directrix.

Equation of Hyperbola in Standard from :

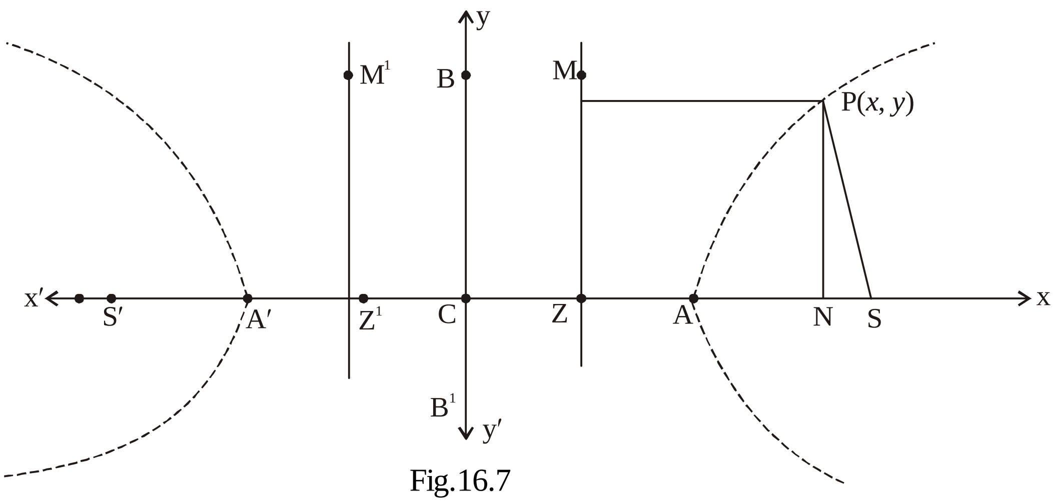


Fig.16.7

Let S be the focus and ZM be the directrix. Draw SZ perpendicular from S on directix we can divide SZ both internally and externally in the ratio $e : 1$ ($e > 1$). Let the points of division be A and A′ as shown in the above figure. Let C be the mid point of AA′. Now take CZ as the x-axis and the perpendicular at C as y-axis.

Let $\quad AA' = 2a$

Now $\quad \dfrac{SA}{AZ} = e\ (e > 1)$ and $\dfrac{SA'}{A'Z} = e\ (e > 1)$.

i.e. $\quad SA = eAZ \qquad$ ...(i)

i.e. $\quad SA' = eA'Z \qquad$ ...(ii)

Adding (i) and (ii) we get

$$SA + SA' = e(AZ + A'Z)$$

$$(CS - CA) + (CS + CA') = eAA'$$

$$\Rightarrow \quad 2CS = e.2a\ (\because CA = CA')$$

$$\Rightarrow \quad CS = ae$$

Hence focus point is $(ae, 0)$.

Subtracting (i) from (ii) we get

$$SA' - SA = e(A'Z - AZ)$$

i.e. $\quad AA' = e\left[(CZ + CA') - (CA - CZ)\right]$

i.e. $\quad AA' = e[2CZ] \quad (\because CA' = CA)$

i.e. $\quad 2a = e(2CZ)$

$$\Rightarrow \quad CZ = \frac{a}{e}$$

$\therefore$ Equation of directrix is $x = \dfrac{a}{e}$.

Let P(x, y) be any point on the hyperbola, PM and PN be the perpendiculars from P on the directrix and x-axis respectively.

Thus, $\frac{SP}{PM} = e \quad \Rightarrow \text{SP} = e\text{PM}$

$\Rightarrow \quad (\text{SP})^2 = e^2(\text{PM})^2$

i.e. $(x-ae)^2+(y-0)^2 = e^2\left(x-\frac{a}{e}\right)^2$

i.e. $x^2+a^2e^2-2aex+y^2 = e^2\left(\frac{e^2x^2+a^2-2aex}{e^2}\right)$

i.e. $x^2+a^2e^2+y^2 = e^2x^2+a^2$

i.e. $(e^2-1)x^2-y^2 = a^2(e^2-1)$

i.e. $\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)} = 1$

Let $a^2(e^2-1) = b^2$

$\therefore \quad \boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1}$

Which is the equation of hyperbola in standard from.

- Now let S′ be the image of S and Z′M′ be the image of ZM w.r.t y-axis. Taking S′ as focus and Z′M′ as directrix, it can be seen that the corresponding equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Hence for every hyperbola, there are two foci and two directrices.
- We have $b^2 = a^2(e^2-1)$ and $e > 1$

$$\Rightarrow \quad e = \sqrt{\frac{a^2+b^2}{a^2}}$$

- If we put $y = 0$ in the equation of hyperbola we get
$x^2 = a^2 \Rightarrow x = \pm\, a$
$\therefore$ Hyperbola cuts x-axis at A(a, 0) and A′($-a$, 0).
- If we put $x = 0$ in the equation of hyperbola we get

$y^2 = -b^2 \Rightarrow y = \pm\sqrt{-1}.b = \pm ib$

Which does not exist in the cartesian plane.

$\therefore$ Hyperbola does not interesct y-axis.

- AA′ = 2a, along the x-axis is called **transverse axis** of the hyperbola and BB′ = 2b, along y-axis is called **conjugate axis** of the hyperbola. Notice that hyperbola does not meet its conjugate axis.
- As in case of ellipse, hyperbola has two foci

  S(ae, 0), S′(–ae, 0) and two directrices $x = \pm\frac{a}{e}$.
- C is called the centre of hyperbola.
- Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola. As in ellipse, it can be proved that the length of the latus rectum of hyperbola is $\frac{2b^2}{a}$.
- Hyperbola is symmetric about both the axes.
- Foci of hyperbola are always on transverse axis. It is the positive term whose denominator gives the transverse axis. For example $\frac{x^2}{9} - \frac{y^2}{16} = 1$ has transverse axis along x-axis and length of transverse axis is 6 units. While $\frac{y^2}{25} - \frac{x^2}{16} = 1$ has transverse axis along y-axis of length 10 unit.
- The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axis of given hyperbola, is called the conjugate hyperbola of the given hyperbola. This equation is of the form $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

In this case : Transverse axis is along y-axis and conjugate axis is along x-axis.

- Length of transverse axis = 2b.
- Length of conjugate axis = 2a
- Length of latus rectum = $\frac{2a^2}{b}$.
- Equations of directrices $y = \pm\frac{b}{e}$.
- Vertices (0, ± b)
- Foci (0, ± be)
- Centre (0, 0)
- Eccentricity (e) = $\sqrt{\frac{b^2 + a^2}{b^2}}$.

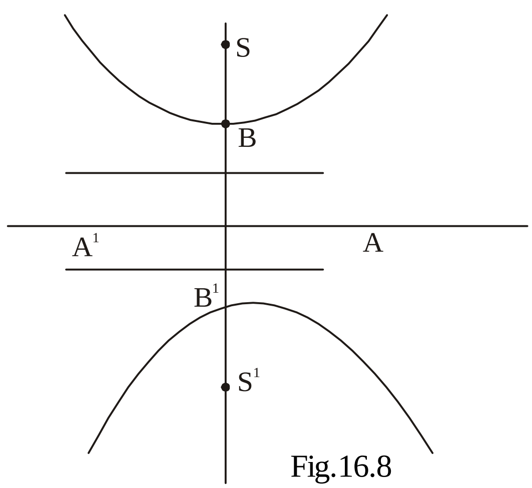


Fig.16.8

16.4.1 RECTANGULAR HYPERBOLA

If in a hyperbola the length of the transverse axis is equal to the length of the conjugate axis, then the hyperbola is called a rectangular hyperbola.

Its equation is $x^2 - y^2 = a^2$ or $y^2 - x^2 = b^2$ ($\because$ $a = b$)

In this case $e = \sqrt{\dfrac{a^2 + a^2}{a^2}}$ or $\sqrt{\dfrac{b^2 + b^2}{b^2}} = \sqrt{2}$

i.e. the eccentricity of rectangualr hyperbola is $\sqrt{2}$.

Example 16.5 For the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, find the following (i) Eccentricity (ii) Foci (iii) Vertices (iv) Directrices (v) Length of transverse axis (vi) Length of conjugate axis (vii) Length of latus rectum (viii) Centre.

Solution : Here $a^2 = 16$ and $b^2 = 9$, $\Rightarrow a = 4$ and $b = 3$.

(i) Eccentricity (e) = $\sqrt{\dfrac{a^2 + b^2}{a^2}} = \sqrt{\dfrac{16+9}{16}} = \dfrac{5}{4}$

(ii) Foci = $(\pm\ ae, 0) = \left(\pm\dfrac{4.5}{4}, 0\right) = (\pm 5, 0)$

(iii) Vertices = $(\pm\, a,\ 0) = (\pm\, 4,\ 0)$

(iv) Directrices $x = \pm\dfrac{a}{e} \Rightarrow x = \pm\dfrac{4}{5/4} \Rightarrow x = \pm\dfrac{16}{5}$.

(v) Length of transverse axis = $2a = 2 \times 4 = 8$.

(vi) Length of conjugate axis = $2a = 2 \times 3 = 6$

(vii) Length of latus rectum = $\dfrac{2b^2}{a} = \dfrac{2\times 9}{4} = \dfrac{9}{2}$.

(viii) Centre = (0, 0)

Example 16.6 Find the equation of hyperbola with vertices (± 2, 0) and foci (± 3, 0)

Solution : Here $a = 2$ and $ae = 3$.

$\therefore$ $e = 3/2$.

We know that $b^2 = a^2(e^2 - 1)$

$\Rightarrow$ $b^2 = 4\left(\dfrac{9}{4} - 1\right) = 5$

$\therefore$ Equation of hyperbola is $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$.

Example 16.7 For hyperbola $\frac{y^2}{9}-\frac{x^2}{27}=1,$ find the following :

(i) Eccentricity (ii) Centre (iii) Foci (iv) Vertices (v) Directrices (vi) Length of transverse axis (vii) Length of conjugate axis (viii) Latus rectum.

Solution : Here $b^2 = 9$ and $a^2 = 27 \Rightarrow b = 3$ and $a = 3\sqrt{3}$.

(i) $e=\sqrt{\frac{27+9}{9}}=\sqrt{4}=2.$ (ii) Centre = (0, 0)

(iii) Foci = $(0,\pm be)=(0,\pm 3.2)=(0,\pm 6).$

(iv) Vertices = $(0, \pm b) = (0, \pm 3)$.

(v) Directrices, $y=\pm\frac{b}{e} \Rightarrow y=\pm\frac{3}{2}.$

(vi) Length of transverse axis = $2b = 2 \times 3 = 6$

(vii) Length of conjugate axis = $2a = 2 \times 3\sqrt{3} = 6\sqrt{3}$

(viii) Length of latus rectum $=\frac{2a^2}{b}=\frac{2\times 27}{3}=18.$

CHECK YOUR PROGRESS 16.3

1. (i) Transverse axis of the hyperbola $\frac{y^2}{25}-\frac{x^2}{16}$ is along ....

   (ii) Eccentricity of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ is ...

   (iii) Eccentricity of rectangular hyperbola is ...

   (iv) Length of latus rectum of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is ...

   (v) Foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is at ...

   (vi) Equation of directrices of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is ...

   (vii) Vertices of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are at ...

2. For the hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, complete the following.

   (i) Eccentricity (e) = ...

   (ii) Centre = ...

(iii) Foci = ...

(iv) Vertices = ...

(v) Equations of directrices, y = ...

(vi) Length of latus rectum = ...

(vii) Length of transverse axis = ...

(viii) Length of conjugate axis = ...

(ix) Transverse axis is along ...

(x) Conjugate axis is along ...

LET US SUM UP

- **Conic Section**

"A conic section is the locus of a point P which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line".

(*i*) **Focus :** The fixed point is called the focus.

(*ii*) **Directrix :** The fixed straight line is called the directrix.

(*iii*) **Axis :** The straight line passing through the focus and pependicular to the directrix is called the axis.

(*iv*) **Eccentricity :** The constant ratio is called the eccentricity.

- (*v*) **Latus Rectum :** The double ordinate passing through the focus and parallel to the directrix is known as latus rectum. (In Fig.16.5 LSL' is the latus rectum).

- **Standard Equation of the Ellipse is :** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(*i*) Major axis = $2a$ (*ii*) Minor axis = $2b$

(*iii*) Equation of directrix is $x=\pm\frac{a}{e}$ (*iv*) Foci : $(\pm ae, 0)$

(*v*) Eccentricity, i.e., e is given by $e^2=1-\frac{b^2}{a^2}$ vi Latus Reotam $=\frac{2b^2}{a}$

- **Standard Equation of the Parabola is :** $y^2=4ax$

(*i*) Vertex is (0,0) (*ii*) Focus is $(a,0)$

(*iii*) Axis of the parabola is $y=0$ (*iv*) Directrix of the parabola is $x+a=0$

(*v*) Latus rectum = $4a$.

- **OTHER FORMS OF THE PARABOLA ARE**

  (*i*) $y^2 = -4ax$ (concave to the left).

  (*ii*) $x^2 = 4ay$ (concave upwards).

  (*iii*) $x^2 = -4ay$ (concave downwards).

- Equation of hyperbola having transverse axis along x-axis and conjugate axis along y-axis is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

  For this hyperbola (i) $e = \sqrt{\frac{a^2 + b^2}{a^2}}.$

  (ii) Centre = (0, 0) (iii) Foci = ($\pm$ *ae*, 0)

  (iv) Vetrices = ($\pm$ *a*, 0)(v) Length of latus rectum = $\frac{2b^2}{a}$

  (vi) Length of transverse axis = 2*a*

  (vii) Length of conjugate axis = 2*b*

  (viii) Equations of directrixes are given by $x = \pm \frac{a}{e}.$

- Equations of hyperbola having transverse axis along y-axis and conjugate axis along x-axis is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$

  For this hyperbola :

  (i) Vertices = (0, $\pm$ *b*)(ii) Centre = (0, 0)

  (iii) Foci = (0, $\pm$ *be*) (iv) $e = \sqrt{\frac{a^2 + b^2}{b^2}}$

  (v) Length of latus rectum = $\frac{2a^2}{b}.$

  (vi) Length of transverse axis = 2*b*.

  (vii) Length of conjugate axis = 2*a*.

  (viii) Equations of directrixes are given by $y = \pm \frac{b}{e}.$

SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=0A7RR0oy2ho
http://www.youtube.com/watch?v=lvAYFUIEpFI
http://www.youtube.com/watch?v=QR2vxfwiHAU
http://www.youtube.com/watch?v=pzSyOTkAsY4
http://www.youtube.com/watch?v=hl58vTCqVIY
http://www.youtube.com/watch?v=lGQw-W1PxBE
http://www.youtube.com/watch?v=S0Fd2Tg2v7M

TERMINAL EXERCISE

1. Find the equation of the ellipse in each of the following cases, when

   (a) focus is (0, 1), directrix is $x + y = 0$ and $e = \frac{1}{2}$.

   (b) focus is (–1,, 1), directrix is $x - y + 3 = 0$ and $e = \frac{1}{2}$.

2. Find the coordinates of the foci and the eccentricity of each of the following ellipses:

   (a) $4x^2 + 9y^2 = 1$ (b) $25x^2 + 4y^2 = 100$

3. Find the equation of the parabola whose focus is (–8, –2) and directrix is $y - 2x + 9 = 0$.

4. Find the equation of the hyperbola whose foci are (± 5, 0) and the length of the transverse axis is 8 units.

5. Find the equation of the hyperbola with vertices at (0, ± 6) and $e = \frac{5}{3}$.

6. Find the eccentricity, length of transverse axis, length of conjugate axis, vertices, foci, equations of directrices, and length of latus rectum of the hyperbola (i) $25x^2 - 9y^2 = 225$ (ii) $16y^2 - 4x^2 = 1$.

7. Find the equation of the hyperbola with foci $(0, \pm\sqrt{10})$, and passing through the point (2, 3).

8. Find the equation of the hyperbola with foci (± 4, 0) and length of latus rectum 12.

ANSWERS

CHECK YOUR PROGRESS 16.1

1. (a) $20x^2 + 36y^2 = 405$

   (b) $x^2 + 2y^2 = 100$

   (c) $8x^2 + 9y^2 = 1152$

2. $\dfrac{\sqrt{3}}{2}$

CHECK YOUR PROGRESS 16.2

1. $(ax - by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0.$

2. $16x^2 + 9y^2 - 94x - 142y - 24xy + 324 = 0$

CHECK YOUR PROGRESS 16.3

1. (i) y-axis (ii) $\dfrac{5}{3}$

   (iii) $\sqrt{2}$ (iv) $\dfrac{2b^2}{a}$

   (v) $(\pm\, ae,\ 0)$ (vi) $x = \pm\dfrac{a}{e}$

   (vii) $(\pm\, a,\ 0)$

2. (i) $\sqrt{\dfrac{b^2 + a^2}{b^2}}$ (ii) $(0,\ 0)$ (iii) $(0,\ \pm\, be)$

   (iv) (0, ± b) (v) $\dfrac{\pm b}{e}$ (vi) $\dfrac{2a^2}{b}$

   (vii) $2b$ (viii) $2a$ (ix) y-axis (x) x-axis

TERMINAL EXERCISE

1. (a) $7x^2 + 7y^2 - 2xy - 16y + 8 = 0$

   (b) $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$

2. (a) $\left(\pm\dfrac{\sqrt{5}}{6}, 0\right);\ \dfrac{\sqrt{5}}{3}$

   (b) $\left(0\ ,\ \pm\sqrt{21}\right);\ \dfrac{\sqrt{21}}{5}$

3. $x^2 + 4y^2 + 4xy + 116x + 2y + 259 = 0$

4. $9x^2 - 16y^2 = 144$

5. $16y^2 - 9x^2 = 576$

6. (i) Eccentricity = $\frac{\sqrt{34}}{3}$, length of transverse axis = 6, length of conjugate axis = 10, vertices (± 3, 0), Foci $(\pm\sqrt{34}, 0)$, equations of directrices $x = \pm\frac{1}{\sqrt{34}}$, latus rectum = $\frac{50}{3}$.

(ii) Eccentricity = $\sqrt{5}$, length of transverse axis = $\frac{1}{2}$, length of conjugate axis = 1, vertices $\left(0, \pm\frac{1}{4}\right)$, Foci $\left(0, \pm\frac{\sqrt{5}}{4}\right)$, equations of directrices, $y = \frac{1}{4\sqrt{5}}$, latus rectum = 2.

7. $y^2 - x^2 = 5$

8. $\frac{x^2}{4} - \frac{y^2}{12} = 1$

17

MEASURES OF DISPERSION

You have learnt various measures of central tendency. Measures of central tendency help us to represent the entire mass of the data by a single value.

Can the **central tendency** describe the data fully and adequately?

In order to understand it, let us consider an example.

The daily income of the workers in two factories are :

Factory A	:	35	45	50	65	70	90	100
Factory B	:	60	65	65	65	65	65	70

Here we observe that in both the groups the mean of the data is the same, namely, 65

(i) In group A, the observations are much more scattered from the mean.

(ii) In group B, almost all the observations are concentrated around the mean.

Certainly, the two groups differ even though they have the same mean.

Thus, there arises a need to differentiate between the groups. We need some other measures which concern with the measure of scatteredness (or spread).

To do this, we study what is known as **measures of dispersion.**

OBJECTIVES

After studying this lesson, you will be able to :

- explain the meaning of dispersion through examples;
- define various measures of dispersion – range, mean deviation, variance and standard deviation;
- calculate mean deviation from the mean of raw and grouped data;
- calculate mean deviation from the median of raw and grouped data.
- calculate variance and standard deviation of raw and grouped data; and
- illustrate the properties of variance and standard deviation.
- Analyses the frequencys distributions with equal means.

EXPECTED BACKGROUND KNOWLEDGE

- Mean of grouped data
- Median of ungrouped data

17.1 MEANING OF DISPERSION

To explain the meaning of dispersion, let us consider an example.

Two sections of 10 students each in class X in a certain school were given a common test in Mathematics (maximum marks 40). The scores of the students are given below :

Section A :	6	9	11	13	15	21	23	28	29	35
Section B:	15	16	16	17	18	19	20	21	23	25

The average score in section A is 19.

The average score in section B is 19.

Let us construct a dot diagram, on the same scale for section A and section B (see Fig. 17.1)

The position of mean is marked by an arrow in the dot diagram.

Section A

Section B

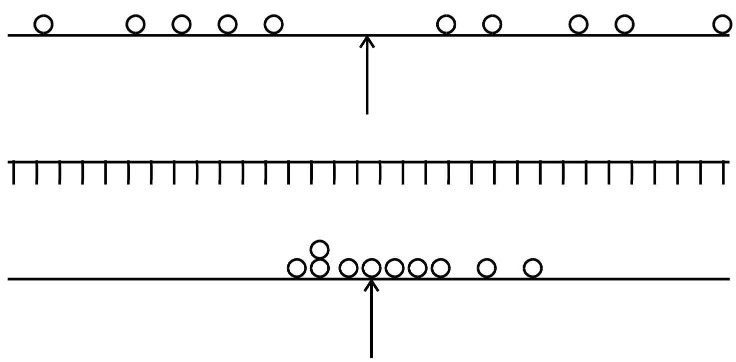


Fig. 17.1

Clearly, the extent of spread or dispersion of the data is different in section A from that of B. The measurement of the scatter of the given data about the average is said to be a measure of dispersion or scatter.

In this lesson, you will read about the following measures of dispersion :

(a) Range

(b) Mean deviation from mean

(c) Mean deviation from median

(d) Variance

(e) Standard deviation

17.2 DEFINITION OF VARIOUS MEASURES OF DISPERSION

(a) **Range :** In the above cited example, we observe that

(i) the scores of all the students in section A are ranging from 6 to 35;

(ii) the scores of the students in section B are ranging from 15 to 25.

The difference between the largest and the smallest scores in section A is 29 (35 – 6)

The difference between the largest and smallest scores in section B is 10 (25 – 15).

Thus, the difference between the largest and the smallest value of a data, is termed as the range of the distribution.

(b) **Mean Deviation from Mean :** In Fig. 17.1, we note that the scores in section B cluster around the mean while in section A the scores are spread away from the mean. Let us take the deviation of each observation from the mean and add all such deviations. If the sum is 'large', the dispersion is 'large'. If, however, the sum is 'small' the dispersion is small.

Let us find the sum of deviations from the mean, i.e., 19 for scores in section A.

Observations (x_i)	Deviations from mean $(x_i - \overline{x})$
6	–13
9	–10
11	–8
13	–6
15	–4
21	+2
23	+4
28	+9
29	+10
35	16
190	0

Here, the sum is zero. It is neither 'large' nor 'small'. Is it a coincidence ?

Let us now find the sum of deviations from the mean, i.e., 19 for scores in section B.

Observations (x_i)	Deviations from mean $(x_i - \overline{x})$
15	–4
16	–3
16	–3
17	–2
18	–1
19	0
20	1
21	2
23	4
25	6
190	0

Again, the sum is zero. Certainly it is not a coincidence. In fact, we have proved earlier that **the sum of the deviations taken from the mean is always zero for any set of data.** Why is the sum always zero ?

On close examination, we find that the signs of some deviations are positive and of some other deviations are negative. Perhaps, this is what makes their sum always zero. In both the cases, we get sum of deviations to be zero, so, we cannot draw any conclusion from the sum of deviations. But this can be avoided if we take only the **absolute value of the deviations** and then take their sum.

If we follow this method, we will obtain a measure (descriptor) called the mean deviation from the mean.

The mean deviation is the sum of the absolute values of the deviations from the mean divided by the number of items, (i.e., the sum of the frequencies).

(c) **Variance :** In the above case, we took the absolute value of the deviations taken from mean to get rid of the negative sign of the deviations. Another method is to square the deviations. Let us, therefore, square the deviations from the mean and then take their sum. If we divide this sum by the number of observations (i.e., the sum of the frequencies), we obtain the average of deviations, which is called variance. **Variance is usually denoted by** σ^2.

(d) **Standard Deviation :** If we take the positive square root of the variance, we obtain the root mean square deviation or simply called standard deviation and is denoted by σ.

17.3 MEAN DEVIATION FROM MEAN OF RAW AND GROUPED DATA

Mean Deviation from mean of raw data = $\dfrac{\sum_{i=1}^{n} |x_i - \overline{x}|}{N}$

Mean deviation from mean of grouped data $= \dfrac{\sum_{i=1}^{n} [f_i |x_i - \overline{x}|]}{N}$

where $N = \sum_{i=1}^{n} f_i, \overline{x} = \dfrac{1}{N} \sum_{i=1}^{n} (f_i x_i)$

The following steps are employed to calculate the mean deviation from mean.

Step 1 : Make a column of deviation from the mean, namely $x_i - \overline{x}$ (In case of grouped data take x_i as the mid value of the class.)

Step 2 : Take absolute value of each deviation and write in the column headed $|x_i - \overline{x}|$.
For calculating the mean deviation from the mean of raw data use

Mean deviation of Mean $= \dfrac{\sum_{i=1}^{n} |x_i - \overline{x}|}{N}$

For grouped data proceed to step 3.

Step 3 : Multiply each entry in step 2 by the corresponding frequency. We obtain $f_i(x_i - \overline{x})$ and write in the column headed $f_i|x_i - \overline{x}|$.

Step 4 : Find the sum of the column in step 3. We obtain $\sum_{i=1}^{n}[f_i|x_i - \bar{x}|]$

Step 5 : Divide the sum obtained in step 4 by N.

Now let us take few examples to explain the above steps.

Example 17.1 Find the mean deviation from the mean of the following data :

Size of items x_i	4	6	8	10	12	14	16
Frequency f_i	2	5	5	3	2	1	4

Mean is 10

Solution :

x_i	f_i	$x_i - \bar{x}$	$\lvert x_i - \bar{x}\rvert$	$f_i\lvert x_i - \bar{x}\rvert$
4	2	–5.7	5.7	11.4
6	4	–3.7	3.7	14.8
8	5	–1.7	1.7	8.5
10	3	0.3	0.3	0.9
12	2	2.3	2.3	4.6
14	1	4.3	4.3	4.3
16	4	6.3	6.3	25.2
	21			69.7

Mean deviation from mean $= \dfrac{\sum[f_i|x_i - \bar{x}|]}{21} = \dfrac{69.7}{21} = 3.319$

Example 17.2 Calculate the mean deviation from mean of the following distribution :

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
No. of Students	5	8	15	16	6

Mean is 27 marks

Solution :

Marks	Class Marks x_i	f_i	$x_i - \bar{x}$	$\lvert x_i - \bar{x}\rvert$	$f_i\lvert x_i - \bar{x}\rvert$
0 – 10	5	5	–22	22	110
10 – 20	15	8	–12	12	96
20 – 30	25	15	–2	2	30
30 – 40	35	16	8	8	128
40 – 50	45	6	18	18	108
Total		50			472

Mean deviation from Mean $= \dfrac{\sum [f_i |x_i - \bar{x}|]}{N} = \dfrac{472}{50}$ Marks $= 9.44$ Marks

CHECK YOUR PROGRESS 17.1

1. The ages of 10 girls are given below :

   3 5 7 8 9 10 12 14 17 18

   What is the range ?

2. The weight of 10 students (in Kg) of class XII are given below :

   45 49 55 43 52 40 62 47 61 58

   What is the range ?

3. Find the mean deviation from mean of the data

   45 55 63 76 67 84 75 48 62 65

   Given mean = 64.

4. Calculate the mean deviation from mean of the following distribution.

Salary (in rupees)	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
No. of employees	4	6	8	12	7	6	4	3

Given mean = Rs. 57.2

5. Calculate the mean deviation for the following data of marks obtained by 40 students in a test

Marks obtained	20	30	40	50	60	70	80	90	100
No. of students	2	4	8	10	8	4	2	1	1

6. The data below presents the earnings of 50 workers of a factory

Earnings (in rupees) :	1200	1300	1400	1500	1600	1800	2000
No. of workers :	4	6	15	12	7	4	2

   Find mean deviation.

7. The distribution of weight of 100 students is given below :

Weight (in Kg)	50 – 55	55 – 60	60 – 65	65 – 70	70 – 75	75 – 80
No. of students	5	13	35	25	17	5

   Calculate the mean deviation.

8. The marks of 50 students in a particular test are :

Marks	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
No. of students	4	6	9	12	8	6	4	1

Find the mean deviation for the above data.

17.4 MEDIAN

17.4.1 MEDIAN OF GROUPED DATA

Median of Discrete Frequency Distribution :

Step 1 : Arrange the data in ascending order.

Step 2 : Find cumulative frequencies

Step 3 : Find $\frac{N}{2}$

Step 4 : The observation whose cumulative frequency is just greater than $\frac{N}{2}$ is the median of the data.

Example 17.3 Find the median of the data

x_i	8	9	10	12	14	16
f_i	6	2	2	2	6	8

Solution : The given data are already in ascending order. Let us now write the cumulative frequencies of observations

x_i	8	9	10	12	14	16
f_i	6	2	2	2	6	8
c.f.	6	8	10	12	18	26

$$N = 26, \qquad \therefore \frac{N}{2} = 13.$$

The observation whose c.f. is just greater than 13 is 14 (whose c.f. is 18)

$\therefore$ Median = 14.

17.4.2 MEDIAN OF CONTINUOUS FREQUENCY DISTRIBUTION

Step 1 : Arrange the data in ascending order

Step 2 : Write cumulative frequencies of the observations

Step 3 : Identify the class whose cumulative frequency is just greater than $\frac{N}{2}$. Call this class-interval as median class.

Step 4 : Find median by the formula

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times i$$

Where

$l \rightarrow$ Lower limit of the median class
$N \rightarrow$ Number of observations $N = \Sigma f_i$
$C \rightarrow$ Cumulative frequency of the class just preceding the median class
$f \rightarrow$ Frequency of the median class
$i \rightarrow$ Width of the median class

Example 17.4 Find the median marks obtained by 50 students from the following distribution :

Marks	0-10	10-20	20-30	30-40	40-50
Number of Students	8	8	14	16	4

Solution : The given intervals are already in ascending order. The following table has the row corresponding to the cumulative frequencies.

Marks	0-10	10-20	20-30	30-40	40-50
Number of students	8	8	14	16	4
Cummulative frequency	8	16	30	46	50

$$N = 50, \quad \frac{N}{2} = 25$$

The class corresponding to the c.f. just greater than 25 is 20-30.

$\therefore$ Median class is 20-30

where $l = 20$, $N = 50$, $C = 16$, $f = 14$, $i = 10$.

$$\therefore \quad \text{Median} = l + \frac{\frac{N}{2} - C}{f} \times i = 20 + \frac{25 - 16}{14} \times 10$$

$$= 20 + \frac{9}{14} \times 10 = 20 + 6.43 = 26.43$$

Example 17.5 Find the median of the following:

Marks	Number of Students
0 – 9	3
10 – 19	5
20 – 29	8
30 – 39	9
40 – 49	13
50 – 59	6

Solution : The given class intervals are inclusive series Before finding the median we have to convert the inclusive series into exclusive series.

Method of converting an inclusive series into exclusive series.

(1) Find the half of the difference between the upper limit of a class and the lower limit of its succeeding (next) class.

(2) Subtract this half from the lower limit and add into the upper limit.

Mark	Exclusive Series	f.	c.f.
0-9	0.5-9.5	3	3
10-19	9.5-19.5	5	8
20-29	19.5-29.5	8	16
30-39	29.5-39.5	9	25
40-49	39.5-49.5	13	38
50-59	49.5-59.5	6	44

$$\frac{N}{2} = \frac{44}{2} = 22$$

$\therefore$ Median class is 29.5 – 39.5 as its c.f. is 25, which is just greater than 22.

Now, l = 29.5, N = 44, C = 16, f = 9, i = 39.5 – 29.5 = 10

$$\therefore \quad \text{Median} = l + \frac{\frac{N}{2} - C}{f} \times i = 29.5 + \frac{22 - 16}{9} \times 10$$

$$= 29.5 + \frac{6}{9} \times 10 = 29.5 + \frac{20}{3}$$

$$= 29.5 + 6.66 = 36.16$$

CHECK YOUR PROGRESS 17.2

Find the median of the following data :

1.

x_i	6	11	16	21	26
f_i	5	3	6	4	7

2.

x_i	5	10	15	20	25
f_i	5	25	29	17	9

3.

Marks	0-5	5-10	10-15	15-20	20-25
Number of Boys	5	9	10	14	12

4.

Age (in years)	17-21	21-26	26-31	31-36	36-41
Number of Boys	5	6	12	7	4

17.5 MEAN DEVIATION FROM MEDIAN

We know that for observations in data the central tendency give us the values about which the data concentrate or cluster. It is also essential to know that how far all observation are, from a measure of central tendency. In other words, in data it is required to know how dispersed the observations are from a given point (or a measure of central tendency). In most of the cases mean deviation from mean and median give us the desired disperson or deviation of the observations. Recall that mean deviation for data is defined as the mean of the absolute values of deviations from '*a*'.

Recall that the deviation of an observation x from a fixed point '*a*' is the difference $x - a$.

So mean deviation about 'a' denoted by M.D (a) is given by

$$\text{M.D. (a)} = \frac{\text{Sum of the absolute values of deviations from 'a'}}{\text{Number of observations}}$$

Methematically we can write

$$\text{M.D.}(a) = \frac{\sum_{i=1}^{n} |x_i - a|}{n}$$

Like wise

$$\text{M.D. (Mean} = \bar{\text{X}}) = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

and

$$\text{M.D.(Median M)} = \frac{1}{n}\sum_{i=1}^{n} |x_i - \text{M}|$$

Example 17.6 Find mean deviation about median for the observation

7, 10, 15, 16, 8, 9, 5, 17, 14

Solution : In order to find median, arrange the given values in ascending order, so we have

5, 7, 8, 9, 10, 14, 15, 16, 17,

Algorithm to find mean deviation about mean/median :

Step 1 : Calculate the mean or median of the data

Step 2 : Find deviations of each observation x_i from mean/median

Step 3 : Find the absolute values of the deviations.
Assolute values can be obtained by dropping the minus sign if it is there

Step 4 : Calculate the mean of the obsolute values of the deviations. This mean will be the required Mean deviation.

$n = 9$,

Median = $\frac{n+1}{2}$th observation

= 5th observation

M = 10.

Deviations of the observation from median i.e. 10 are

	5–10	7–10	8–10	9–10	10–10	14–10	15–10	16–10	17–10
i.e x_i-M are	–5	–3	–2	–1	0	4	5	6	7

Absolute values of the deviations i.e. $|x_i - \text{M}|$ are

5, 3, 2, 1, 0, 4, 5, 6, 7

Now M.D. (M) = $\dfrac{\sum_{i=1}^{n} |x_i - \text{M}|}{n}$

$$= \frac{5+3+2+1+0+4+5+6+7}{10} = \frac{33}{10} = 3.3.$$

17.5.1 MEAN DEVIATION OF GROUPED DATA FROM MEDIAN

Recall that data presented in the following form are called grouped data

(a) Discrete frequency distribution

Observation :	x_1	x_2	x_3	...	x_n
Frequencies :	f_1	f_2	f_3	...	f_n

(b) Continuous frequency distribution :

Observations	$l_1 - u_1$	$l_2 – u_2$	$l_3 - u_3$	...	$l_n – u_n$
Frequencies	f_1	f_2	f_3	...	f_n

For example, marks obtained by 50 students

Marks	0-5	5-10	10-15	15-20	20-25	25-30
Number of Students	8	6	12	10	10	4

Let us now learn to find mean deviation about median by following examples.

Example 17.7 Find the mean deviation about the median for the following data :

x_i	25	20	15	10	5
f_i	7	4	6	3	5
c.f.	7	11	17	20	25

Here N = 25, and we know that median is the $\frac{25+1}{2}$ = 13th observation. This observation lies in the C.f 17, for which corresponding observation is 15.

$\therefore$ Median M = 15

Now deviations and their absolute values are given in following table.

x_i	f_i	x_i – M	$\|x_i$ – M$\|$	$f_i\|x_i$ – M$\|$
25	7	25 – 15 = 10	10	7 × 10 = 70
20	4	20 – 15 = 5	5	4 × 5 = 20
15	6	15 – 15 = 0	0	6 × 0 = 0
10	3	10 – 15 = – 5	5	3 × 5 = 15
5	5	5 – 15 = – 10	10	5 × 10 = 50
	N= Σf_i = 25			$\Sigma f_i \|x_i - M\|$ = 155

$$\therefore \quad \text{Mean Deviation (M)} = \frac{\sum_{i=1}^{n} f_i |x_i - M|}{\sum_{i=1}^{n} f_i} = \frac{155}{25} = 6.2$$

Example 17.8 Find the mean deviation about median for the following data :

Heights (in cm)	95-105	105-115	115-125	125-135	135-145	145-155
Number of Girls	9	15	23	30	13	10

Solution : Let us first find median :

Height (in cm)	Number of Girls (f)	Cumulative frequncy (c.f)
95-105	9	9
105-115	15	24
115-125	23	47
125-135	30	77
135-145	13	90
145-155	10	100

N = 100 $\Rightarrow \frac{N+1}{2} = \frac{101}{2} = 50.5$

$\frac{N}{2} = 50.5$ lies in c.f. 77.

$\therefore$ Median class is corresponding to the c.f. 77 i.e., 125 – 135

Now, Median = $l + \dfrac{\frac{N}{2} - C}{f} \times i$

where l = lower limit of the median class

N = sum of frequencies

C = c.f. of the class just preceding the median class

f = frequency of the median class

and i = width or class-size of the median class

Here, l = 125, N = 100, C = 47, f = 30, i = 10

$$\therefore \quad M = 125 + \frac{50-47}{30} \times 10 = 125 + \frac{3}{3} = 126$$

To find mean deviation let us form the following table :

Height (in cm)	Number of Girls (f)	Mid-value of the heights	Absolute Deviation (x_i – M|)	$f_i\|x_i$ – M|
95-105	9	100	|100-126| = 26	9 × 26 = 234
105-115	15	110	|110-126| = 16	15 × 16 = 240
115-125	23	120	|120 – 126| = 6	23 × 6 = 138
125-135	30	130	|130-126| = 4	30 × 4 = 120
135-145	13	140	|140-126| = 14	13 × 14 = 182
145-155	10	150	|150-126| = 24	10 × 24 = 240
	Σf_i = 100			Σf_i |xi – M| = 1154

$$\therefore \quad \text{Mean Deviation (Median)} = \text{M.D.(M)} = \frac{\sum_{i=1}^{n} f_i |x_i - M|}{\sum_{i=1}^{n} f_i} = \frac{1154}{100} = 11.54.$$

17.5.2 STEP TO FIND MEAN DEVIATION FROM MEDIAN OF A CONTINUOUS FREQUENCY DISTRIBUTION.

Step 1 : Arrange the intervals in ascending order

Step 2 : Write cumulative frequencies

Step 3 : Identify the median class, as the class having c.f. just greater than $\frac{N}{2}$, where N is the total number of obsservations (i.e. sum of all frequencies)

Step 4 : Find the corresponding values for the median class and put in the formula :

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times i$$

where

$l \rightarrow$ lower limit of the madian class

$N \rightarrow$ Sum of frequencies

$C \rightarrow$ c.f. of the class just preceding the median class

$f \rightarrow$ frequency of the median class

$i \rightarrow$ width of the median class

Step 5 : Now form the table for following columns :

Given intervals	Frequencies	Mid-value x_i	Absolute Deviation from Median $\|x_i - M)$	$f_i\|x_i - M\|$

Step 6 : Now calculate

$$\text{M.D.(M)} = \frac{\sum_{i=1}^{n} f_i \,|\, x_i - M \,|}{\sum_{i=1}^{n} f_i}$$

CHECK YOUR PROGRESS 17.3

Find the mean deviation about median of the following data.

1.

x_i	11	12	13	14	16	17	18
f_i	2	3	2	3	1	2	1

2.

x_i	3	6	7	9	11	13
f_i	3	9	11	8	9	6

3.

Weight (in kg)	40-42	42-44	44-46	46-48	48-50
No. of Students	9	13	24	28	6

4.

Age (in years)	0-1	1-2	2-3	3-4	4-5
No. of Children given polio drops	100	155	210	315	65

17.6 VARIANCE AND STANDARD DEVIATION OF RAW DATA

If there are n observations, $x_1, x_2, ..., x_n$, then

$$\text{Variance}\left(\sigma^2\right) = \frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + ... + (x_n - \overline{x})^2}{n}$$

or

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}; \text{ where } \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The standard deviation, denoted by σ, is the positive square root of σ^2. Thus

$$\sigma = +\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}}$$

The following steps are employed to calculate the variance and hence the standard deviation of raw data. The mean is assumed to have been calculated already.

Step 1 : Make a column of deviations from the mean, namely, $x_i - \overline{x}$.

Step 2 (check) : Sum of deviations from mean must be zero, i.e., $\sum_{i=1}^{n}(x_i - \overline{x}) = 0$

Step 3: Square each deviation and write in the column headed $(x_i - \overline{x})^2$.

Step 4 : Find the sum of the column in step 3.

Step 5 : Divide the sum obtained in step 4 by the number of observations. We obtain σ^2.

Step 6 : Take the positive square root of σ^2. We obtain σ (Standard deviation).

Example 17.9 The daily sale of sugar in a certain grocery shop is given below :

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
75 kg	120 kg	12 kg	50 kg	70.5 kg	140.5 kg

The average daily sale is 78 Kg. Calculate the variance and the standard deviation of the above data.

Solution : $\overline{x} = 78$ kg (Given)

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
75	– 3	9
120	42	1764
12	– 66	4356
50	– 28	784
70.5	– 7.5	56.25
140.5	62.5	3906.25
	0	10875.50

Thus $$\sigma^2 = \frac{\sum_{i=1} (x_i - \overline{x})^2}{n} = \frac{10875.50}{6} = 1812.58 \text{ (approx.)}$$

and $\sigma = 42.57$ (approx.)

Example 17.10 The marks of 10 students of section A in a test in English are given below :

7 10 12 13 15 20 21 28 29 35

Determine the variance and the standard deviation.

Solution : Here $\overline{x} = \frac{\sum x_i}{10} = \frac{190}{10} = 19$

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
7	–12	144
10	–9	81
12	–7	49
13	–6	36
15	–4	16
20	+1	1
21	+2	4
28	+9	81
29	+10	100
35	+16	256
	0	768

Thus $\sigma^2 = \frac{768}{10} = 76.8$ and $\sigma = +\sqrt{76.8} = 8.76$ (approx)

CHECK YOUR PROGRESS 17. 4

1. The salary of 10 employees (in rupees) in a factory (per day) is

   50 60 65 70 80 45 75 90 95 100

   Calculate the variance and standard deviation.

2. The marks of 10 students of class X in a test in English are given below :

   9 10 15 16 18 20 25 30 32 35

   Determine the variance and the standard deviation.

3. The data on relative humidity (in %) for the first ten days of a month in a city are given below:

   90 97 92 95 93 95 85 83 85 75

   Calculate the variance and standard deviation for the above data.

4. Find the standard deviation for the data

   4 6 8 10 12 14 16

5. Find the variance and the standard deviation for the data

   4 7 9 10 11 13 16

6. Find the standard deviation for the data.

   40 40 40 60 65 65 70 70 75 75 75 80 85 90 90 100

17. 7 STANDARD DEVIATION AND VARIANCE OF RAW DATA AN ALTERNATE METHOD

If $\overline{x}$ is in decimals, taking deviations from $\overline{x}$ and squaring each deviation involves even more decimals and the computation becomes tedious. We give below an alternative formula for computing σ^2. In this formula, we by pass the calculation of $\overline{x}$.

We know

$$\sigma^2 = \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{n} = \sum_{i=1}^{n} \frac{x_i^2 - 2x_i\overline{x} + \overline{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \frac{2\overline{x}\sum_{i=1}^{n} x_i}{n} + \overline{x}^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \overline{x}^2, \quad \left(\because \overline{x} = \frac{\sum x_i}{n}\right)$$

i.e.

$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

And $\sigma = +\sqrt{\sigma^2}$

The steps to be employed in calculation of σ^2 and, hence σ by this method are as follows :

Step 1 : Make a column of squares of observations i.e. x_i^2.

Step 2 : Find the sum of the column in step 1. We obtain $\sum_{i=1}^{n} x_i^2$

Step 3 : Substitute the values of $\sum_{i=1}^{n} x_i^2$, n and $\sum_{i=1}^{n} x_i$ in the above formula. We obtain σ^2.

Step 4 : Take the positive sauare root of σ^2. We obtain σ.

Example 17.11 We refer to Example 17.10 of this lesson and re-calculate the variance and standard deviation by this method.

Solution :

x_i	x_i^2
7	49
10	100
12	144
13	169
15	225
20	400
21	441
28	784
29	841
35	1225
190	4378

$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

$$= \frac{4378 - \frac{(190)^2}{10}}{10}$$

$$= \frac{4378 - 3610}{10} = \frac{768}{10} = 76.8$$

and $\sigma = +\sqrt{76.8} = 8.76$ (approx)

We observe that we get the same value of σ^2 and σ by either methods.

17.8 STANDARD DEVIATION AND VARIANCE OF GROUPED DATA : METHOD - I

We are given k classes and their corresponding frequencies. We will denote the variance and the standard deviation of grouped data by σ_g^2 and σ_g respectively. The formulae are given below :

$$\sigma_g^2 = \frac{\sum_{i=1}^{K}\left[f_i\,(x_i - \bar{x})^2\right]}{N}, \quad N = \sum_{i=1}^{K} f_i \quad \text{and} \quad \sigma_g = +\sqrt{\sigma_g^2}$$

The following steps are employed to calculate σ_g^2 and, hence σ_g : (The mean is assumed to have been calculated already).

Step 1 : Make a column of class marks of the given classes, namely x_i

Step 2 : Make a column of deviations of class marks from the mean, namely, $x_i - \bar{x}$. Of course the sum of these deviations need not be zero, since x_i's are no more the original observations.

Step 3 : Make a column of squares of deviations obtained in step 2, i.e., $(x_i - \bar{x})^2$ and write in the column headed by $(x_i - \bar{x})^2$.

Step 4 : Multiply each entry in step 3 by the corresponding frequency.

We obtain $f_i\,(x_i - \bar{x})^2$.

Step 5 : Find the sum of the column in step 4. We obtain $\sum_{i=1}^{k}\left[f_i\,(x_i - \bar{x})^2\right]$

Step 6 : Divide the sum obtained in step 5 by N (total no. of frequencies). We obtain σ_g^2.

Step 7 : $\sigma_g = +\sqrt{\sigma_g^2}$

Example 17.12 In a study to test the effectiveness of a new variety of wheat, an experiment was performed with 50 experimental fields and the following results were obtained :

Yield per Hectare (in quintals)	Number of Fields
31 – 35	2
36 – 40	3
41 – 45	8
46 – 50	12
51 – 55	16
56 – 60	5
61 – 65	2
66 – 70	2

The mean yield per hectare is 50 quintals. Determine the variance and the standard deviation of the above distribution.

Solution :

Yield per Hectare (in quintal)	No. of Fields	Class Marks	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
31 – 35	2	33	– 17	289	578
36 – 40	3	38	– 12	144	432
41 – 45	8	43	– 7	49	392
46 – 50	12	48	– 2	4	48
51 – 55	16	53	+3	9	144
56 – 60	5	58	+8	64	320
61 – 65	2	63	+13	169	338
66 – 70	2	68	+18	324	648
Total	50				2900

Thus $\sigma_g^2 = \dfrac{\sum_{i=1}^{n}\left[f_i(x_i - \bar{x})^2\right]}{N} = \dfrac{2900}{50} = 58$ and $\sigma_g = +\sqrt{58} = 7.61$ (approx)

17.9 STANDARD DEVIATION AND VARIANCE OF GROUPED DATA :– METHOD - II

If $\bar{x}$ is not given or if $\bar{x}$ is in decimals in which case the calculations become rather tedious, we employ the alternative formula for the calculation of σ_g^2 as given below:

$$\sigma_g^2 = \frac{\sum_{i=1}^{k}\left[f_i x_i^2\right] - \dfrac{\left(\sum_{i=1}^{k}\left[f_i x_i\right]\right)^2}{N}}{N}, \qquad N = \sum_{i=1}^{k} f_i$$

and $\qquad \sigma_g = +\sqrt{\sigma_g^2}$

The following steps are employed in calculating σ_g^2, and, hence σ_g by this method:

Step 1 : Make a column of class marks of the given classes, namely, x_i.

Step 2 : Find the product of each class mark with the corresponding frequency. Write the product in the column $x_i f_i$.

Step 3 : Sum the entries obtained in step 2. We obtain $\sum_{i=1}^{k}(f_i x_i)$.

Step 4 : Make a column of squares of the class marks of the given classes, namely, x_i^2.

Step 5 : Find the product of each entry in step 4 with the corresponding frequency. We obtain $f_i {x_i}^2$.

Step 6 : Find the sum of the entries obtained in step 5. We obtain $\sum_{i=1}^{k}\left(f_i x_i^2\right)$.

Step 7 : Substitute the values of $\sum_{i=1}^{k}\left(f_i x_i^2\right)$, N and $\left(\sum_{i=1}^{k}(f_i x_i)\right)$ in the formula and obtain σ_g^2.

Step 8 : $\sigma_g = +\sqrt{\sigma_g^2}$.

Example 17.13 Determine the variance and standard deviation for the data given in Example 17.12 by this method.

Solution :

Yields per Hectare (in quintals)	f_i	x_i	$f_i x_i$	${x_i}^2$	$f_i {x_i}^2$
31 – 35	2	33	66	1089	2178
36 – 40	3	38	114	1444	4332
41 – 45	8	43	344	1849	14792
46 – 50	12	48	576	2304	27648
51 – 55	16	53	848	2809	44944
56 – 60	5	58	290	3364	16820
61 – 65	2	63	126	3969	7938
66 – 77	2	68	136	4624	9248
Total	50		2500		127900

Substituting the values of $\sum_{i=1}^{k}\left(f_i x_i^2\right)$, N and $\sum_{i=1}^{k}\left(f_i x_i\right)$ in the formula, we obtain

$$\sigma_g^2 = \frac{127900 - \dfrac{(2500)^2}{50}}{50} = \frac{2900}{50} = 58$$

and $\quad \sigma_g = +\sqrt{58} = 7.61 \text{ (approx.)}$

Again, we observe that we get the same value of σ_g^2, by either of the methods.

CHECK YOUR PROGRESS 17.5

1. In a study on effectiveness of a medicine over a group of patients, the following results were obtained :

Percentage of relief	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
No. of patients	10	10	25	15	40

   Find the variance and standard deviation.

2. In a study on ages of mothers at the first child birth in a village, the following data were available :

Age (in years) at first child birth	18 – 20	20 – 22	22 – 24	24 – 26	26 – 28	28 – 30	30 – 32
No. of mothers	130	110	80	74	50	40	16

   Find the variance and the standard deviation.

3. The daily salaries of 30 workers are given below:

Daily salary (In Rs.)	0 – 50	50 – 100	100 – 150	150 – 200	200 – 250	250 – 300
No. of workers	3	4	5	7	8	3

   Find variance and standard deviation for the above data.

17.10 STANDARD DEVIATION AND VARIANCE: STEP DEVIATION METHOD

In Example 17.12, we have seen that the calculations were very complicated. In order to simplify the calculations, we use another method called the step deviation method. In most of the frequency distributions, we shall be concerned with the equal classes. Let us denote, the class size by h.

Now we not only take the deviation of each class mark from the arbitrary chosen 'a' but also divide each deviation by h. Let

$$u_i = \frac{x_i - a}{h} \qquad .....(1)$$

Then $$x_i = hu_i + a \qquad .....(2)$$

We know that $$\bar{x} = h\bar{u} + a \qquad .....(3)$$

Subtracting (3) from (2) , we get

$$x_i - \bar{x} = h(u_i - \bar{u}) \qquad .....(4)$$

In (4) , squaring both sides and multiplying by f_i and summing over k, we get

$$\sum_{i=1}^{k}\left[f_i (x_i - \bar{x})^2\right] = h^2 \sum_{i=1}^{k}\left[f_i (u_i - \bar{u})^2\right] \qquad .....(5)$$

Dividing both sides of (5) by N, we get

$$\frac{\sum_{i=1}^{k}\left[f_i (x_i - \bar{x})^2\right]}{N} = \frac{h^2}{N}\sum_{i=1}^{k}\left[f_i (u_i - \bar{u})^2\right]$$

i.e. $$\sigma_x^2 = h^2 \sigma_u^2 \qquad .....(6)$$

where $\sigma_x{}^2$ is the variance of the original data and $\sigma_u{}^2$ is the variance of the coded data or coded variance. $\sigma_u{}^2$ can be calculated by using the formula which involves the mean, namely,

$$\sigma_u^2 = \frac{1}{N}\sum_{i=1}^{k}\left[f_i (u_i - \bar{u})^2\right], \qquad N = \sum_{i=1}^{k} f_i \qquad .....(7)$$

or by using the formula which does not involve the mean, namely,

$$\sigma_u^2 = \frac{\sum_{i=1}^{k}\left[f_i u_i^2\right] - \frac{\left(\sum_{i=1}^{k}\left[f_i u_i\right]\right)^2}{N}}{N}, \qquad N = \sum_{i=1}^{k} f_i \qquad .....(8)$$

Example 17.14 We refer to the Example 17.12 again and find the variance and standard deviation using the coded variance.

Solution : Here h = 5 and let a = 48.

Yield per Hectare (in quintal)	Number of fields f_i	Class marks x_i	$u_i = \frac{x_i - 48}{5}$	$f_i u_i$	$u_i{}^2$	$f_i u_i{}^2$
31 – 35	2	33	–3	–6	9	18
36 – 40	3	38	–2	–6	4	12

41 – 45	8	43	– 1	– 8	1	8
46 – 50	12	48	0	0	0	0
51 – 55	16	53	+1	16	1	16
56 – 60	5	58	+2	10	4	20
61 – 65	2	63	+3	6	9	18
66 – 70	2	68	+4	8	16	32
Total	50			20		124

Thus

$$\sigma_u^2 = \frac{\sum_{i=1}^{k} f_i u_i^2 - \frac{\left(\sum_{i=1}^{k} f_i u_i\right)^2}{N}}{N}$$

$$= \frac{124 - \frac{(20)^2}{50}}{50} = \frac{124 - 8}{50} \quad \text{or} \quad \sigma_u^2 = \frac{58}{25}$$

Variance of the original data will be

$$\sigma_x^2 = h^2 \sigma_u^2 = 25 \times \frac{58}{25} = 58$$

and

$$\sigma_x = +\sqrt{58}$$

$$= 7.61 \text{ (approx)}$$

We, of course, get the same variance, and hence, standard deviation as before.

Example 17.15 Find the standard deviation for the following distribution giving wages of 230 persons.

Wages (in Rs)	No. of persons	Wages (in Rs)	No. of persons
70 – 80	12	110 – 120	50
80 – 90	18	120 – 130	45
90 – 100	35	130 – 140	20
100 – 110	42	140 – 150	8

Solution :

Wages (in Rs.)	No. of persons f_i	class mark x_i	$u_i = \frac{x_i - 105}{10}$	u_i^2	$f_i u_i$	$f_i u_i^2$
70 – 80	12	75	–3	9	–36	108
80 – 90	18	85	–2	4	–36	72
90 – 100	35	95	–1	1	–35	35
100 – 110	42	105	0	0	0	0
110 – 120	50	115	+1	1	50	50
120 – 130	45	125	+2	4	90	180
130 – 140	20	135	+3	9	60	180
140 – 150	8	145	+4	16	32	128
Total	230				125	753

$$\sigma^2 = h^2\left[\frac{1}{N}\sum\left[f_i u_i^2\right] - \left(\frac{1}{N}\sum\left[f_i u_i\right]\right)^2\right]$$

$$= 100\left[\frac{753}{230} - \left(\frac{125}{230}\right)^2\right] = 100\,(3.27 - 0.29) = 298$$

$\therefore \quad \sigma = +\sqrt{298} = 17.3$ (approx)

CHECK YOUR PROGRESS 17.6

1. The data written below gives the daily earnings of 400 workers of a flour mill.

Weekly earning (in Rs.)	No. of Workers
80 – 100	16
100 – 120	20
120 – 140	25
140 – 160	40
160 – 180	80
180 – 200	65
200 – 220	60
220 – 240	35
240 – 260	30
260 – 280	20
280 – 300	9

Calculate the variance and standard deviation using step deviation method.

2. The data on ages of teachers working in a school of a city are given below:

Age (in years)	20 – 25	25 – 30	30 – 35	35 – 40
No. of teachers	25	110	75	120
Age (in years)	40 – 45	45 – 50	50 – 55	55 – 60
No. of teachers	100	90	50	30

Calculate the variance and standard deviation using step deviation method.

3. Calculate the variance and standard deviation using step deviation method of the following data :

Age (in years)	25 – 30	30 – 35	35 – 40
No. of persons	70	51	47
Age (in years)	40 – 50	45 – 50	50 – 55
No. of persons	31	29	22

17.11 PROPERTIES OF VARIANCE AND STANDARD DEVIA TION

Property I : The variance is independent of change of origin.

To verify this property let us consider the example given below.

Example : 17.16 The marks of 10 students in a particular examination are as follows:

10 12 15 12 16 20 13 17 15 10

Later, it was decided that 5 bonus marks will be awarded to each student. Compare the variance and standard deviation in the two cases.

Solution : ***Case –I***

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
10	2	20	– 4	16	32
12	2	24	– 2	4	8
13	1	13	– 1	1	1
15	2	30	1	1	2
16	1	16	2	4	4
17	1	17	3	9	9
20	1	20	6	36	36
	10	140			92

Here $$\bar{x} = \frac{140}{10} = 14$$

Variance $= \frac{\sum\left[f_i (x_i - \bar{x})^2\right]}{10}$

$= \frac{92}{10} = 9.2$

Standard deviation $= +\sqrt{9.2} = 3.03$

Case – II (By adding 5 marks to each x_i)

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i (x_i - \bar{x})^2$
15	2	30	–4	16	32
17	2	34	–2	4	8
18	1	18	–1	1	1
20	2	40	1	1	2
21	1	21	2	4	4
22	1	22	3	9	9
25	1	25	6	36	36
	10	190			92

$$\bar{x} = \frac{190}{10} = 19$$

$\therefore$ Variance $= \frac{92}{10} = 9.2$

Standard deviation $= +\sqrt{9.2} = 3.03$

Thus, we see that there is no change in variance and standard deviation of the given data if the origin is changed i.e., if a constant is added to each observation.

Property II : The variance is not independent of the change of scale.

Example 17.17 In the above example, if each observation is multiplied by 2, then discuss the change in variance and standard deviation.

Solution : In case-I of the above example , we have variance = 9.2, standard deviation = 3.03.

Now, let us calculate the variance and the Standard deviation when each observation is multiplied by 2.

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i (x_i - \bar{x})^2$
20	2	40	–8	64	128
24	2	48	–4	16	32
26	1	26	–2	4	4
30	2	60	2	4	8
32	1	32	4	16	16
34	1	34	6	36	36
40	1	40	12	144	144
	10	280			368

$$\overline{x} = \frac{280}{10} = 28, \text{ Variance} = \frac{368}{10} = 36.8$$

$$\text{Standard deviation} = +\sqrt{36.8} = 6.06$$

Here we observe that, the variance is four times the original one and consequently the standard deviation is doubled.

In a similar way we can verify that if each observation is divided by a constant then the variance of the new observations gets divided by the square of the same constant and consequently the standard deviation of the new observations gets divided by the same constant.

Property III : Prove that the standard deviation is the least possible root mean square deviation.

Proof : Let $\overline{x} - a = d$

By definition, we have

$$s^2 = \frac{1}{N}\sum\left[f_i(x_i - a)^2\right] = \frac{1}{N}\sum\left[f_i(x_i - \overline{x} + \overline{x} - a)^2\right]$$

$$= \frac{1}{N}\sum f_i\left[(x_i - \overline{x})^2 + 2(x_i - \overline{x})(\overline{x} - a) + (\overline{x} - a)^2\right]$$

$$= \frac{1}{N}\sum f_i(x_i - \overline{x})^2 + \frac{2}{N}(\overline{x} - a)\sum f_i(x_i - \overline{x}) + \frac{(\overline{x} - a)^2}{N}\sum f_i$$

$$= \sigma^2 + 0 + d^2$$

$\therefore$ The algebraic sum of deviations from the mean is zero

or $$s^2 = \sigma^2 + d^2$$

Clearly s^2 will be least when d = 0 i.e., when $a = \overline{x}$.

Hence the root mean square deviation is the least when deviations are measured from the mean i.e., the standard deviation is the least possible root mean square deviation.

Property IV : The standard deviations of two sets containing n_1, and n_2 numbers are σ_1 and σ_2 respectively being measured from their respective means m_1 and m_2. If the two sets are grouped together as one of $(n_1 + n_2)$ numbers, then the standard deviation σ of this set, measured from its mean m is given by

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(m_1 - m_2)^2$$

Example 17.18 The means of two samples of sizes 50 and 100 respectively are 54.1 and 50.3; the standard deviations are 8 and 7. Find the standard deviation of the sample of size 150 by combining the two samples.

Solution : Here we have

$$n_1 = 50, n_2 = 100, m_1 = 54.1, m_2 = 50.3$$

$$\sigma_1 = 8 \text{ and } \sigma_2 = 7$$

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{(n_1 + n_2)} + \frac{n_1 n_2}{(n_1 + n_2)^2}(m_1 - m_2)^2$$

$$= \frac{(50 \times 64) + (100 \times 49)}{150} + \frac{50 \times 100}{(150)^2}(54.1 - 50.3)^2$$

$$= \frac{3200 + 4900}{150} + \frac{2}{9}(3.8)^2 = 57.21$$

$\therefore$ $\sigma = 7.56$ (approx)

Example 17.19 Find the mean deviation (M.D) from the mean and the standard deviation (S.D) of the A.P.

$$a, a + d, a + 2\,d, \ldots\ldots, a + 2n.d$$

and prove that the latter is greater than the former.

Solution : The number of items in the A.P. is (2n + 1)

$\therefore$ $\bar{x} = a + nd$

Mean deviation about the mean

$$= \frac{1}{(2n+1)}\sum_{r=0}^{2n}\left|(a + rd) - (a + nd)\right|$$

$$= \frac{1}{(2n+1)}.2\left[nd + (n-1)d + \ldots\ldots + d\right]$$

$$= \frac{2}{(2n+1)}\left[1 + 2 + \ldots\ldots + (n-1) + n\right]d$$

$$= \frac{2n(n+1)}{(2n+1)2}.d \qquad = \frac{n(n+1)d}{(2n+1)} \qquad \ldots\ldots(1)$$

Now $$\sigma^2 = \frac{1}{(2n+1)}\sum_{r=0}^{2n}\left[(a + rd) - (a + nd)\right]^2$$

$$= \frac{2d^2}{(2n+1)}\left[n^2 + (n-1)^2 + \ldots + 2^2 + 1^2\right]$$

$$= \frac{2d^2}{(2n+1)} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)d^2}{3}$$

$$\therefore \qquad \sigma = d.\sqrt{\left(\frac{n(n+1)}{3}\right)} \qquad .....(2)$$

We have further, (2) > (1)

if $$d\sqrt{\left(\frac{n(n+1)}{3}\right)} > \frac{n(n+1)}{(2n+1)}d$$

or if $$(2n+1)^2 > 3n(n+1)$$

or if $n^2 + n + 1 > 0$, which is true for $n > 0$

Hence the result.

Example 17.20 Show that for any discrete distribution the standard deviation is not less than the mean deviation from the mean.

Solution : We are required to show that

S.D. $\geq$ M.D. from mean

or $$(\text{S. D})^2 \geq (\text{M.D. from mean})^2$$

i.e. $$\frac{1}{N}\sum\left[f_i(x_i - \overline{x})^2\right] \geq \left[\frac{1}{N}\sum\left[f_i\left|(x_i - \overline{x})\right|\right]\right]^2$$

or $$\frac{1}{N}\sum\left[f_i d_i^2\right] \geq \left[\frac{1}{N}\sum\left[f_i|d_i|\right]\right]^2, \text{ where } d_i = x_i - \overline{x}$$

or $$N\sum\left(f_i d_i^2\right) \geq \left[\sum\{f_i|d_i|\}\right]^2$$

or $$(f_1 + f_2 + ...)\left(f_1 d_1^2 + f_2 d_2^2 + ...\right) \geq [f_1|d_1| + f_2|d_2| + ...]^2$$

or $$f_1 f_2\left(d_1^2 + d_2^2\right) + ... \geq 2f_1 f_2\, d_1 d_2 + ...$$

or $$f_1 f_2 (d_1 - d_2)^2 + ... \geq 0$$

which is true being the sum of perfect squares.

17.12 ANALYSIS OF FREQUENCY DISTRIBUTIONS WITH EQUAL MEANS

The variability of two series with same mean can be compared when the measures of variation are absolute and are free of units. For this, coefficient of variation (C.V.) is obtained which is defined as

$$\text{C.V.} = \frac{\sigma}{\overline{x}} \times 100, \ \overline{x} \neq 0$$

where σ and $\overline{x}$ are standard deviation and mean of the data. The coefficients of variation are compared to compare the variability of two series. The series with greater C.V. is said to be more variable than the other. The series having less C.V. is said to be more consistent than the other.

For series with same means, we can have

$$\text{C.V. (1st distribution)} = \frac{\sigma_1}{\overline{x}} \times 100 \qquad \ldots(1)$$

$$\text{C.V. (2nd distribution)} = \frac{\sigma_2}{\overline{x}} \times 100 \qquad \ldots(2)$$

where σ_1, σ_2 are standard deviation of the Ist and 2nd distribution respectively, $\overline{x}$ is the equal mean of the distributions.

From (1) and (2), we can conclude that two C.V.'s can be compared on the basis of the values of σ_1 and σ_2 only.

Example 17.21 The standard deviation of two distributions are 21 and 14 and their equal mean is 35. Which of the distributions is more variable?

Solution : Let

$$\sigma_1 = \text{Standard dev. of 1st series} = 21$$

$$\sigma_2 = \text{Standard dev. of 2nd series} = 14$$

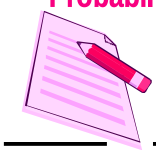
$$\overline{x} = 35$$

$$\text{C.V. (Series I)} = \frac{\sigma_1}{\overline{x}} \times 100 = \frac{21}{35} \times 100 = 60$$

$$\text{C.V. (Series II)} = \frac{\sigma_1}{\overline{x}} \times 100 = \frac{14}{35} \times 100 = 40$$

C.V. of series I > C.V. of series II

$\Rightarrow$ Series with S.D = 21 is more variable.



Example 17.22 Monthly wages paid to workers in two factories A and B and other data are given below :

	Factory A	**Factory B**
Mean of monthly wages	` 15550	` 15550
Variance of the distribution of wages	100	121

Which factory A or B shows greater variablility in individual wages?

Solution : Given

$$\sigma_A = \sqrt{\text{variance}} = \sqrt{100} = 10$$

$$\sigma_B = \sqrt{\text{variance}} = \sqrt{121} = 11$$

$$\overline{x} = \text{` } 15550$$

Now,

$$\text{C.V. (A)} = \frac{\sigma_A}{\overline{x}} \times 100 = \frac{10}{15550} \times 100$$

$$= 0.064$$

$$\text{C.V.(B)} = \frac{\sigma_B}{\overline{x}} \times 100 = \frac{11}{15550} \times 100 = 0.07$$

Clearly C.V. (B) > C.V.(A)

∴ Factory B has greater variability in the individual wages.

Example 17.23 Which of the following series X or Y is more consistent?

X	58	52	50	51	49	35	54	52	53	56
Y	101	104	103	104	107	106	105	105	107	108

Solution : From the given data we have following table

X	Y	$D_i = X - \overline{X}$	D_i^2	$d_i = Y - \overline{Y}$	d_i^2
58	101	7	49	–4	16
52	104	1	1	–1	1
50	103	–1	1	–2	4
51	104	0	0	–1	1
49	107	–2	4	2	4
35	106	–16	256	1	1
54	105	3	9	0	0
52	105	1	1	0	0
53	107	2	4	2	4
56	108	5	25	3	9
$\Sigma X = 510$	$\Sigma Y = 1050$		$\Sigma D_i^2 = 350$		$\Sigma d_i^2 = 40$

Now, $\bar{X} = \frac{\Sigma X_i}{10} = \frac{510}{10} = 51$

$$\bar{Y} = \frac{\Sigma Y_i}{10} = \frac{1050}{10} = 105$$

$$\sigma_x = \sqrt{\frac{\Sigma(X-\bar{X})^2}{N}} = \sqrt{\frac{\Sigma D_i^2}{N}} = \sqrt{\frac{350}{10}}$$

$$= 5.9$$

$$\sigma_y = \sqrt{\frac{\Sigma(Y-\bar{Y})^2}{N}} = \sqrt{\frac{\Sigma d_i^2}{N}} = \sqrt{\frac{40}{10}} = 2$$

Now, $\text{C.V.(X)} = \frac{\sigma_X}{X} \times 100 = \frac{5.9}{51} \times 100 = 11.5$

$$\text{C.V.(Y)} = \frac{\sigma_Y}{Y} \times 100 = \frac{2}{105} \times 100 = 1.9$$

Clearly C.V.(Y) < C.V.(X) $\therefore$ Series Y is more consistent.

CHECK YOUR PROGRESS 17.7

1. From the data given below which section is more variable?

Marks	0-10	10-20	20-30	30-40	40-50
Section A	9	10	40	33	8
Section B	8	15	43	25	9

2. Which of the factory give better consistent wages to workers?

Wages (in `) per day	100-150	150-200	200-250	250-300	300-350
Factory A	35	45	50	42	28
Factory B	16	50	55	13	46

3. Two schools show following results of board examination in a year

	School A	School B
Average Marks Obtained	250	225
No. of Students Appeared	62	62
Variance of distribution of marks	2.25	2.56

Which school has greater variability in individual marks?

LET US SUM UP

- Range : The difference between the largest and the smallest value of the given data.

- Mean deviation from mean $= \dfrac{\sum_{i=1}^{n}(f_i|x_i - \overline{x}|)}{N}$

where $N = \sum_{i=1}^{n} f_i$, $\overline{x} = \dfrac{1}{N}\sum_{i=1}^{n}(f_i x_i)$

- Mean deviation from median $= \dfrac{\sum_{i=1}^{n} fi|xi - m|}{N}$

Where $N = \dfrac{\sum_{i=1}^{n} fi}{N}$,

$$M = l + \frac{\frac{N}{2} - C}{f} \times i$$

- Variance $\left(\sigma^2\right) = \dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}$ [for raw data]

- Standard derivation $(\sigma) = +\sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}}$

- Variance for grouped data

$\sigma_g^2 = \dfrac{\sum_{i=1}^{k}\left[f_i(x_i - \overline{x})^2\right]}{N}$, x_i is the mid value of the class.

Also, $\sigma_x{}^2 = h^2\sigma_u{}^2$ and $\sigma_u{}^2 = \dfrac{1}{N}\sum_{i=1}^{k}\left[f_i\left(u_i - \overline{u}^2\right)\right]$

$$N = \sum_{i=1}^{k} f_i$$

or $$\sigma_u^{\,2} = \frac{\sum_{i=1}^{k}\left(f_i u_i^{\,2}\right) - \frac{\left[\sum_{i=1}^{k}\left(f_i u_i^{\,2}\right)\right]^2}{N}}{N} \quad \text{where } N = \sum_{i=1}^{k} f_i$$

- Standard deviation for grouped data $\sigma_g = +\sqrt{\sigma_g^2}$
- If two frequency distributions have same mean, then the distribution with greater Coefficient of variation (C.V) is said to be more variable than the other.

SUPPORTIVE WEB SITES

http:// en.wikipedia.org/wiki/Statistical_**dispersion**

simon.cs.vt.edu/SoSci/converted/**Dispersion**_I/activity.html

TERMINAL EXERCISE

1. Find the mean deviation for the following data of marks obtained (out of 100) by 10 students in a test

   55 45 63 76 67 84 75 48 62 65

2. The data below presents the earnings of 50 labourers of a factory

Earnings (in Rs.)	1200	1300	1400	1500	1600	1800
No. of Labourers	4	7	15	12	7	5

   Calculate mean deviation.

3. The salary per day of 50 employees of a factory is given by the following data.

Salary (in Rs.)	20 – 30	30 – 40	40 – 50	50 – 60
No. of employees	4	6	8	12
Salary (in rupees)	60 – 70	70 – 80	80 – 90	90 – 100
No. of employees	7	6	4	3

   Calculate mean deviation.

4. Find the batting average and mean deviation for the following data of scores of 50 innings of a cricket player:

Run Scored	0 – 20	20 – 40	40 – 60	60 – 80
No. of Innings	6	10	12	18
Run scored	80 – 100	100 – 120		
No. of innings	3	1		

5. The marks of 10 students in a test of Mathematics are given below:

6	10	12	13	15	20	24	28	30	32

Find the variance and standard deviation of the above data.

6. The following table gives the masses in grams to the nearest gram, of a sample of 10 eggs.

46	51	48	62	54	56	58	60	71	75

Calculate the standard deviation of the masses of this sample.

7. The weekly income (in rupees) of 50 workers of a factory are given below:

Income	400	425	450	500	550	600	650
No of workers	5	7	9	12	7	6	4

Find the variance and standard deviation of the above data.

8. Find the variance and standard deviation for the following data:

Class	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
Frequency	7	8	25	15	45

9. Find the standard deviation of the distribution in which the values of x are 1,2,......, N. The frequency of each being one.

10. The following values are calculated in respect of heights and weights of students :

	Weight	Height
Mean	52.5 Kg	160.5 cm
Standard Dev.	11.5	12.2

Which of the attribute weight or height show greater variation?

11. The following are the wickets taken by a bowler in 20 matches, for Player A

No. of Wickets	0	1	2	3	4
No. of Matches	2	6	7	4	1

For the bowler B, mean number of wickets taken in 20 matches is 1.6 with standard deviation 1.25. Which of the players is more consistent?

Find the median of the following distributions (12-14) :

12.

x_i	14	20	26	29	34	46
f_i	4	6	7	8	9	6

13.

Age (in years)	15-19	20-24	25-29	30-34	35-39
Number	8	7	9	11	5

14.

Height (in cm)	95-104	105-114	115-124	125-134	135-144
Number of Boys	10	8	18	8	16

Find mean deviation from median (15-18) :

15.

x_i	5	15	25	35	45	55
f_i	5	23	30	20	16	6

16.

x_i	105	107	109	111	113	115
f_i	8	6	2	2	2	6

17.

Income (per month) (` in '000)	0-5	6-10	11-15	16-20	21-25
Number of Persons	5	6	12	14	26

18.

Age (in years)	0-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40
No. of Persons	5	6	12	14	26	32	16	29

ANSWERS

CHECK YOUR PROGRESS 17.1

1. 15 2. 22 3. 9.4 4. 15.44

5. 13.7 6. 136 7. 5.01 8. 14.4

CHECK YOUR PROGRESS 17.2

1. 16 2. 15 3. 15.35 marks 4. 28 years

CHECK YOUR PROGRESS 17.3

1. 1.85 2. 2.36 3. 3..73 4. 0.977

CHECK YOUR PROGRESS 17.4

1. Variance = 311, Standard deviation = 17.63
2. Variance = 72.9, Standard deviation = 8.5
3. Variance = 42.6, Standard deviation = 6.53
4. Standard deviation = 4
5. Variance = 13.14, Standard deviation = 3.62
6. Standard deviation = 17.6

CHECK YOUR PROGRESS 17.5

1. Variance = 734.96, Standard deviation = 27.1
2. Variance = 12.16, Standard deviation = 3.49
3. Variance = 5489 , Standard deviation = 74.09

CHECK YOUR PROGRESS 17.6

1. Variance = 2194, Standard deviation = 46.84
2. Variance = 86.5 , Standard deviation = 9.3
3. Variance = 67.08 , Standard deviation = 8.19

CHECK YOUR PROGRESS 17.7

1. Section A
2. Factory A
3. School B

TERMINAL EXERCISE

1. 9.4   2. 124.48   3. 15.44   4. 52, 19.8

5. Variance = 72.29, Standard Deviation = 8.5   6. 8.8

7. Variance = 5581.25,Standard Deviation = 74.7

8. Variance = 840, Standard Deviation = 28.9

9. Standard deviation $= \sqrt{\dfrac{N^2 - 1}{12}}$

10. Weight   11. Player B   12. 29   13. 27.27

14. 121.16   15. 10.3   16. 3.38   17. 5.2

18. 0.62

18

RANDOM EXPERIMENTS AND EVENTS

In day-to-day life we see that before commencement of a cricket match two captains go for a toss. Tossing of a coin is an activity and getting either a 'Head' or a "Tail' are two possible outcomes. (Assuming that the coin does not stand on the edge). If we throw a die (of course fair die) the possible outcomes of this activity could be any one of its faces having numerals, namely 1,2,3,4,5 and 6..... at the top face.

An activity that yields a result or an outcome is called an experiment. Normally there are variety of outcomes of an experiment and it is a matter of chance as to which one of these occurs when an experiment is performed. In this lesson, we propose to study various experiments and their outcomes.

OBJECTIVES

After studying this lesson, you will be able to :

- explain the meaning of a random experiments and cite examples thereof;
- explain the role of chance in such random experiments;
- define a sample space corresponding to an experiment;
- write a sample space corresponding to a given experiment; and
- differentiate between various types of events such as equally likely, mutually exclusive, exhaustive, independent and dependent events.

EXPECTED BACKGROUND KNOWLEDGE

- Basic concepts of probability

18.1 RANDOM EXPERIMENT

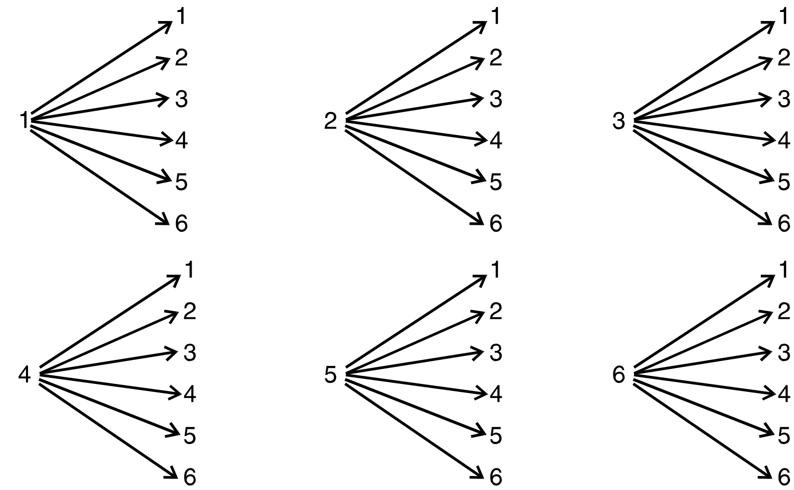
Let us consider the following activities :

(i) Toss a coin and note the outcomes. There are two possible outcomes, either a head (H) or a tail (T).

(ii) In throwing a fair die, there are six possible outcomes, that is, any one of the six faces 1,2,...... 6.... may come on top.

(iii) Toss two coins simultaneously and note down the possible outcomes. There are four possible outcomes, HH,HT,TH,TT.

(iv) Throw two dice and there are 36 possible outcomes which are represented as below :



i.e. outcomes are (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

$$(2,1),(2,2),\ldots,(2,6)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$(6,1),(6,2),\ldots,(6,6)$$

Each of the above mentioned activities fulfil the following two conditions.

(a) The activity can be repeated number of times under identical conditions.

(b) Outcome of an activity is not predictable beforehand, since the chance play a role and each outcome has the same chance of being selection. Thus, due to the chance playing a role, an activity is

(i) repeated under identical conditions, and

(ii) whose outcome is not predictable beforehand is called a random experiment.

Example 18.1 Is drawing a card from well shuffled deck of cards, a random experiment ?

Solution :

(a) The experiment can be repeated, as the deck of cards can be shuffled every time before drawing a card.

(b) Any of the 52 cards can be drawn and hence the outcome is not predictable beforehand.

Hence, this is a random experiment.

Example 18.2 Selecting a chair from 100 chairs without preference is a random experiment. Justify.

Solution :

(a) The experiment can be repeated under identical conditions.

(b) As the selection of the chair is without preference, every chair has equal chances of selection. Hence, the outcome is not predictable beforehand. Thus, it is a random experiment.

Can you think of any other activities which are not random in nature.

Let us consider some activities which are not random experiments.

(i) Birth of Manish : Obviously this activity, that is, the birth of an individual is not repeatable and hence is not a random experiment.

(ii) Multiplying 4 and 8 on a calculator.

Although this activity can be repeated under identical conditions, the outcome is always 32. Hence, the activity is not a random experiment.

18.2 SAMPLE SPACE

We throw a die once, what are possible outcomes ? Clearly, a die can fall with any of its faces at the top. The number on each of the faces is, therefore, a possible outcome. We write the set S of all possible outcomes as , S = {1, 2, 3, 4, 5, 6}

Again, if we toss a coin, the possible outcomes for this experiment are either a head or a tail. We write the set S of all possible outcomes as , S = {H, T}.

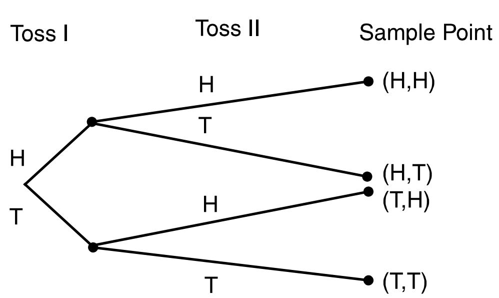
The set S associated with an experiment satisfying the following properties :

(i) each element of S denotes a possible outcome of the experiment.

(ii) any trial results in an outcome that corresponds to one and only one element of the set S is called the sample space of the experiment and the elements are called sample points. Sample space is generally denoted by S.

Example 18.3 Write the sample space in two tosses of a coin.

Solution : Let H denote a head and T denote a tail in the experiment of tossing of a coin.



S = { (H, H), (H, T), (T, H), (T, T) }.

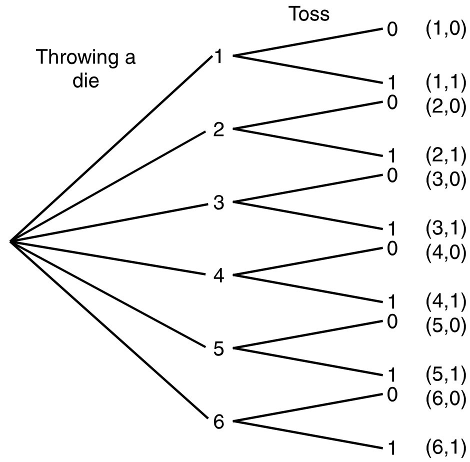
Note : If two coins are tossed simultaneously then the sample space S can be written as

S = { H H, H T, T H, T T}.

Example 18.4 Consider an experiment of rolling a fair die and then tossing a coin.

Write the sample space.

Solution : In rolling a die possible outcomes are 1, 2, 3, 4, 5 and 6. On tossing a coin the possible outcomes are either a head or a tail. Let H (head) = 0 and T (tail) = 1.

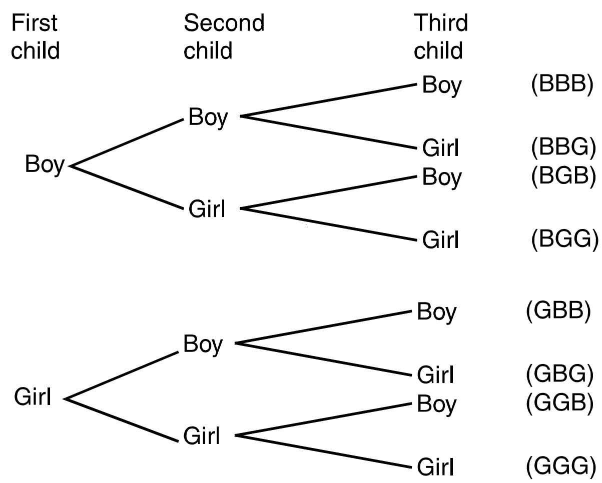


S = {(1, 0), (1, 1), (2, 0), (2, 1),(3, 0), (3, 1),(4, 0), (4, 1), (5, 0), (5, 1), (6, 0), (6, 1)}

$\therefore \quad n(S) = 6 \times 2 = 12$

Example 18.5 Suppose we take all the different families with exactly 3 children. The experiment consists in asking them the sex (or genders) of the first, second and third chid. Write down the sample space.

Solution : Let us write 'B' for boy and 'G' for girl and construct the following tree diagram.



The sample space is

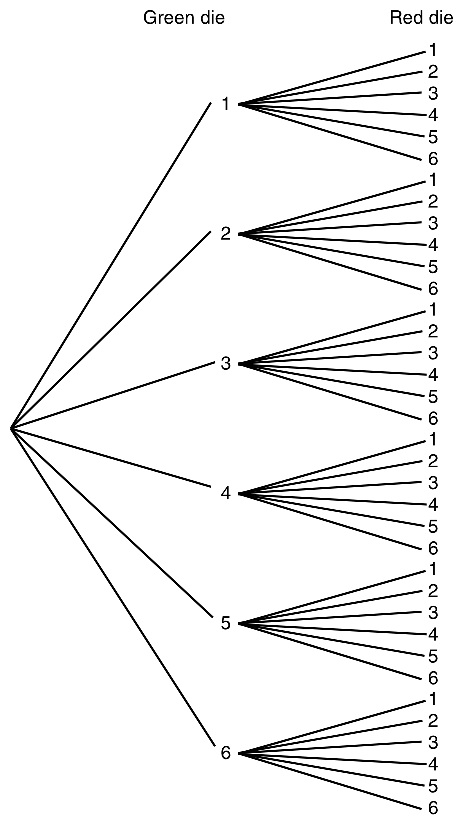
S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

The advantage of writing the sample space in the above form is that a question such as "Was the second child a girl" ? or " How many families have first child a boy ?" and so forth can be answered immediately.

$$n(S) = 2 \times 2 \times 2 = 8$$

Example 18.6 Consider an experiment in which one die is green and the other is red. When these two dice are rolled, what will be the sample space ?

Solution : This experiment can be displayed in the form of a tree diagram, as shown below :



Let g_i and r_j denote, the number that comes up on the green die and red die respectively. Then an out-come can be represented by an ordered pair (g_i, r_j), where i and j can assume any of the values 1, 2, 3, 4, 5, 6.

Thus, a sample space S for this experiment is the set, $S = \{(g_i, r_j) : 1 \le i \le 6, 1 \le j \le 6\}$.

Also, notice that the multiplication principle (principle of counting) shows that the number of elements in S is 36, since there are 6 choices for g and 6 choices for r, and $6 \times 6 = 36$

$$\therefore\ n(S) = 36$$

Example 18.7 Write the sample space for each of the following experiments :

(i) A coin is tossed three times and the result at each toss is noted.

(ii) From five players A, B, C, D and E , two players are selected for a match.

(iii) Six seeds are sown and the number of seeds germinating is noted.

(iv) A coin is tossed twice. If the second throw results in a head, a die is thrown, otherwise a coin is tossed.

Solution :

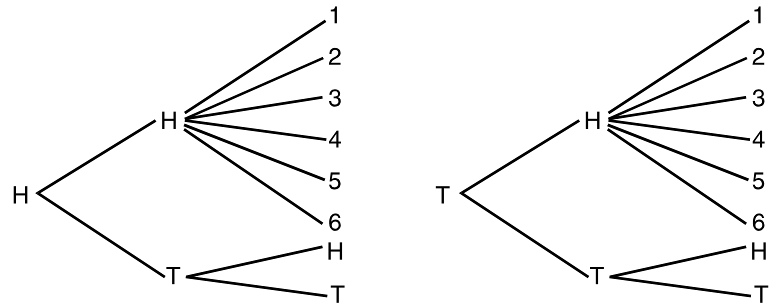
(i) S = { TTT, TTH, THT, HTT, HHT, HTH, THH, HHH}

number of elements in the sample space is $2 \times 2 \times 2 = 8$

(ii) S= { AB, AC, AD, AE, BC, BD, BE, CD, CE, DE}. Here n (S) = 10

(iii) S = { 0, 1, 2, 3, 4, 5, 6}. Here n (S) = 7

(iv) This experiment can be displayed in the form of a tree-diagram as shown below :



Thus S = {HH1, HH2, HH3, HH4, HH5, HH6, HTH, HTT, TH1, TH2,TH3,TH4, TH5, TH6, TTH, TTT}

i.e. there are 16 outcomes of this experiment.

18.3. DEFINITION OF VARIOUS TERMS

Event : Let us consider the example of tossing a coin. In this experiment, we may be interested in 'getting a head'. Then the outcome 'head' is an event.

In an experiment of throwing a die, our interest may be in, 'getting an even number'. Then the outcomes 2, 4 or 6 constitute the event. We have seen that an experiment which, though repeated under identical conditions, does not give unique results but may result in any one of the several possible outcomes, which constitute the sample space.

Some outcomes of the sample space satisfy a specified description, which we call an 'event'.

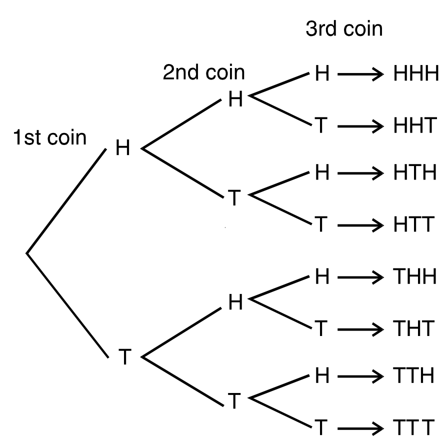
We often use the capital letters A, B, C etc. to represent the events.

Example 18.8 Let E denote the experiment of tossing three coins at a time. List all possible outcomes and the events that

(i) the number of heads exceeds the number of tails.

(ii) getting two heads.

Solution :



The sample space S is

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$$= \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\} \text{ (say)}$$

If E_1 is the event that the number of heads exceeds the number of tails, and E_2 the event getting two heads. Then

$$E_1 = \{ w_1 , w_2 , w_3 , w_5 \}$$

and $$E_2 = \{ w_2 , w_3 , w_5 \}$$

18.3.1 Equally Likely Events

Outcomes of a trial are said to be equally likely if taking into consideration all the relevant evidences there is no reason to expect one in preference to the other.

Examples :

(i) In tossing an unbiased coin, getting head or tail are equally likely events.

(ii) In throwing a fair die, all the six faces are equally likely to come.

(iii) In drawing a card from a well shuffled deck of 52 cards, all the 52 cards are equally likely to come.

18.3.2 Mutually Exclusive Events

Events are said to be mutually exclusive if the happening of any one of the them preludes the happening of all others, i.e., if no two or more of them can happen simultaneously in the same trial.

Examples :

(i) In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive. If any one of these faces comes at the top, the possibility of others, in the same trial is ruled out.

(ii) When two coins are tossed, the event that both should come up tails and the event that there must be at least one head are mutually exclusive.

Mathematically events are said to be mutually exclusive if their intersection is a null set (i.e., empty)

18.3.3 Exhaustive Events

If we have a collection of events with the property that no matter what the outcome of the experiment, one of the events in the collection must occur, then we say that the events in the collection are exhaustive events.

For example, when a die is rolled, the event of getting an even number and the event of getting an odd number are exhaustive events. Or when two coins are tossed the event that at least one head will come up and the event that at least one tail will come up are exhaustive events.

Mathematically a collection of events is said to be exhaustive if the union of these events is the complete sample space.

18.3.4 Independent and Dependent Events

A set of events is said to be independent if the happening of any one of the events does not affect the happening of others. If, on the other hand, the happening of any one of the events influence the happening of the other, the events are said to be dependent.

Examples :

(i) In tossing an unbiased coin the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

(ii) If we draw a card from a pack of well shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however, if the first card drawn is not replaced then the second card is dependent on the first draw (in the sense that it cannot be the card drawn the first time).

CHECK YOUR PROGRESS 18.1

1. Selecting a student from a school without preference is a random experiment. Justify.
2. Adding two numbers on a calculator is not a random experiment. Justify.
3. Write the sample space of tossing three coins at a time.
4. Write the sample space of tossing a coin and a die.
5. Two dice are thrown simultaneously, and we are interested to get six on top of each of the die. Are the two events mutually exclusive or not ?
6. Two dice are thrown simultaneously. The events A, B, C, D are as below :

   A : Getting an even number on the first die.

   B : Getting an odd number on the first die.

   C : Getting the sum of the number on the dice < 7.

   D : Getting the sum of the number on the dice > 7.

   State whether the following statements are True or False.

   (i) A and B are mutually exclusive.

   (ii) A and B are mutually exclusive and exhaustive.

   (iii) A and C are mutually exclusive.

   (iv) C and D are mutually exclusive and exhaustive.
7. A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. There will be how many sample points, in its sample space?
8. In a single rolling with two dice, write the sample space and its elements.
9. Suppose we take all the different families with exactly 2 children. The experiment consists in asking them the sex of the first and second child.

   Write down the sample space.

LET US SUM UP

- An activity that yields a result or an outcome is called an experiment.
- An activity repeated number of times under identical conditions and outcome of activity is not predictable is called Random Experiment.
- The set of possible outcomes of a random experiment is called sample space and elements of the set are called sample points.

- Some outcomes of the sample space satisfy a specified description, which is called an Event.
- Events are said to be Equally likely, when we have no preference for one rather than the other.
- If happening of an event prevents the happening of another event, then they are called Mutually Exclusive Events.
- The total number of possible outcomes in any trial is known as Exhaustive Events.
- A set of events is said to be Independent events, if the happening of any one of the events does not effect the happening of other events, otherwise they are called dependent events.

SUPPORTIVE WEB SITES

www.math.uah.edu/stat/prob/**Events**.html

http://en.wikipedia.org/wiki/**Experiment**_(probability_theory)

TERMINAL EXERCISE

1. A tea set has four cups and saucers. If the cups are placed at random on the saucers, write the sample space.
2. If four coins are tossed, write the sample space.
3. If n coins are tossed simultaneously, there will be how many sample points ?

   [Hint : try for n = 1,2, 3,4, ...]
4. In a single throw of two dice, how many sample points are there ?

ANSWERS

CHECK YOUR PROGRESS 18.1

1. Both properties are satisfied 2. Outcome is predictable

3. S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

4. {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T4, T6} 5. No.

6. (i) True (ii) True (ii) False (iv) True 7. 15

8. $\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)$
$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)$
$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

9. {MM, MF, FM, FF}

TERMINAL EXERCISE

1. $\{C_1S_1, C_1S_2, C_1S_3, C_1S_4, C_2S_1, C_2S_2, C_2S_3, C_2S_4,$
$C_3S_1, C_3S_2, C_3S_3, C_3S_4, C_4S_1, C_4S_2, C_4S_3, C_4S_4\}$

2. $2^4 = 16$, { HHHH, HHHT, HHTH, HTHH, HHTT, HTHT, HTTH, HTTT,
THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

3. 2^n

4. $6^2 = 36$

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PROBABILITY

In our daily life, we often used phrases such as 'It may rain today', or 'India may win the match' or ' I may be selected for this post.' These phrases involve an element of uncertainty. How can we measure this uncertainty? A measure of this uncertainty is provided by a branch of Mathematics, called the theory of probability. Probability Theory is designed to measure the degree of uncertainty regarding the happening of a given event. The dictionary meaning of probability is ' likely though not certain to occur. Thus, when a coin is tossed, a head is likely to occur but may not occur. Similarly, when a die is thrown, it may or may not show the number 6.

In this lesson we shall discuss some basic concepts of probability, addition theorem, dependent and independent events, multiplication theorem, Baye's theorem, ramdom variable, its probability distribution and binomial distribution.

OBJECTIVES

After studying this lesson, you will be able to :

- define probability of occurance of an event;
- cite through examples that probability of occurance of an event is a non-negative fraction, not greater than one;
- use permutation and combinations in solving problems in probability;
- state and establish the addition theorems on probability and the conditions under which each holds;
- generalize the addition theorem of probability for mutually exclusive events;
- understand multiplication law for independent and dependent events and solve problems releated to them.
- understand conditional probability and solve problems releated to it.
- understand Baye's theorem and solve questions related to it.
- define random variable and find its probability distribution.
- understand and find, mean and variance of random variable.
- understand binomial distribution and solve questions based on it.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of random experiments and events.
- The meaning of sample space.

- A standard deck of playing cards consists of 52 cards divided into 4 suits of 13 cards each : spades, hearts, diamonds, clubs and cards in each suit are - ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3 and 2. Kings, Queens and Jacks are called face cards and the other cards are called number cards.

19.1 EVENTS AND THEIR PROBABILITY

In the previous lesson, we have learnt whether an activity is a random experiment or not. The study of probability always refers to random experiments. Hence, from now onwards, the word experiment will be used for a random experiment only. In the preceeding lesson, we have defined different types of events such as equally likely, mutually exclusive, exhaustive, independent and dependent events and cited examples of the above mentioned events.

Here we are interested in the chance that a particular event will occur, when an experiment is performed. Let us consider some examples.

What are the chances of getting a ' Head' in tossing an unbiased coin ? There are only two equally likely outcomes, namely head and tail. In our day to day language, we say that the coin has chance 1 in 2 of showing up a head. In technical language, we say that the probability of getting a head is $\frac{1}{2}$.

Similarly, in the experiment of rolling a die, there are six equally likely outcomes 1, 2,3,4,5 or 6. The face with number '1' (say) has chance 1 in 6 of appearing on the top. Thus, we say that the probability of getting 1 is $\frac{1}{6}$.

In the above experiment, suppose we are interested in finding the probability of getting even number on the top, when a die is rolled. Clearly, the possible numbers are 2, 4 and 6 and the chance of getting an even number is 3 in 6. Thus, we say that the probability of getting an even number is $\frac{3}{6}$, i.e., $\frac{1}{2}$.

The above discussion suggests the following definition of probability.

If an experiment with '*n*' exhaustive, mutually exclusive and equally likely outcomes, *m* outcomes are favourable to the happening of an event A, the probability '*p*' of happening of A is given by

$$p = P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \frac{m}{n} \quad \text{....(i)}$$

Since the number of cases favourable to the non-happening of the event A are $n - m$, the probability 'q' that 'A' will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n}$$

$$= 1 - p \quad \text{[Using (i)]}$$

$$\therefore \quad p + q = 1.$$

Obviously, *p* as well as *q* are non-negative and cannot exceed unity.

i.e., $0 \le p \le 1, \quad 0 \le q \le 1$

Thus, the probability of occurrence of an event lies between 0 and 1[including 0 and 1].

Remarks

1. Probability '*p*' of the happening of an event is known as the probability of success and the probability '*q*' of the non-happening of the event as the probability of failure.
2. Probability of an impossible event is 0 and that of a sure event is 1

   if P (A) = 1, the event A is certainly going to happen and

   if P (A) = 0, the event is certainly not going to happen.
3. The number (*m*) of favourable outcomes to an event cannot be greater than the total number of outcomes (n).

Let us consider some examples

Example 19.1 In a simultaneous toss of two coins, find the probability of

(i) getting 2 heads (ii) exactly 1 head

Solution : Here, the possible outcomes are

HH, HT, TH, TT.

i.e., Total number of possible outcomes = 4.

(i) Number of outcomes favourable to the event (2 heads) = 1 (i.e., HH).

$$\therefore \quad P\,(\,2 \text{ heads}\,) = \frac{1}{4}.$$

(ii) Now the event consisting of exactly one head has two favourable cases,

namely HT and TH . $\therefore \quad P\,(\,\text{exactly one head}\,) = \frac{2}{4} = \frac{1}{2}.$

Example 19.2 In a single throw of two dice, what is the probability that the sum is 9?

Solution : The number of possible outcomes is $6 \times 6 = 36$. We write them as given below :

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Now, how do we get a total of 9. We have :

$$3 + 6 = 9,\ 4 + 5 = 9,\ 5 + 4 = 9,\ 6 + 3 = 9$$

In other words, the outcomes (3, 6), (4, 5), (5, 4) and (6, 3) are favourable to the said event, i.e., the number of favourable outcomes is 4.

Hence, $P(\text{a total of } 9) = \frac{4}{36} = \frac{1}{9}$

Example 19.3 What is the chance that a leap year, selected at random, will contain 53 Sundays?

Solution : A leap year consists of 366 days consisting of 52 weeks and 2 extra days. These two extra days can occur in the following possible ways.

(i) Sunday and Monday
(ii) Monday and Tuesday
(iii) Tuesday and Wednesday
(iv) Wednesday and Thursday
(v) Thursday and Friday
(vi) Friday and Saturday
(vii) Saturday and Sunday

Out of the above seven possibilities, two outcomes,

e.g., (i) and (vii), are favourable to the event

$\therefore \quad P(53 \text{ Sundays}) = \frac{2}{7}$

CHECK YOUR PROGRESS 19.1

1. A die is rolled once. Find the probability of getting 3.
2. A coin is tossed once. What is the probability of getting the tail ?
3. What is the probability of the die coming up with a number greater than 3 ?
4. In a simultaneous toss of two coins, find the probability of getting ' at least' one tail.
5. From a bag containing 15 red and 10 blue balls, a ball is drawn 'at random'. What is the probability of drawing (i) a red ball ? (ii) a blue ball ?
6. If two dice are thrown, what is the probability that the sum is (i) 6 ? (ii) 8? (iii) 10? (iv) 12?
7. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is divisible by 3 or by 4 ?
8. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is greater than 10 ?
9. What is the probability of getting a red card from a well shuffled deck of 52 cards ?
10. If a card is selected from a well shuffled deck of 52 cards, what is the probability of drawing

    (i) a spade ? (ii) a king ? (iii) a king of spade ?
11. A pair of dice is thrown. Find the probability of getting

(i) a sum as a prime number (ii) a doublet, i.e., the same number on both dice

(iii) a multiple of 2 on one die and a multiple of 3 on the other.

12. Three coins are tossed simultaneously. Find the probability of getting

(i) no head (ii) at least one head (iii) all heads

19.2. CALCULATION OF PROBABILITY USING COMBINATORICS (PERMUTATIONS AND COMBINATIONS)

In the preceding section, we calculated the probability of an event by listing down all the possible outcomes and the outcomes favourable to the event. This is possible when the number of outcomes is small, otherwise it becomes difficult and time consuming process. In general, we do not require the actual listing of the outcomes, but require only the total number of possible outcomes and the number of outcomes favourable to the event. In many cases, these can be found by applying the knowledge of permutations and combinations, which you have already studied.

Let us consider the following examples :

Example 19.4 A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue ?

Solution : Total number of balls = 3 + 6 + 7 = 16

Now, out of 16 balls, 2 can be drawn in ${}^{16}C_2$ ways.

$$\therefore \text{ Exhaustive number of cases} = {}^{16}C_2 = \frac{16 \times 15}{2} = 120$$

Out of 6 white balls, 1 ball can be drawn in ${}^{6}C_1$ ways and out of 7 blue balls, one can be drawn is ${}^{7}C_1$ ways. Since each of the former case is associated with each of the later case, therefore total number of favourable cases are ${}^{6}C_1 \times {}^{7}C_1 = 6 \times 7 = 42$.

$$\therefore \text{ Required probability} = \frac{42}{120} = \frac{7}{20}$$

Remarks

When two or more balls are drawn from a bag containing several balls, there are two ways in which these balls can be drawn.

(i) Without replacement : The ball first drawn is not put back in the bag, when the second ball is drawn. The third ball is also drawn without putting back the balls drawn earlier and so on. Obviously, the case of drawing the balls without replacement is the same as drawing them together.

(ii) With replacement : In this case, the ball drawn is put back in the bag before drawing the next ball. Here the number of balls in the bag remains the same, every time a ball is drawn.

In these types of problems, unless stated otherwise, we consider the problem of without replacement.

Example 19.5 Six cards are drawn at random from a pack of 52 cards. What is the probability that 3 will be red and 3 black?

Solution : Six cards can be drawn from the pack of 52 cards in ${}^{52}C_6$ ways.

i.e., Total number of possible outcomes = ${}^{52}C_6$

3 red cards can be drawn in ${}^{26}C_3$ ways and

3 black cards can be drawn in ${}^{26}C_3$ ways.

$\therefore$ Total number of favourable cases = ${}^{26}C_3 \times {}^{26}C_3$

Hence, the required probability $= \dfrac{{}^{26}C_3 \times {}^{26}C_3}{{}^{52}C_6} = \dfrac{13000}{39151}$

Example 19.6 Four persons are chosen at random from a group of 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is $\frac{10}{21}$.

Solution : Total number of persons in the group = 3 + 2 + 4 = 9. Four persons are chosen at random. If two of the chosen persons are children, then the remaining two can be chosen from 5 persons (3 men + 2 women).

Number of ways in which 2 children can be selected from 4, children = ${}^{4}C_2 = \dfrac{4\times 3}{1\times 2} = 6$

Number of ways in which remaining of the two persons can be selected

from 5 persons = ${}^{5}C_2 = \dfrac{5\times 4}{1\times 2} = 10$

Total number of ways in which 4 persons can be selected out of

9 persons = ${}^{9}C_4 = \dfrac{9\times 8\times 7\times 6}{1\times 2\times 3\times 4} = 126$

Hence, the required probability $= \dfrac{{}^{4}C_2 \times {}^{5}C_2}{{}^{9}C_4} = \dfrac{6\times 10}{126} = \dfrac{10}{21}$

CHECK YOUR PROGRESS 19.2

1. A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn at random are both white?
2. A bag contains 5 red and 8 blue balls. What is the probability that two balls drawn are red and blue ?
3. A bag contains 20 white and 30 black balls. Find the probability of getting 2 white balls, when two balls are drawn at random

   (a) with replacement (b) without replacement

4. Three cards are drawn from a well-shuffled pack of 52 cards. Find the probability that all the three cards are jacks.
5. Two cards are drawn from a well-shuffled pack of 52 cards. Show that the chances of drawing both aces is $\frac{1}{221}$.
6. In a group of 10 outstanding students in a school, there are 6 boys and 4 girls. Three students are to be selected out of these at random for a debate competition. Find the probability that

   (i) one is boy and two are girls. (ii) all are boys. (iii) all are girls.
7. Out of 21 tickets marked with numbers from 1 to 21, three are drawn at random. Find the probability that the numbers on them are in A.P.
8. Two cards are drawn at random from 8 cards numbered 1 to 8. What is the probability that the sum of the numbers is odd, if the cards are drawn together ?
9. A team of 5 players is to be selected from a group of 6 boys and 8 girls. If the selection is made randomly, find the probability that there are 2 boys and 3 girls in the team.
10. An integer is chosen at random from the first 200 positive integers.Find the probability that the integer is divisible by 6 or 8.

19.3 EVENT RELATIONS

19.3.1 Complement of an event

Let us consider the example of throwing a fair die. The sample space of this experiment is

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

If A be the event of getting an even number, then the sample points 2, 4 and 6 are favourable to the event A.

The remaining sample points 1, 3 and 5 are not favourable to the event A. Therefore, these will occur when the event A will not occur.

In an experiment, the outcomes which are not favourable to the event A are called complement of A and defined as follows :

'The outcomes favourable to the complement of an event A consists of all those outcomes which are not favourable to the event A, and are denoted by 'not' A or by $\overline{A}$.

19.3.2 Event 'A or B'

Let us consider the example of throwing a die. A is an event of getting a multiple of 2 and B be another event of getting a multiple of 3.

The outcomes 2, 4 and 6 are favourable to the event A and the outcomes 3 and 6 are favourable to the event B.

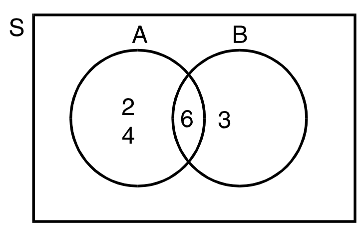


Fig. 19.1

The happening of event A or B is $A \cup B = \{ 2, 3, 4, 6 \}$

Again, if A be the event of getting an even number and B is another event of getting an odd number, then $A = \{ 2, 4, 6 \}, B = \{ 1, 3, 5 \}$

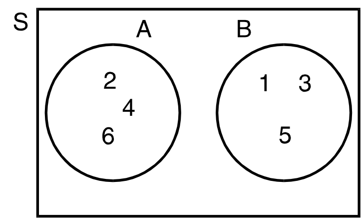


Fig. 19.2

$A \cup B = \{1, 2, 3, 4, 5, 6 \}$

Here, it may be observed that if A and B are two events, then the event 'A or B' $(A \cup B)$ will consist of the outcomes which are either favourable to the event A or to the event B or to both the events.

Thus, the event 'A or B' occurs, if either A or B or both occur.

19.3.3 Event "A and B'

Recall the example of throwing a die in which A is the event of getting a multiple of 2 and B is the event of getting a multiple of 3. The outcomes favourable to A are 2, 4, 6 and the outcomes favourable to B are 3 , 6.

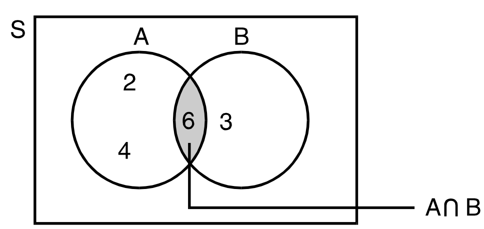


Fig. 19.3

Here, we observe that the outcome 6 is favourable to both the events A and B.

Draw a card from a well shuffled deck of 52 cards. A and B are two events defined as

A : a red card, B : a king

We know that there are 26 red cards and 4 kings in a deck of cards. Out of these 4 kings, two are red.

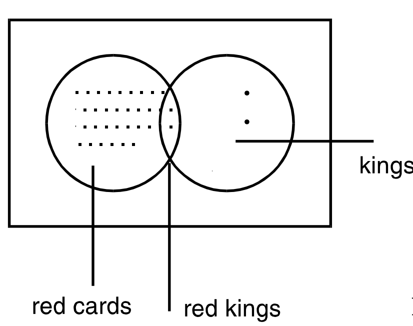


Fig. 19.4

Here, we see that the two red kings are favourable to both the events.

Hence, the event ' A and B' consists of all those outcomes which are favourable to both the events A and B. That is, the event 'A and B' occurs, when both the events A and B occur simultaneously. Symbolically, it is denoted as $A \cap B$.

19.4 ADDITIVE LAW OF PROBABILITY

Let A be the event of getting an odd number and B be the event of getting a prime number in a single throw of a die. What will be the probability that it is either an odd number or a prime number ?

In a single throw of a die, the sample space would be

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

The outcomes favourable to the events A and B are

$$A = \{ 1, 3, 5 \}, B = \{ 2, 3, 5 \}$$

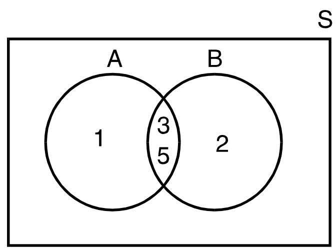


Fig. 19.5

The outcomes favourable to the event 'A or B' are

$$A \cup B = \{1, 2, 3, 5 \}.$$

Thus, the probability of getting either an odd number or a prime number will be

$$P(A \text{ or } B) = \frac{4}{6} = \frac{2}{3}$$

To discover an alternate method, we can proceed as follows :

The outcomes favourable to the event A are 1, 3 and 5. $\therefore \; P(A) = \frac{3}{6}$

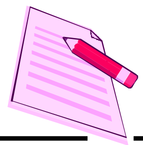
Similarly, $P(B) = \frac{3}{6}$

The outcomes favourable to the event 'A and B' are 3 and 5. $\therefore \; P(A \text{ and } B) = \frac{2}{6}$

Now, $P(A) + P(B) - P(A \text{ and } B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6}$

$$\frac{4}{6} = \frac{2}{3} = P(A \text{ or } B)$$

Thus, we state the following law, called additive rule, which provides a technique for finding the probability of the union of two events, when they are not disjoint.



For any two events A and B of a sample space S,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad .....(ii)$$

Example 19.7 A card is drawn from a well-shuffled deck of 52 cards. What is the prob ability that it is either a spade or a king ?

Solution : If a card is drawn at random from a well-shuffled deck of cards, the likelyhood of any of the 52 cards being drawn is the same. Obviously, the sample space consists of 52 sample points.

If A and B denote the events of drawing a 'spade card' and a 'king' respectively, then the event A consists of 13 sample points, whereas the event B consists of 4 sample points. Therefore,

$$P(A) = \frac{13}{52}, \qquad P(B) = \frac{4}{52}$$

The compound event $(A \cap B)$ consists of only one sample point, viz.; king of spade. So,

$$P(A \cap B) = \frac{1}{52}$$

Hence, the probability that the card drawn is either a spade or a king is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example 19.8 In an experiment with throwing 2 fair dice, consider the events

A : The sum of numbers on the faces is 8

B : Doubles are thrown.

What is the probability of getting A or B ?

Solution : In a throw of two dice, the sample space consists of $6 \times 6 = 36$ sample points.

The favourable outcomes to the event A (the sum of the numbers on the faces is 8) are

$$A = \{ (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) \}$$

The favourable outcomes to the event B (Double means both dice have the same number) are

$$B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$$

$\therefore \quad A \cap B = \{ (4, 4) \}.$

Now $P(A) = \frac{5}{36}, \quad P(B) = \frac{6}{36}, \quad P(A \cap B) = \frac{1}{36}$

Thus, the probability of A or B is

$$P(A \cup B) = \frac{5}{36} + \frac{6}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$$

19.5 ADDITIVE LAW OF PROBABILITY FOR MUTUALLY EXCLUSIVE EVENTS

We know that the events A and B are mutually exclusive, if and only if they have no outcomes in common. That is, for mutually exclusive events,

$$P\,(A \text{ and } B) = 0$$

Substituting this value in the additive law of probability, we get the following law :

$$P\,(A \text{ or } B) = P\,(A) + P\,(B) \qquad .....(iii)$$

Example 19.9 In a single throw of two dice, find the probability of a total of 9 or 11.

Solution : Clearly, the events - a total of 9 and a total of 11 are mutually exclusive.

Now $\quad P\,(\text{a total of } 9) = P\,[(3, 6), (4, 5), (5, 4), (6, 3)] = \dfrac{4}{36}$

$$P\,(\text{a total of } 11) = P\,[(5, 6), (6, 5)] = \frac{2}{36}$$

Thus, $\quad P\,(\text{a total of 9 or 11}) = \dfrac{4}{36} + \dfrac{2}{36} = \dfrac{1}{6}$

Example 19.10 Prove that the probability of the non-occurrence of an event A is 1 – P (A).

i.e., $\quad P\,(\text{not } A) = 1 - P\,(A) \quad$ or, $\quad P\,(\bar{A}) = 1 - P\,(A)$.

Solution : We know that the probability of the sample space S in any experiment is 1.

Now, it is clear that if in an experiment an event A occurs, then the event ($\bar{A}$) cannot occur simultaneously, i.e., the two events are mutually exclusive.

Also, the sample points of the two mutually exclusive events together constitute the sample space S. That is,

$$A \cup \bar{A} = S$$

Thus, $\quad P\left(A \cup \bar{A}\right) = P(S)$

$$\Rightarrow P\,(A) + P(\bar{A}) = 1 \;(\because A \text{ and } \bar{A} \text{ are mutually exclusive and S is sample space})$$

$$\Rightarrow P\left(\bar{A}\right) = 1 - P\,(A),$$

which proves the result.

This is called the law of complementation.

Law of complimentation : $P\left(\bar{A}\right) = 1 - P\,(A)$

$$\frac{P(\bar{A})}{P(A)} \text{ or } P\,(A) \text{ to } P\,(\bar{A}).$$

Example 19.11 The probability of the event that it will rain is 0.3. Find the odds in favour of rain and odds against rain.

Solution : Let A be the event that it will rain. $\therefore$ $P(A) = .3$

By law of complementation, $P\left(\bar{A}\right) = 1 - .3 = .7$.

Now, the odds in favour of rain are $\frac{0.3}{0.7}$ or 3 to 7 (or 3 : 7).

The odds against rain are $\frac{0.7}{0.3}$ or 7 to 3.

When either the odds in favour of A or the odds against A are given, we can obtain the probability of that event by using the following formulae

If the odds in favour of A are *a* to *b*, then $P(A) = \frac{a}{a+b}$.

If the odds against A are a to b, then $P(A) = \frac{b}{a+b}$.

This can be proved very easily.

Suppose the odds in favour of A are *a* to *b*. Then, by the definition of odds,

$$\frac{P(A)}{P(\bar{A})} = \frac{a}{b}.$$

From the law of complimentation, $P\left(\bar{A}\right) = 1 - P(A)$

Therefore, $\frac{P(A)}{1 - P(A)} = \frac{a}{b}$ or $b\,P(A) = a - a\,P(A)$

or $(a + b)\,P(A) = a$ or $P(A) = \frac{a}{a+b}$

Similarly, we can prove that $P(A) = \frac{b}{a+b}$

when the odds against A are b to a.

Example 19.12 Are the following probability assignments consistent ? Justify your answer.

(a) $P(A) = P(B) = 0.6$, $P(A \text{ and } B) = 0.05$

(b) $P(A) = 0.5$, $P(B) = 0.4$, $P(A \text{ and } B) = 0.1$

(c) $P(A) = 0.2$, $P(B) = 0.7$, $P(A \text{ and } B) = 0.4$

Solution : (a) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$$= 0.6 + 0.6 - 0.05 = 1.15$$

Since P (A or B) > 1 is not possible, hence the given probabilities are not consistent.

(b) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$= 0.5 + 0.4 - 0.1 = 0.8$

which is less than 1.

As the number of outcomes favourable to event 'A and B' should always be less than or equal to those favourable to the event A,

Therefore, $P(A \text{ and } B) \le P(A)$

and similarly $P(A \text{ and } B) \le P(B)$

In this case, P (A and B) = 0.1, which is less than both P (A) = 0.5 and P (B) = 0.4. Hence, the assigned probabilities are consistent.

(c) In this case, P (A and B) = 0.4, which is more than P (A) = 0.2.

$[\because P(A \text{ and } B) \le P(A)]$

Hence, the assigned probabilities are not consistent.

Example 19.13 An urn contains 8 white balls and 2 green balls. A sample of three balls is selected at random. What is the probability that the sample contains at least one green ball ?

Solution : Urn contains 8 white balls and 2 green balls.

$\therefore$ Total number of balls in the urn = 10

Three balls can be drawn in ${}^{10}C_3$ ways = 120 ways.

Let A be the event " at least one green ball is selected".

Let us determine the number of different outcomes in A. These outcomes contain either one green ball or two green balls.

There are ${}^{2}C_1$ ways to select a green ball from 2 green balls and for this remaining two white balls can be selected in ${}^{8}C_2$ ways.

Hence, the number of outcomes favourable to one green ball

$$= {}^{2}C_1 \times {}^{8}C_2 = 2 \times 28 = 56$$

Similarly, the number of outcomes favourable to two green balls

$$= {}^{2}C_2 \times {}^{8}C_1 = 1 \times 8 = 8$$

Hence, the probability of at least one green ball is

P (at least one green ball)

$$= P(\text{one green ball}) + P(\text{two green balls})$$

$$= \frac{56}{120} + \frac{8}{120} = \frac{64}{120} = \frac{8}{15}$$

Example 19.14 Two balls are drawn at random with replacement from a bag containing 5 blue and 10 red balls. Find the probability that both the balls are either blue or red.

Solution : Let the event A consists of getting both blue balls and the event B is getting both red balls. Evidently A and B are mutually exclusive events.

By fundamental principle of counting, the number of outcomes favourable to $A = 5 \times 5 = 25.$

Similarly, the number of outcomes favourable to B = 10 × 10 = 100.

Total number of possible outcomes = 15 × 15 = 225.

$$\therefore \qquad P(A) = \frac{25}{225} = \frac{1}{9} \text{ and } P(B) = \frac{100}{225} = \frac{4}{9}.$$

Since the events A and B are mutually exclusive, therefore

$$P(A \text{ or } B) = P(A) + P(B)$$

$$= \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$$

Thus, $P(\text{both blue or both red balls}) = \frac{5}{9}$

CHECK YOUR PROGRESS 19.3

1. A card is drawn from a well-shuffled pack of cards. Find the probability that it is a queen or a card of heart.
2. In a single throw of two dice, find the probability of a total of 7 or 12.
3. The odds in favour of winning of Indian cricket team in 2010 world cup are 9 to 7. What is the probability that Indian team wins ?
4. The odds against the team A winning the league match are 5 to 7. What is the probability that the team A wins the league match.
5. Two dice are thrown. Getting two numbers whose sum is divisible by 4 or 5 is considered a success. Find the probability of success.
6. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement. What is the probability that both the cards are either black or red ?
7. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that the card is an ace or a black card.
8. Two dice are thrown once. Find the probability of getting a multiple of 3 on the first die or a total of 8.
9. (a) In a single throw of two dice, find the probability of a total of 5 or 7.

   (b) A and B are two mutually exclusive events such that P (A) = 0.3 and P (B) = 0.4. Calculate P (A or B).
10. A box contains 12 light bulbs of which 5 are defective. All the bulbs look alike and have equal probability of being chosen. Three bulbs are picked up at random. What is the probability that at least 2 are defective ?
11. Two dice are rolled once. Find the probability

(a) that the numbers on the two dice are different,

(b) that the total is at least 3.

12. A couple have three children. What is the probability that among the children, there will be at least one boy or at least one girl ?

13. Find the odds in favour and against each event for the given probability

(a) P (A) = .7 (b) $P(A) = \frac{4}{5}$

14. Determine the probability of A for the given odds

(a) 7 to 2 in favour of A (b) 10 to 7 against A.

15. If two dice are thrown, what is the probability that the sum is

(a) greater than 4 and less than 9 ?

(b) neither 5 nor 8 ?

16. Which of the following probability assignments are inconsistent ? Give reasons.

(a) P (A) = 0.5 , P (B) = 0.3 , P (A and B) = 0.4

(b) P (A) = P (B) = 0.4, P (A and B) = 0.2

(c) P (A) = 0.85, P (B) = 0.8, P (A and B) = 0.61

17. Two balls are drawn at random from a bag containing 5 white and 10 green balls. Find the probability that the sample contains at least one white ball.

18. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement. What is the probability that both cards are of the same suit?

Thus, the probability of simultaneous occurrence of two independent events is the product of their separate probabilities.

19.6 MULTIPLICATION LAW OF PROBABILITY FOR INDEPENDENT EVENTS

Let us recall the definition of independent events.

Two events A and B are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence (and hence non-occurrence) of the other.

Can you think of some examples of independent events ?

The event of getting 'H' on first coin and the event of getting 'T' on the second coin in a simultaneous toss of two coins are independent events.

What about the event of getting 'H' on the first toss and event of getting 'T' on the second toss in two successive tosses of a coin ? They are also independent events.

Let us consider the event of 'drawing an ace' and the event of 'drawing a king' in two successive draws of a card from a well-shuffled deck of cards without replacement.

Are these independent events ?

No, these are not independent events, because we draw an ace in the first draw with probability

$\frac{4}{52}$. Now, we do not replace the card and draw a king from the remaining 51 cards and this affect the probability of getting a king in the second draw, i.e., the probability of getting a king in the second draw without replacement will be $\frac{4}{51}$.

Note : If the cards are drawn with replacement, then the two events become independent.
Is there any rule by which we can say that the events are independent ?
How to find the probability of simultaneous occurrence of two independent events?
If A and B are independent events, then

$$P\ (A\ \text{and}\ B) = P(A)\ .\ P(B)$$

or

$$P(A \cap B) = P(A).\ P(B)$$

Thus, the probability of simultaneous occurrence of two independent events is the product of their separate probabilities.

Note : The above law can be extended to more than two independent events, i.e.,

$$P(A \cap B \cap C...) = P(A) \cdot P(B) \cdot P(C)...$$

On the other hand, if the probability of the event 'A' and 'B' is equal to the product of the probabilities of the events A and B, then we say that the events A and B are independent.

Example 19.15 A die is tossed twice. Find the probability of a number greater than 4 on each throw.

Solution : Let us denote by A, the event 'a number greater than 4' on first throw. B be the event 'a number greater than 4' in the second throw. Clearly A and B are independent events.

In the first throw, there are two outcomes, namely, 5 and 6 favourable to the event A.

$$\therefore \qquad P(A) = \frac{2}{6} = \frac{1}{3}$$

Similarly, $$P(B) = \frac{1}{3}$$

Hence, $P(A\ \text{and}\ B) = P(A)\ .\ P(B) = \frac{1}{3}.\frac{1}{3} = \frac{1}{9}$.

Example 19.16 Arun and Tarun appear for an interview for two vacancies. The probability of Arun's selection is $\frac{1}{3}$ and that of Tarun's selection is $\frac{1}{5}$. Find the probability that

(a) both of them will be selected. (b) none of them is selected.

(c) at least one of them is selected. (d) only one of them is selected.

Solution : Probability of Arun's selection $= P(A) = \frac{1}{3}$

Probability of Tarun's selection $= P(T) = \frac{1}{5}$

(a) P (both of them will be selected) = P (A) P (T)

$$= \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$$

(b) P (none of them is selected)

$$= P(\bar{A})P(\bar{T}) = \left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

(c) P (at least one of them is selected)

$$= 1 - P \text{ (None of them is selected)}$$

$$= 1 - P(\bar{A})P(\bar{T}) = 1 - \left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$= 1 - \left(\frac{2}{3} \times \frac{4}{5}\right) = 1 - \frac{8}{15} = \frac{7}{15}$$

(d) P (only one of of them is selected)

$$= P(A)P(\bar{T}) + P(\bar{A})\ P(T)$$

$$= \frac{1}{3} \times \frac{4}{5} + \frac{2}{3} \times \frac{1}{5} = \frac{6}{15} = \frac{2}{5}$$

Example 19.17 A problem in statistics is given to three students, whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that problem will be solved ?

Solution : Let p_1, p_2 and p_3 be the probabilities of three persons of solving the problem.

Here, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$ and $p_3 = \frac{1}{4}$.

The problem will be solved, if at least one of them solves the problem.

$\therefore$ P (at least one of them solves the problem)

= 1 – P (None of them solves the problem) .....(1)

Now, the probability that none of them solves the problem will be

P (none of them solves the problem) $= (1-p_1)(1-p_2)(1-p_3)$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

Putting this value in (1), we get

$$\text{P (at least one of them solves the problem)} = 1 - \frac{1}{4} = \frac{3}{4}$$

Hence, the probability that the problem will be solved is $\frac{3}{4}$.

Example 19.18 Two balls are drawn at random with replacement from a box containing 15 red and 10 white balls. Calculate the probability that

(a) both balls are red.

(b) first ball is red and the second is white.

(c) one of them is white and the other is red.

Solution :

(a) Let A be the event that first drawn ball is red and B be the event that the second ball drawn is red. Then as the balls drawn are with replacement,

therefore $\quad P(A) = \frac{15}{25} = \frac{3}{5}, \; P(B) = \frac{3}{5}$

As A and B are independent events

therefore $\quad$ P (both red) = P (A and B)

$$= P(A) \times P(B) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$$

(b) Let A : First ball drawn is red.

B : Second ball drawn is white.

$\therefore \quad P(\text{A and B}) = P(A) \times P(B) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}.$

(c) If WR denotes the event of getting a white ball in the first draw and a red ball in the second draw and the event RW of getting a red ball in the first draw and a white ball in the second draw. Then as 'RW' and WR' are mutually exclusive events, therefore

$\therefore \quad$ P (a white and a red ball)

$$= P(\text{WR or RW})$$

$$= P(WR) + P(RW)$$

$$= P(W)\,P(R) + P(R)\,P(W)$$

$$= \frac{2}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5}$$

$$= \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$$

Example 19.19 A dice is thrown 3 times. Getting a number '5 or 6' is a success. Find the probability of getting

(a) 3 successes (b) exactly 2 successes (c) at most 2 successes (d) at least 2 successes.

Solution : Let S denote the success in a trial and F denote the ' not success' i.e. failure. Therefore,

$$P(S) = \frac{2}{6} = \frac{1}{3},\ P(F) = 1 - \frac{1}{3} = \frac{2}{3}$$

(a) As the trials are independent, by multiplication theorem for independent events,

$$P(SSS) = P(S)\ P(S)\ P(S) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$$P(SSF) = P(S)\,P(S)\,P(F) = \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$$

Since the two successes can occur in ${}^3C_2$ ways

$$\therefore \quad P(\text{exactly two successes}) = {}^3C_2 \times \frac{2}{27} = \frac{2}{9}$$

(c) $P(\text{at most two successes}) = 1 - P(3\text{ successes}) = 1 - \frac{1}{27} = \frac{26}{27}$

(d) P (at least two successes) = P (exactly 2 successes) + P (3 successes)

$$= \frac{2}{9} + \frac{1}{27} = \frac{7}{27}$$

Example 19.20 A card is drawn from a pack of 52 cards so that each card is equally likely to be selected. Which of the following events are independent ?

(i) A : the card drawn is a spade

B : the card drawn is an ace

(ii) A : the card drawn is black

B : the card drawn is a king

(iii) A : the card drawn is a king or a queen

B : the card drawn is a queen or a jack

Solution : (i) There are 13 cards of spade in a pack. $P(A) = \frac{13}{52} = \frac{1}{4}$

There are four aces in the pack. $P(B) = \frac{4}{52} = \frac{1}{13}$

$$A \cap B = \{\text{ an ace of spade }\}$$

$\therefore \quad P(A \cap B) = \frac{1}{52}$

Now $\quad P(A)\ P(B) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$

Since $\quad P(A \cap B) = P(A).\ P(B)$

Hence, the events A and B are independent.

(ii) There are 26 black cards in a pack.

$\therefore \quad P(A) = \frac{26}{52} = \frac{1}{2}$

There are four kings in the pack. $\therefore \quad P(B) = \frac{4}{52} = \frac{1}{13}$

$$A \cap B = \{\text{ 2 black kings }\} \therefore \ P(A \cap B) = \frac{2}{52} = \frac{1}{26}$$

Now, $\quad P(A) \times P(B) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$

Since $\quad P(A \cap B) = P(A)\ P(B)$

Hence, the events A and B are independent.

(iii) There are 4 kings and 4 queens in a pack of cards.

$\therefore$ Total number of outcomes favourable to the event A is 8.

$\therefore \quad P(A) = \frac{8}{52} = \frac{2}{13}$

Similarly, $\quad P(B) = \frac{2}{13},\quad A \cap B = \{\text{ 4 queens }\}$

$\therefore \quad P(A \cap B) = \frac{4}{52} = \frac{1}{13}$

$\therefore \quad P(A) \times P(B) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169}$

Here, $\quad P(A \cap B) \neq P(A).P(B)$

Hence, the events A and B are not independent.

CHECK YOUR PROGRESS 19.4

1. A husband and wife appear in an interview for two vacancies in the same department. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that

(a) Only one of them will be selected ?

(b) Both of them will be selected ?

(c) None of them will be selected ?

(d) At least one of them will be selected ?

2. Probabilities of solving a specific problem independently by Raju and Soma are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved.

(b) exactly one of them solves the problem.

3. A die is rolled twice. Find the probability of a number greater than 3 on each throw.

4. Sita appears in the interview for two posts A and B, selection for which are independent. The probability of her selection for post A is $\frac{1}{5}$ and for post B is $\frac{1}{7}$. Find the probability that she is selected for

(a) both the posts

(b) at least one of the posts.

5. The probabilities of A, B and C solving a problem are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$ respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.

6. A draws two cards with replacement from a well-shuffled deck of cards and at the same time B throws a pair of dice. What is the probability that

(a) A gets both cards of the same suit and B gets a total of 6 ?

(b) A gets two jacks and B gets a doublet ?

7. Suppose it is 9 to 7 against a person A who is now 35 years of age living till he is 65 and 3:2 against a person B now 45 living till he is 75. Find the chance that at least one of these persons will be alive 30 years hence.

8. A bag contains 13 balls numbered from 1 to 13. Suppose an even number is considered a 'success'. Two balls are drawn with replacement, from the bag. Find the probability of getting

(a) Two successes (b) exactly one success

(c) at least one success (d) no success

9. One card is drawn from a well-shuffled deck of 52 cards so that each card is equally likely to be selected. Which of the following events are independent ?

(a) A : The drawn card is red

B : The drawn card is a queen

(b) A : The drawn card is a heart B: The drawn card is a face card

19.7 CONDITIONAL PROBABILITY

Suppose that a fair die is thrown and the score noted. Let A be the event, the score is 'even'. Then

$$A = \{2, 4, 6\}, \therefore \; P(A) = \frac{3}{6} = \frac{1}{2}.$$

Now suppose we are told that the score is greater than 3. With this additional information what will be P (A) ?

Let B be the event, 'the score is greater than 3'. Then B is {4, 5, 6 }. When we say that B has occurred, the event 'the score is less than or equal to 3' is no longer possible. Hence the sample space has changed from 6 to 3 points only. Out of these three points 4, 5 and 6; 4 and 6 are even scores.

Thus, given that B has occurred, P (A) must be $\frac{2}{3}$.

Let us denote the probability of A given that B has already occurred by $P(A \mid B)$.

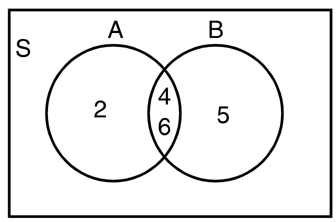


FIg. 19.7

Again, consider the experiment of drawing a single card from a deck of 52 cards. We are interested in the event A consisting of the outcome that a black ace is drawn.

Since we may assume that there are 52 equally likely possible outcomes and there are two black aces in the deck, so we have

$$P(A) = \frac{2}{52}.$$

However, suppose a card is drawn and we are informed that it is a spade. How should this information be used to reappraise the likelihood of the event A ?

Clearly, since the event B "A spade has been drawn " has occurred, the event "not spade" is no longer possible. Hence, the sample space has changed from 52 playing cards to 13 spade cards. The number of black aces that can be drawn has now been reduced to 1.

Therefore, we must compute the probability of event A relative to the new sample space B.

Let us analyze the situation more carefully.

The event A is " a black ace is drawn'. We have computed the probability of the event A knowing that B has occurred. This means that we are computing a probability relative to a new sample space B. That is, B is treated as the universal set. We should consider only that part of A which is included in B.

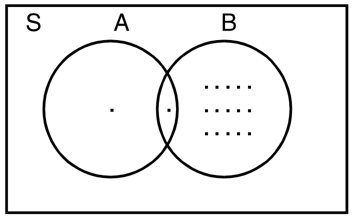


Fig. 19.8

Hence, we consider $A \cap B$ (see figure 31.8).

Thus, the probability of A. given B, is the ratio of the number of entries in $A \cap B$ to the number of entries in B. Since $n(A \cap B) = 1$ and $n(B) = 13$,

then $$P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{13}$$

Notice that $$n(A \cap B) = 1 \Rightarrow P(A \cap B) = \frac{1}{52}$$

$$n(B) = 13 \Rightarrow P(B) = \frac{13}{52}$$

$$\therefore \quad P(A \mid B) = \frac{1}{13} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{P(A \cap B)}{P(B)}.$$

This leads to the definition of conditional probability as given below :

Let A an B be two events defined on a sample space S. Let P(B) > 0, then the conditional probability of A, provided B has already occurred, is denoted by P(A|B) and mathematically written as :

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Similarly, $$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

The symbol P(A | B) is usually read as "the probability of A given B".

Example 19.21 Consider all families "with two children (not twins). Assume that all the elements of the sample space {BB, BG, GB,GG} are equally likely. (Here, for instance, BG denotes the birth sequence "boy girls"). Let A be the event {BB} and B be the event that 'at least one boy'. Calculate P (A | B).

Solution : Here, $A = \{BB\}$, $B = \{BB, BG, GB\}$

$$A \cap B = \{BB\} \therefore \quad P(A \cap B) = \frac{1}{4}$$

$$P(B) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Hence, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{4}}{\frac{3}{4}} = \dfrac{1}{3}$.

Example 19.22 Assume that a certain school contains equal number of female and male students. 5 % of the male population is football players. Find the probability that a randomly selected student is a football player male.

Solution : Let M = Male

F = Football player

We wish to calculate $P(M \cap F)$. From the given data,

$$P(M) = \frac{1}{2} \quad (\because \text{ School contains equal number of male and female students})$$

$$P(F|M) = 0.05$$

But from definition of conditional probability, we have

$$P(F \mid M) = \frac{P(M \cap F)}{P(M)}$$

$\Rightarrow$ $$P(M \cap F) = P(M) \times P(F \mid M)$$

$$= \frac{1}{2} \times 0.05 = 0.025$$

Example 19.23 If A and B are two events, such that $P(A) = 0.8$, $P(B) = 0.6$, $P(A \cap B) = 0.5$, find the value of

(i) $P(A \cup B)$ (ii) $P(B \mid A)$ (iii) $P(A \mid B)$.

Solution :(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.8 + 0.6 - 0.5 = 0.9$$

(ii) $$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.5}{0.8} = \frac{5}{8}$$

(iii) $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.5}{0.6} = \frac{5}{6}$$

Example 19.24 A coin is tossed until a head appears or until it has been tossed three times. Given that head does not occur on the first toss, what is the probability that coin is tossed three times ?

Solution : Here, it is given that head does not occur on the first toss. That is, we may get the head on the second toss or on the third toss or even no head.

Let B be the event, " no heads on first toss".

Then $B = \{TH, TTH, TTT\}$

These events are mutually exclusive.

$$P(B) = P(TH) + P(TTH) + P(TTT) \qquad .....(1)$$

Now $P(TH) = \frac{1}{4}$ ($\because$ This event has the sample space of four outcomes)

and $P(TTH) = P(TTT) = \frac{1}{8}$ ($\because$ This event has the sample space of eight outcomes)

Putting these values in (1), we get

$$P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Let A be the event "coin is tossed three times".

Then $A = \{TTH, TTT\}$

$\therefore$ We have to find $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Here, $A \cap B = A$, $\therefore$ $P(A|B) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

CHECK YOUR PROGRESS 19.5

1. A sequence of two cards is drawn at random (without replacement) from a well-shuffled deck of 52 cards. What is the probability that the first card is red and the second card is black ?
2. Consider a three child family for which the sample space is

   { BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG }

   Let A be the event " the family has exactly 2 boys " and B be the event " the first child is a boy". What is the probability that the family has 2 boys, given that first child is a boy ?
3. Two cards are drawn at random without replacement from a deck of 52 cards. What is the probability that the first card is a diamond and the second card is red ?
4. If A and B are events with $P(A) = 0.4$, $P(B) = 0.2$, $P(A \cap B) = 0.1$, find the probability of A given B. Also find $P(B|A)$.
5. From a box containing 4 white balls, 3 yellow balls and 1 green ball, two balls are drawn one at a time without replacement. Find the probability that one white and one yellow ball is drawn.

19.8 THEOREMS ON MULTIPLICATION LAW OF PROBABILITY AND CONDITIONAL PROBABILITY.

Theorem 1 : For two events A and B,

$$P(A \cap B) = P(A).P(B|A),$$

and $$P(A \cap B) = P(B).P(A|B),$$

where P (B|A) represents the conditional probability of occurrence of B, when the event A has already occurred and P (A|B) is the conditional probability of happening of A, given that B has already happened.

Proof : Let n (S) denote the total number of equally likely cases, n (A) denote the cases favourable to the event A, n (B) denote the cases favourable to B and $n(A \cap B)$ denote the cases favourable to both A and B.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)}, \quad P(B) = \frac{n(B)}{n(S)}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} \qquad .....(1)$$

For the conditional event A|B , the favourable outcomes must be one of the sample points of B, i.e., for the event A|B, the sample space is B and out of the n (B) sample points, $n(A \cap B)$ pertain to the occurrence of the event A, Hence,

$$P(A|B) = \frac{n(A \cap B)}{n(B)}$$

Rewriting (1), we get $$P(A \cap B) = \frac{n(B)}{n(S)} \cdot \frac{n(A \cap B)}{n(B)} = P(B).P(A|B)$$

Similarly, we can prove

$$P(A \cap B) = P(A).P(B|A)$$

Note : If A and B are independent events, then

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

$$P(A \cap B) = P(A).P(B)$$

Theorem 2 : Two events A and B of the sample space S are independent, if and only if

$$P(A \cap B) = P(A).P(B)$$

Proof : If A and B are independent events,

then $$P(A|B) = P(A)$$

We know that $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \quad P(A \cap B) = P(A)P(B)$$

Hence, if A and B are independent events, then the probability of 'A and B' is equal to the product of the probability of A and probability of B.

Conversely, if $P(A \cap B) = P(A).P(B)$, then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ gives}$$

$$P(A \mid B) = \frac{P(A)P(B)}{P(B)} = P(A)$$

That is, A and B are independent events.

19.9 INTRODUCTION TO BAYES' THEOREM

In conditional probability we have learnt to find probability of an event with the condition that some other event has already occurred. Consider an experiment of selecting one coin out of three coins : If I with P(H) = $\frac{1}{3}$ and P(T) = $\frac{2}{3}$, II with P(H) = $\frac{3}{4}$ and P(T) = $\frac{1}{4}$ and III with P(H) = $\frac{1}{2}$, P(T) = $\frac{1}{2}$ (a normal coin).

After randomly selecting one of the coins, it is tossed. We can find the probability of selecting one coin $\left(\text{i.e. } \frac{1}{3}\right)$ and can also find the probability of any outcome i.e. head or tail; given the coin selected. But can we find the probability that coin selected is coin I, II or III when it is known that the head occurred as outcome? For this we have to find the probability of an event which occurred prior to the given event. Such probability can be obtained by using Bayes' theorem, named after famous mathematician, Johan Bayes Let us first learn some basic definition before taking up Baye's theorem

Mutually exclusive and exhaustive events.

For a sample space S, the set of events E_1, E_2, ... E_n is said to mutually exclusive and exhaustive if

(i) $E_i \cap E_j = \phi, \forall\, i \neq j = 1, 2, \ldots\ldots n$ i.e. none of two events can occur together.

(ii) $E_i \cup E_2 \cup \ldots . \cup E_n = S$, all outcomes of S have been taken up in the events E_1, E_2...E_n

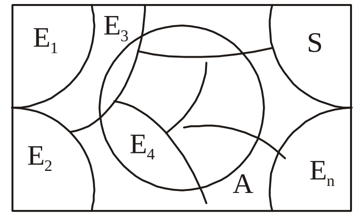
(iii) $P(E_i) > 0$ for all $i = 1, 2, \ldots\ldots n$

19.10 : THEOREM OF TOTAL PROBABILITY

Let E_1, E_2, ..., E_n are mutually exclusive and exhaustive events for a sample space S with $P(E_i) > 0$, $\forall\, i = 1, 2, \ldots n$. Let A be any event associated with S, then

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/E_n)$$

$$= \sum_{i=1}^{n} P(E_i) P(A/E_i)$$



$U(A \cap E_n)$

Fig. 19.9

Proof : The events E_i and A are shown in the venn-diagram

Given $S = E_1 \cup E_2, \cup E_3, \ldots \cup E_n$ and $E_i \cap E_j \neq \phi$.

We can write

$$\begin{aligned} A &= A \cap S \\ &= A \cup (E_1 \cup E_2 \cup \ldots \cup E_n) \\ &= (A \cap E_1) \cap (A \cap E_2) \cup (A \cap E_3) \ldots (A \cap E_n) \end{aligned}$$

Since all E_i, are mutually exclusive, so $A \cap E_1$, $A \cap E_2$ ... will also be mutually exclusive

$$\Rightarrow \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \ldots + P(A \cap E_n)$$

$$= P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/En)$$

By using the multiplication rule of probability,

$$P(A) = \sum_{i=1}^{n} P(E_i) P(A/E_i)$$

19.11 : BAYE'S THEOREM

If $E_1, E_2, \ldots E_n$ are non-empty mutually exclusive and exhaustive events (i.e. $P(E_i) > 0 \ \forall i$) of a sample space S and A be any event of non-zero probability then

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)} \ \forall \ i = 1, 2, \ldots n$$

Proof : By law of total probabilities we know that

$$P(A) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i) \qquad \ldots(i)$$

Also by law of multiplication of probabilities we have

$$P(E_i|A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i) P(A/E_i)} \text{ by using (i)}$$

This gives the proof of the Baye's theorem let us now apply the result of Baye's theorem to find probabilities.

Example 19.25 Given three identical coins (in shape and size) with following specifications

Coin I : with $P(H) = \frac{1}{3}$, $P(T) = \frac{2}{3}$

Coin II : with $P(H) = \frac{3}{4}$ $P(T) = \frac{1}{4}$

Coin III : with $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$ (normal coin).

A Coin is selected at random and tossed. The out come found to be head. What is the probability that the selected coin was coin III?

Solution : Let E_1, E_2, E_3 be the events that coins I, II or III is selected, respectively.

Then $\qquad P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, Let A be the event 'the coin drawn 'has head on tossing'.

Then $\qquad P(A/E_1) = $ P(a head on coin I) $= \frac{1}{3}$

$P(A/E_2) = $ P(a head on coin II) $= \frac{3}{4}$

$P(A/E_3) = $ P(a head on coin III) $= \frac{1}{2}$

Now the probability that the coin tossed is Coin III = $P(E_3/A)$

$$= \frac{P(E_3)\,P(A/E_3)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{3}{4}+\frac{1}{3}\times\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{3}+\frac{3}{4}+\frac{1}{2}} = \frac{6}{4+9+6} = \frac{6}{19}$$

Example 19.26 Bag I contains 4 red and 3 black balls while another bag II contains 6 red and 5 black balls. One of the bags is selected at random and a ball is drawn from it. Find the probability that the ball is drawn from Bag II, if it is known that the ball drawn is red.

Solution : Let E_1 and E_2 be the events of selecting Bag I and Bag II, respectively and A be the event of selecting a red ball.

Then, $\qquad P(E_1) = P(E_2) = \frac{1}{2}$

Also, $\qquad P(A/E_1) = $ P(drawing a red ball from Bag I) $= \frac{4}{7}$

$P(A/E_2) = $ P(drawing a red ball from Bag II) $= \frac{6}{11}$

Now, By Baye's theorem

P(bag selected is Bag II when it is known that red ball is drawn) = $P(E_2/A)$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1)+P(E_2)P(A/E_2)} = \frac{\frac{1}{2}\times\frac{4}{7}}{\frac{1}{2}\times\frac{6}{11}+\frac{1}{2}\times\frac{4}{7}}$$

$$= \frac{4/7}{\frac{6}{11}+\frac{4}{7}} = \frac{22}{43}$$

CHECK YOUR PROGRESS 19.6

1. Urn I contains 3 blue and 4 white balls and another Urn II contains 4 blue and 3 white balls. One Urn was selected at random and a ball was drawn from the selected Urn. The ball was found to be white. What is the probability that the ball was drawn from Urn-II?
2. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% by machine B were defective. All the items are put in one stock pile and then one item is randomly drawn from this and is found to be defective. Find the probability that the defective item was produced by machine A?
3. By examining the chest x-ray, the probability that T.B is detected when a person is actually suffering from it is 0.99.

   The probability that the doctor, diagnoses in correctly that a person has TB, on the basis of the x-ray is 0.001. In a certain city, 1 in 10000 persons suffer from TB. A person selected at random is diagnosed to have TB. What is the probability that person has actually TB?

19.12 : PROBABILITY DISTRIBUTION OF RANDOM VARIABLE

19.12.1 Variables : In earlier section you have learnt to find probabilities of various events with certain conditions. Let us now consider the case of tossing a coin four times. The outcomes can be shown in a sample space as :

S = {HHHH, HHHT, HHTH, HTHH, THHH, THHT, HHTT, HTTH, TTHH, HTHT, THTH, HTTT, THTT, TTHT, TTTH, TTTT}

On this sample space we can talk about various number associated with each outcome. For example, for each outcome, there is a number corresponding to number of heads we can call this number as X.

Clearly

X(HHHH) = 4, X(HHHT) = 3, X(HHTH) = 3

X(THHH) = 3, X(HHTT) = 2, X(HTTH) = 2

X(TTHH) = 2, X(HTHT) = 2, X(THTH) = 2

X(THHT) = 2, X(HTTT) = 1, X(THTT) = 1

X(TTHT) = 1, X(TTTH) = 1, X(TTTT) = 0

We find for each out come there corresponds values of X ranging from 0 to 4.

Such a variable X is called a random variable.

19.12.2 Definition

A random variable is a function whose domain is the sample space of a random experiment and range is real number values.

Example 19.27 Two dice are thrown simultaneously. Write the value of the random variable X : sum of number appearing on the upper faces of the dice.

Solution : The sample space of the experiment contains 36 elements.

S = {(1, 1), (1, 2), (1, 3) .... (1, 6)

(2, 1), (2, 2), (2, 3) .... (2, 6)

....

....

....

(6, 1), (6, 2), (6, 3) .... (6, 6)}

Clearly for each pair the sum of numbers appear ranging from 2 to 12. So the random variable x has the following values.

X(1, 1) = 2

X((1, 2), (2, 1)) = 3

X((1, 3), (2, 2), (3, 1)) = 4

X((1, 4), (2, 3), (3, 2), (4, 1) = 5

X((1, 5), (2, 4), (3, 3), (4, 2), (5, 1) = 6

X((1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), = 7

X((2, 6), (3, 5), (4, 4), (5, 3), (6, 2) = 8

X((3, 6), (4, 5), (5, 4), (6, 3) = 9

X((4, 6), (5, 5), (6, 4) = 10

X((5, 6), (6, 5)) = 11

X((6, 6)) = 12

19.12.3 Probability Distribution of a Random Variable

Let us now look at the experiment of drawing two cards successively with replacement from a well shuffled deck of 52 cards. Let us concentrate on the number of aces that can be there when two cards are successively drawn. Let it be denoted by X. Clearly X can take the values 0, 1 or 2.

The sample space for the experiment is given by S = {(Ace, Ace), (Ace, Non Ace), (Non Ace, Ace), (Non Ace, Non Ace)}

For X(Ace, Ace) = 2

X{(Ace, Non Ace) or (Non Ace, Ace)} = 1

and X{(Non Ace, Non Ace)} = 0

The probability that X can take the value 2 is P(Ace, Ace) = $\frac{4}{52}\times\frac{4}{52}$ as probability of an Ace is drawing one card is $\frac{4}{52}$.

Similarly

P(X = 1) =P[(Ace, non Ace) or (Non Ace, ace)]

=P(Ace, non Ace) + P(Non Ace, Ace)

$$= \frac{4}{52}\times\frac{48}{52}+\frac{48}{52}\times\frac{4}{52} = \frac{12}{169}+\frac{2}{169}=\frac{24}{169}$$

and P(X = 0) =P(Non Ace, Non Ace) = $\frac{48}{52}\times\frac{48}{52}=\frac{144}{169}$

The description given by the values of the random variable with the corresponding probabilities is called probability distribution.

19.12.4 Definition : The probability distribution of a random variable X is the distribution of probabilities to each value of X. A probability distribution of a random variable X is represented as

X_i	:	x_1	x_2	x_3	....	x_n
$P(X_i)$	:	P_1	P_2	P_3	....	P_n

where $P_i > 0, \sum_{i=1}^{n} P_i = 1, \forall\, i = 1, 2, 3, ...n$.

The real numbers x_1, x_2, ... x_n are the possible values of X and P_i is the probability of the random variable X_i taking the value X_i denoted as

$$P(X = x_i) = P_i$$

Thus the probability distribution of number of aces when two cards are successively drawn, with replacement from a deck of 52 cards is given by

X	:	0	1	2
$P(x_i)$	:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Note that in a probability distribution all probabilities must be between 0 and 1 and sum of all probabilities must be 1.

$$\Sigma P_i = \frac{144}{169}+\frac{24}{169}+\frac{1}{169}=\frac{144+24+1}{169}=1$$

Example 19.28 Check whether the distribution given below is a probability distribution or not

X	2	1	0	–1	–2
P(X)	0.1	0.2	0.3	0.2	0.2

Solution : All probabilities P(X) are positive and less than 1.

Also,
$$\Sigma P(x_i) = 0.1 + 0.2 + 0.3 + 0.2 + 0.2 = 1.0$$

Hence, the given distribution is probability distribution of a the random variable X.

Example 19.29 A random variable X has the following probability distribution :

X	–1	–2	–3	–4	–5	–6
P(X)	$\frac{1}{3}$	k	$\frac{1}{4}$	$2k$	$\frac{1}{6}$	$\frac{k}{4}$

Find (1) k (2) $P(X > -4)$ (3) $P(X < -4)$

Solution : (1) The sum of probabilities in the given distribution, must be 1.

$$\Rightarrow \quad \frac{1}{3} + k + \frac{1}{4} + 2k + \frac{1}{6} + \frac{k}{4} = 1$$

$$\Rightarrow \quad \frac{4 + 12k + 3 + 24k + 2 + 3k}{12} = 1$$

$$39k + 9 = 12$$

$$\Rightarrow \quad 39k = 3$$

$$\therefore \quad k = \frac{1}{13}$$

(2)
$$P(X > -4) = P(x = -3) + P(x = -2) + P(x = -1)$$

$$= \frac{1}{4} + k + \frac{1}{3} = \frac{1}{4} + \frac{1}{13} + \frac{1}{3} = \frac{103}{156}$$

(3)
$$P(X < -4) = P(x = -5) + P(x = -6)$$

$$= \frac{1}{6} + \frac{k}{4} = \frac{1}{6} + \frac{1}{13 \times 4} = \frac{29}{156}$$

Example 19.30 Find the probability distribution of number of tails in the simultaneous tosses of three coins.

Solution : The sample space for simultaneous toss of three coins is given by

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let X be the number of tails.

Clearly X can take values, 0, 1, 2 or 3.

Now,

$$P(X = 0) = P(HHH) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$$

$$P(X = 1) = P(HHT \text{ or } HTH \text{ or } THH)$$

$$= P(HHT) + P(HTH) + P(THH)$$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

$$P(X = 2) = P(HTT \text{ or } THT \text{ or } TTH)$$

$$= P(HTT) + P(THT) + P(TTH) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

and $P(X = 3) = P(TTT) = \frac{1}{8}$.

Hence, the required probability distribution is

X	:	0	1	2	3
P(X)	:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

CHECK YOUR PROGRESS 19.7

1. State which of the following are not probability distribution of a random variable. Justify your answer

(a)

x	100	200	300
P(x)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(b)

Y	0	1	2	3	4	5
P(y)	0.1	0.2	0.3	0.4	0.5	0.6

(c)

x_i	–1	–2	0	2	1
P_i	0.2	0.15	–0.5	0.45	0.7

(d)

x_i	2	3	4	5
P_i	0.4	0.1	0.2	0.2

2. Find the probability distribution of

(a) Number of red balls when two balls drawn are one after other with replacement from a bag containing 4 red and 3 white balls.

(b) Number of sixes when two dice are thrown simultaneously

(c) Number of doublets when two dice are thrown simultaneously

19.13 : MEAN AND VARIANCE OF A RANDOM VARIABLE

19.13.1 Mean

The mean of a random variable is denoted by μ and is defined as

$$\mu = \sum_{i=1}^{n} x_i P_i \,,$$

where $\Sigma P_i = 1,\ P_i > 0,\ \forall\, i = 1, 2, \ldots n.$

In other words we can say that the mean of a random variable is the sum of the product of values of the variables with corresponding probabilities. Mean of a random variable X is also called Expectation of the random variable 'X', denoted by E(x)

So $$E(x) = \mu = \sum_{i=1}^{n} x_i P_i \,.$$

19.13.2 : VARIANCE

Recall in frequently distribution we have studied that variance is a measure of dispersion or variability in the values. The similar meaning is attached to variance of a random variable.

Definition : Let a probability distribution be given as

X_i	:	x_1	x_2	x_3	...	x_n
$P(X_i)$	:	P_1	P_2	P_3	...	P_n

Let $\mu = E(x)$ be the mean of x.

Then the varaiance of X, denoted by var(x) or σ_x^2 is defined as

$$\sigma_x^2 = \text{Var}(x) = \sum_{i=1}^{n} (x_i - \mu)^2 P_i$$

$$= \sum_{i=1}^{n} (x_1^2 p_i + \mu^2 p_i - 2\mu\, x_i p_i) = \sum_{i=1}^{n} (x_i^2 P_i + \mu^2 P_i - 2\mu x_i\ p_i)$$

$$= \sum_{i=1}^{n} x_i^2 p_i + \sum_{i=1}^{n} \mu^2 P_i - \sum_{i=1}^{n} 2\mu x_i\ p_i = \sum_{i=1}^{n} x_i^2 p_i + \mu^2 \sum_{i=1}^{n} P_i - 2\mu \sum_{i=1}^{n} x_i\ p_i$$

$$= \sum_{i=1}^{n} x_i^2 p_i + \mu^2 .1 - 2\,\mu.\mu = \sum_{i=1}^{n} x_i^2 p_i - \mu^2 \quad \left(\because\ \mu = \sum_{i=1}^{n} x_i^2 p_i \text{ and } \sum_{i=1}^{n} p_i = 1\right)$$

$$= \sum_{i=1}^{n} x_i^2 p_i - \left(\sum_{i=1}^{n} x_i p_i \right)^2$$

We can also write

$$\text{var}(x) = E(X^2) - [E(X)]^2$$

Example 19.31 Find the mean and variance of the following distribution

x	-2	-1	0	1	2
$P(x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Solution : Given distribution is

X_i	-2	-1	0	1	2
$P(X_i)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$X_iP(X_i)$	$-\frac{2}{8}$	$-\frac{2}{8}$	0	$\frac{1}{8}$	$\frac{2}{8}$
$x^2P(X_i)$	$\frac{4}{8}$	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{4}{8}$

Now, $$\mu = \Sigma P(X_i)X_i = -\frac{2}{8} - \frac{2}{8} + 0 + \frac{1}{8} + \frac{2}{8} = -\frac{1}{8}$$

$$\text{Var}(x) = \Sigma X_i^2 P(X_i) - [\Sigma\, P(X_i)X_i]^2$$

$$= \left[\frac{4}{8} + \frac{2}{8} + 0 + \frac{1}{8} + \frac{4}{8}\right] - \left(-\frac{1}{8}\right)^2$$

$$= \frac{11}{8} - \frac{1}{64} = \frac{87}{64}$$

CHECK YOUR PROGRESS 19.8

1. Find mean and variance in each of the following distributions

(a)

X	:	1	2	3	4
P(X)	:	0.3	0.2	0.4	0.1

(b)

y_i	-2	-1	0	1	2
$P(y_i)$	0.1	0.2	0.3	0.25	0.15

2. Find the mean number of heads in three tosses of a fair coin.
3. Let X denote the difference of two numbers obtained on throwing two fair dice. Find the mean and variance of X. (Take absolute value of the difference)
4. Find the mean of the numbers of tails obtained when a biased coin having 25% chances of head and 75% of tail, is tossed two times.
5. Find the mean and variance of the number of sixes when two dice are thrown.

19.14 BERNOULLI TRIALS

When an experiment is repeated under similar conditions, each repeat is called a trial of the experiment. For example, if a coin is tossed three times, we say that there are three trials of the tossing of the coin.

A particular event may be called success of a trial. Clearly non-happening of the event may be termed as a failure. For, example in throwing a die, if the occurrence of a number less then 4 is named as success then the non-occurrence of a number less than 4 is named as failure. Thus, each trial can have two outcomes namely, success or failure.

Two or more trials of a random experiment can be performed in two ways :

1. The probability of success or failure remain constant in each trial. For example tossing a coin *n* number of times, but in each trial probability of getting head is $\frac{1}{2}$. Such trials are called independent trials.
2. The probability of success/failure varies with each trial. For example in drawing card from a deck of cards one after the other without replacement, in such trials if success is taken to be drawing a card of spade, the probability of success in respective trials will change.

i.e.	Trial	1st	2nd	3rd, ...
	Probability	$\frac{13}{52}$	$\frac{12}{51}$	$\frac{11}{50}$, ...

The trials of first type i.e. independent trials with two out comes success or failure are called Bernoulli trials.

Definition : Trials of a random experiment are called Bernoulli trials, if each trial has exactly two outcomes and trials are finite and independent.

19.15 : BINOMIAL DISTRIBUTION

The probability distribution of number successes in Bernoulli trials of a random experiment may be obtained by the expansion of $(q + p)^n$ where

p = prob. of success in each trial

q = $1 - p$, = prob. of failure

n = number of trials

Such a probability distribution is called Binomial Distribution. In other words we can say that in n Bernoulli trials of a random experiment, the number of successes can have the value, 0, 1, 2, 3, ...n.

So the Binomial Distribution of number of success : X, is given by

P(X = 0) = 1st term of the expansion of $(q + p)^n$

P(X= 1) = 2nd term of the expansion of $(q + p)^n$

$\vdots$

P(X = r) = (r + 1)th term of expansion of $(q + p)^n$

$\vdots$

P(X = n) = (n + 1)th term of expansion of (q + p)n

We know that

$$(q + p)^n = {}^nC_0\, q^n + {}^nC_1\, q^{n-1}\, p + {}^nC_2\, q^{n-2}\, p^2 + \ldots + {}^nC_r\, q^{n-r}\, p^r + \ldots + {}^nC_n p^n$$

$$\Rightarrow \quad P(x = 0) = {}^nC_0\, q^n$$

$$P(x = 1) = {}^nC_1\, q^{n-1}\, p$$

$$P(x = 2) = {}^nC_2\, q^{n-2}\, p^2$$

$$\vdots$$

$$P(X = r) = {}^nC_r\, q^{n-r}\, p^r$$

$$\vdots$$

$$P(X = n) = {}^nC_n\, p^n.$$

A Binomial distribution with n Bernoulli trials and probability of success in each trial as P, is denoted by B(n, p)

Let us now understand Binomial Distribution with following examples.

Example 19.32 Write the Binomial Distribution of number of successes in 3 Bernoulli trials.

Solution : Let p = prob. of success (S) in each trial

q = prob. of failure (F) in each trial

Clearly $q = 1 - p$

Number of successes in three trials can take the values 0, 1, 2 or 3

The sample space for three trials ....

S = {SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF}

where S and F denote success and failure.

Now P(S = 0) = P(FFF) = P(F) P(F) P(F) = $q.q.q = q^3$

P(S = 1) = P(SFF, FSF or FFS) = P(SFF) + P(FSF) + P(FFS)

= P(S).P(F) P(F) + P(F).P(S).P(F) + P(F) P(F) P(S)

= $p.q.q. + q.p.q + q.q.p = 3\, q^2 p$

$$P(S = 2) = P(SSF \text{ or } SFS \text{ or } FSS)$$
$$= P(SSF) + P(SFS) + P(FSS)$$
$$= P(S).P(S).P(F) + P(S)\,P(F)P(S) + P(F)\,P(S)\,P(S)$$
$$= p.p.q + p.q.p + q.p.p = 3qp^2$$
$$P(S = 3) = P(SSS) = P(S)\,.\,P(S)\,.\,P(S) = p.p.p = p^3$$

Hence the prob. distribution of number of successes is

X_i	:	0	1	2	3
$P(X_i)$	:	q^3	$3q^2p$	$3qp^2$	p^3

Also $(q + p)^3 = q^3 + 3q^2p + 3p^2q + p^3$

Note that probabilities of 0, 1, 2 or 3 successes are respectively the 1st, 2nd, 3rd and 4th term in the expansion of $(q + p)^3$.

Example 19.33 A die is thrown 5 times. If getting 'an even number' is a success, what is the probability of.

(a) 5 successes

(b) at least 4 successes

(c) at most 3 successes?

Solution : Given X : "an even number"

Then p =P(an even number) $= \frac{3}{6} = \frac{1}{2}$

$$q = P(\text{not an even number}) = \frac{3}{6} = \frac{1}{2}$$

Since the trials of throwing die are Bernoulli trials.

So, $P(r \text{ successes}) = {}^nC_r\, q^{n-r} p^r$

Here, $n = 5 = {}^5C_r \left(\frac{1}{2}\right)^{5-2} \left(\frac{1}{2}\right)^2 = {}^5C_r \left(\frac{1}{2}\right)^5$

(*a*) Now P(5 successes) $= {}^5C_5 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

(b) P(at least 3 successes)

= P(3 success or 4 successes or 5 successes)

= P(3 successes) + P(4 successes) + P(5 successes).

$$= {}^5C_3 \left(\frac{1}{2}\right)^5 + {}^5C_4 \left(\frac{1}{2}\right)^5 + {}^5C_5 \left(\frac{1}{2}\right)^5$$

$$= \left(\frac{1}{2}\right)^5 \left(\frac{5\times4\times3}{3\times2\times1}+5+1\right) = \left(\frac{1}{2}\right)^5 (10+5+1) = \frac{16}{32} = \frac{1}{2}$$

(c) P(at most 3 successes)

= P(0 successes or 1 success or 2 success or 3 successes)

= P(0 successes) + P(1 success) + P(2 successes) + P(3 successes)

$$= {}^5C_0\left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5$$

$$= \frac{1}{32} + 5\times\frac{1}{32} + \frac{5\times4}{2\times1}\times\frac{1}{32} + \frac{5\times4\times3}{3\times2\times1}\times\frac{1}{32}$$

$$= \frac{1}{32}[1+5+10+10] = \frac{26}{32} = \frac{13}{16}.$$

CHECK YOUR PROGRESS 19.9

1. Find the following probabilities when a fair coin is tossed 10 times.
   (a) exactly 6 heads
   (b) at least 6 heads
   (c) at most 6 heads
2. A pair of dice is thrown 4 times. If getting a doublet (1, 1), (2, 2)... etc. is considered a success, find the probability of two successes.
3. From a bag containing 3 red and 4 black sells, five balls are drawn successively with replacement. If getting "a black ball" is considered "success", find the probability of getting 3 successes.
4. In a lot of bulbs manufactured in a factory, 5% are defective. What is the probability that a sample of 10 bulbs will include not more than one defective bulb?
5. Probability that a CFL produced by a factory will fuse after 1 year of use is 0.01. Find the probability that out of 5 such CFL's.
   (a) none
   (b) not more than one
   (c) more than one
   (d) at least one

   will fuse after 1 year of use.

LET US SUM UP

- **Complement of an event :** The complement of an event A consists of all those outcomes which are not favourable to the event A, and is denoted by 'not A' or by $\bar{A}$.

- **Event 'A or B' :** The event 'A or B' occurs if either A or B or both occur.
- **Event 'A and B' :** The event 'A and B' consists of all those outcomes which are favourable to both the events A and B.
- **Addition Law of Probability :** For any two events A and B of a sample space S

  $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

- **Additive Law of Probability for Mutually Exclusive Events :** If A and B are two mutually exclusive events, then

  $$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B).$$

- **Odds in Favour of an Event :** If the odds for A are a to b, then $P(A) = \dfrac{a}{a+b}$

  If odds against A are a to b, then $P(A) = \dfrac{b}{a+b}$

- Two events are mutually exclusive, if occurrence of one precludes the possibility of simultaneous occurrence of the other.
- Two events are independent, if the occurence of one does not affect the occurence of other. If A and B are independent events, then P (A and B) = P (A). P (B) or $P(A \cap B) = P(A).P(B)$
- For two dependent events $P(A \cap B) = P(A).P\left(B/A\right) \, where \, P(A) > 0$

  or

  $P(A \cap B) = P(B) \, where \, P\left(A/B\right) / P(B) > 0$

- Conditional Probability $P\left(\dfrac{A}{B}\right) = \dfrac{A(A \cap B)}{P(B)}$ and $P\left(B/A\right) = \dfrac{P(A \cap B)}{P(A)}$
- Theorem of Total Probability

  $$P(A) = P(E_1).\, P\left(\frac{A}{E_1}\right) + P(E_2).\, P\left(\frac{A}{E_2}\right) + \cdots + P(E_n).P\left(\frac{A}{E_n}\right)$$

- Baye's Theorem: If $B_1, B_2 \cdots B_n$ are mutually exclusive events and A is any event that

  occurs with B_1 or B_2 or B_n then $P\left(\dfrac{Bi}{A}\right) = \dfrac{P(Bi), P\left(A/Bi\right)}{\sum_{i=1}^{n} P(Bi), P\left(A/Bi\right)}, i = 1, 2, \cdots n$

- Mean and Variance of a Random Varibale

  $$\mu = E(x) = \frac{n}{z} XiPi \, , \; \sigma^2 = \frac{n}{z}(xi - \mu)^2 = \frac{n}{z} xi^2 pi - \mu^2$$

- Binomial Distribution, $P(x = r) = nc_r \, p^r . q^{n-r}$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Probability

http://mathworld.wolfram.com/Probability

http://en.wikipedia.org/wiki/probability_distribution

http://en.wikipedia.org/wiki/Probability_theory

http://en.wikipedia.org/wiki/Bernoulli_distribution

TERMINAL EXERCISE

1. In a simultaneous toss of four coins, what is the probability of getting
   (a) exactly three heads ?
   (b) at least three heads ?
   (c) atmost three heads ?
2. Two dice are thrown once. Find the probability of getting an odd number on the first die or a sum of seven.
3. An integer is chosen at random from first two hundred integers. What is the probability that the integer chosen is divisible by 6 or 8 ?
4. A bag contains 13 balls numbered from 1 to 13. A ball is drawn at random. What is the probability that the number obtained it is divisible by either 2 or 3 ?
5. Find the probability of getting 2 or 3 heads, when a coin is tossed four times.
6. Are the following probability assignments consistent ? Justify your answer.
   (a) P (A) = 0.6, P (B) = 0.5 , P (A and B) = 0.4
   (b) P (A) = 0.2, P (B) = 0.3 , P (A and B) = 0.4
   (c) P (A) = P (B) = 0.7, P (A and B) = 0.2
7. A box contains 25 tickets numbered 1 to 25. Two tickets are drawn at random. What is the probability that the product of the numbers is even ?
8. A drawer contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or is a bolt ?
9. A lady buys a dozen eggs, of which two turn out to be bad. She chose four eggs to scramble for breakfast. Find the chances that she chooses
   (a) all good eggs
   (b) three good and one bad eggs
   (c) two good and two bad eggs (d) at least one bad egg.

10. Two cards are drawn at random without replacement from a well-shuffled deck of 52 cards. Find the probability that the cards are both red or both kings.

11. Let A and B be two events such that $P(\vec{A})=\frac{1}{2}, P(\vec{B})=\frac{2}{3}, P(A\vec{\cap}B)=\frac{1}{4}$, Find $P(A/B)$ and $P(B/A)$.

12. A bag contains 10 black and 5 white balls. Two balls are drawn from the bag successively whithout replacement. Find the probability that both the balls drawn are black.

13. Find the probability distribution of X; where X denotes the sum of numbers obtained when two dice are rolled.

14. An urn contains 4 black, 2 red and 2 white balls. Two balls (one after the other without replacement) are drawn randomly from the urn. Find the probability distribution of number of black balls.

15. Find the mean and variance of number of kings when two cards are simultaneously drawn from a deck of 52 cards.

16. Ten bolts are drawn successively with replacement from a bag containing 5% defective bolts. Find the probability that there is at least one defective bolt.

17. Find the mean of the Binomial $B\left(4, \frac{1}{3}\right)$.

18. A die is thrown again and again until three sixes are obtained. Find the probability of getting the third six in the sixth throw.

19. How many times must a man toss a fair coin so that the probability of having atleast one head is more than 90%?

20. Find the probability of getting 5 exactly twice in seven throws of a die.

21. Find the mean number of heads in three tosses of a fair coin.

22. A factory produces nuts, by using three machine A, B and C, manufacturing 20%, 40% and 40% of the nuts. 5%, 4% and 2% of their outputs are respectively found to be defective, A nut is drawn of randomly from the product and is found to be defective. What is the probability that it is manufactured by the machine C?

MODULE - V
Statistics and Probability

Notes

ANSWERS

CHECK YOUR PROGRESS 19.1

1. $\frac{1}{6}$ 2. $\frac{1}{2}$ 3. $\frac{1}{2}$ 4. $\frac{3}{4}$

5. (i) $\frac{3}{5}$ (ii) $\frac{2}{5}$

6. (i) $\frac{5}{36}$ (ii) $\frac{5}{36}$ (iii) $\frac{1}{12}$ (iv) $\frac{1}{36}$

7. $\frac{5}{9}$ 8. $\frac{1}{12}$ 9. $\frac{1}{2}$

10. (i) $\frac{1}{4}$ (ii) $\frac{1}{13}$ (iii) $\frac{1}{52}$

11. (i) $\frac{5}{12}$ (ii) $\frac{1}{6}$ (iii) $\frac{11}{36}$

12. (i) $\frac{1}{8}$ (ii) $\frac{7}{8}$ (iii) $\frac{1}{8}$

CHECK YOUR PROGRESS 19.2

1. $\frac{1}{8}$ 2. $\frac{20}{39}$ 3.(a) $\frac{4}{25}$ (b) $\frac{38}{245}$

4. $\frac{1}{5525}$ 6. (i) $\frac{3}{10}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{30}$

7. $\frac{10}{133}$ 8. $\frac{4}{7}$ 9. $\frac{60}{143}$ 10. $\frac{1}{4}$

CHECK YOUR PROGRESS 19.3

1. $\frac{4}{13}$ 2. $\frac{7}{36}$ 3. $\frac{9}{16}$ 4. $\frac{7}{12}$

5. $\frac{4}{9}$ 6. $\frac{1}{2}$ 7. $\frac{7}{13}$ 8. $\frac{5}{12}$

9. (a) $\frac{5}{18}$ (b) 0.7 10. $\frac{4}{11}$

11. (a) $\frac{5}{6}$ (b) $\frac{35}{36}$ 12. $\frac{3}{4}$

13. (a) The odds for A are 7 to 3 . The odds against A are 3 to 7

13. (b) The odds for A are 4 to 1 and The odds against A are 1 to 4

14. (a) $\frac{7}{9}$ (b) $\frac{7}{17}$ 15. (a) $\frac{5}{9}$ (b) $\frac{3}{4}$ 16. (a), (c) 17. $\frac{4}{7}$ 18. $\frac{1}{4}$

CHECK YOUR PROGRESS 19.4

1. (a) $\frac{2}{7}$ (b) $\frac{1}{35}$ (c) $\frac{24}{35}$ (d) $\frac{11}{35}$ 2. (a) $\frac{2}{3}$ (b) $\frac{1}{2}$ 3. $\frac{1}{4}$

4. (a) $\frac{1}{35}$ (b) $\frac{11}{35}$ 5. $\frac{1}{2}$ 6. (a) $\frac{5}{144}$ (b) $\frac{1}{1014}$ 7. $\frac{53}{80}$

8. (a) $\frac{36}{169}$ (b) $\frac{84}{169}$ (c) $\frac{120}{169}$ (d) $\frac{49}{169}$

9. (a) Independent (b) Independent

CHECK YOUR PROGRESS 19.5

1. $\frac{13}{51}$ 2. $\frac{1}{2}$ 3. $\frac{25}{204}$ 4. $\frac{1}{2}, \frac{1}{4}$ 5. $\frac{3}{7}$

CHECK YOUR PROGRESS 19.6

1. $\frac{3}{7}$ 2. $\frac{3}{4}$ 3. $\frac{10}{111}$

CHECK YOUR PROGRESS 19.7

1. (a) Yes (b) No, as ΣP_i is not 1

(c) No, as one of the P_i is –ve (d) No as ΣP_i is not 1

2. (a)

x	:	0	1	2
$P(x)$	:	$\frac{9}{49}$	$\frac{24}{49}$	$\frac{16}{49}$

(b)

X_i	:	0	1	2
$P(X_i)$	:	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

CHECK YOUR PROGRESS 19.8

1. (a) $\mu = 2.3$, Var $= 1.01$ (b) $\mu = 0.15$, Var $= 0.4275$

2. $\mu = \frac{3}{2}$

3.

X	:	0	1	2	3	4	5
$P(X_i)$	:	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

4. Mean $\mu = \frac{3}{2}$, Var. $(X_i) = \frac{3}{8}$ 5. Mean $= \frac{1}{3}$, Var. $= \frac{5}{18}$

CHECK YOUR PROGRESS 19.9

1. (i) $\frac{105}{512}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$

2. $\frac{25}{216}$ 3. $\frac{90 \times 64}{7^5}$ 4. $\left(\frac{29}{20}\right)\left(\frac{19}{20}\right)^9$

5. (a) $\left(\frac{99}{100}\right)^5$ (b) $\left(\frac{99}{100}\right)^5 + 5.\frac{99^4}{100^5}$

(c) $1-\left\{\left(\frac{99}{100}\right)^5 + \frac{5 \times 99^4}{100^5}\right\}$ (d) $1-\left(\frac{99}{100}\right)^5$

TERMINAL EXERCISE

1. (a) $\frac{1}{4}$ (b) $\frac{5}{16}$ (c) $\frac{15}{16}$ 2. $\frac{7}{12}$ 3. $\frac{1}{4}$

4. $\frac{8}{13}$ 5. $\frac{5}{8}$ 6. Only (a) is consistent 7. $\frac{456}{625}$

8. $\frac{5}{8}$ 9. (a) $\frac{14}{33}$ (b) $\frac{16}{33}$ (c) $\frac{1}{11}$ (d) $\frac{19}{33}$

10. $\frac{55}{221}$ 11. $\frac{3}{4}, \frac{1}{2}$ 12. $\frac{3}{7}$

13.

X_i :	2	3	4	5	6	7	8	9	10	11	12
$P(X_i)$:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

14.

x :	0	1	2
$P(x)$:	$\frac{3}{14}$	$\frac{4}{7}$	$\frac{3}{14}$

15. Mean $= \frac{34}{221}$, variance $= \frac{6800}{(221)^2}$. 16. $1-\left(\frac{19}{20}\right)^{10}$ 17. $\frac{4}{3}$

18. $\frac{625}{23328}$ 19. $n = 4$ 20. $\frac{7}{12} \times \left(\frac{5}{6}\right)^5$

21. 1.5 22. $\frac{4}{17}$