# 25

311en25

# LIMIT AND CONTINUITY

Consider the function $f(x)=\dfrac{x^2-1}{x-1}$

You can see that the function f(x) is not defined at $x = 1$ as $x-1$ is in the denominator. Take the value of x very nearly equal to but not equal to 1 as given in the tables below. In this case $x-1 \neq 0$ as $x \neq 1$.

$\therefore$ We can write $f(x)=\dfrac{x^2-1}{x-1}=\dfrac{(x+1)(x-1)}{(x-1)}=x+1$, because $x-1\neq 0$ and so division by $(x-1)$ is possible.

Table -1

| x | f(x) |
|---|---|
| 0.5 | 1.5 |
| 0.6 | 1.6 |
| 0.7 | 1.7 |
| 0.8 | 1.8 |
| 0.9 | 1.9 |
| 0.91 | 1.91 |
| : | : |
| : | : |
| 0.99 | 1.99 |
| : | : |
| : | : |
| 0.9999 | 1.9999 |

Table - 2

| x | f(x) |
|---|---|
| 1.9 | 2.9 |
| 1.8 | 2.8 |
| 1.7 | 2.7 |
| 1.6 | 2.6 |
| 1.5 | 2.5 |
| : | : |
| : | : |
| 1.1 | 2.1 |
| 1.01 | 2.01 |
| 1.001 | 2.001 |
| : | : |
| : | : |
| 1.00001 | 2.00001 |

In the above tables, you can see that as x gets closer to 1, the corresponding value of f (x) also gets closer to 2.

However, in this case f(x) is not defined at $x = 1$. The idea can be expressed by saying that the limiting value of f(x) is 2 when x approaches to 1.

Let us consider another function $f(x) = 2x$. Here, we are interested to see its behavior near the point 1 and at $x = 1$. We find that as x gets nearer to 1, the corresponding value of f (x) gets closer to 2 at $x = 1$ and the value of f (x) is also 2.

So from the above findings, what more can we say about the behaviour of the function near $x = 2$ and at $x = 2$ ?

In this lesson we propose to study the behaviour of a function near and at a particular point where the function may or may not be defined.

## OBJECTIVES

After studying this lesson, you will be able to :

- define limit of a function
- derive standard limits of a function
- evaluate limit using different methods and standard limits.
- define and interprete geometrically the continuity of a function at a point;
- define the continuity of a function in an interval;
- determine the continuity or otherwise of a function at a point; and
- state and use the theorems on continuity of functions with the help of examples.

## EXPECTED BACKGROUND KNOWLEDGE

- Concept of a function
- Drawing the graph of a function
- Concept of trigonometric function
- Concepts of exponential and logarithmic functions

## 25.1 LIMIT OF A FUNCTION

In the introduction, we considered the function $f(x) = \dfrac{x^2 - 1}{x - 1}$. We have seen that as x approaches l, f (x) approaches 2. In general, if a function f (x) approaches L when x approaches 'a', we say that L is the limiting value of f (x)

Symbolically it is written as

$$\lim_{x \to a} f(x) = L$$

Now let us find the limiting value of the function $(5x - 3)$ when x approaches 0.

i.e. $$\lim_{x \to 0} (5x - 3)$$

For finding this limit, we assign values to x from left and also from right of 0.

| x | −0.1 | −0.01 | −0.001 | −0.0001 .......... |
|---|---|---|---|---|
| 5x − 3 | −3.5 | −3.05 | −3.005 | −3.0005 .......... |

| x | 0.1 | 0.01 | 0.001 | 0.0001 .......... |
|---|---|---|---|---|
| 5x − 3 | −2.5 | −2.95 | −2.995 | −2.9995 ......... |

It is clear from the above that the limit of $(5x-3)$ as $x \to 0$ is -3

i.e., $$\lim_{x\to 0}(5x-3) = -3$$

This is illustrated graphically in the Fig. 20.1

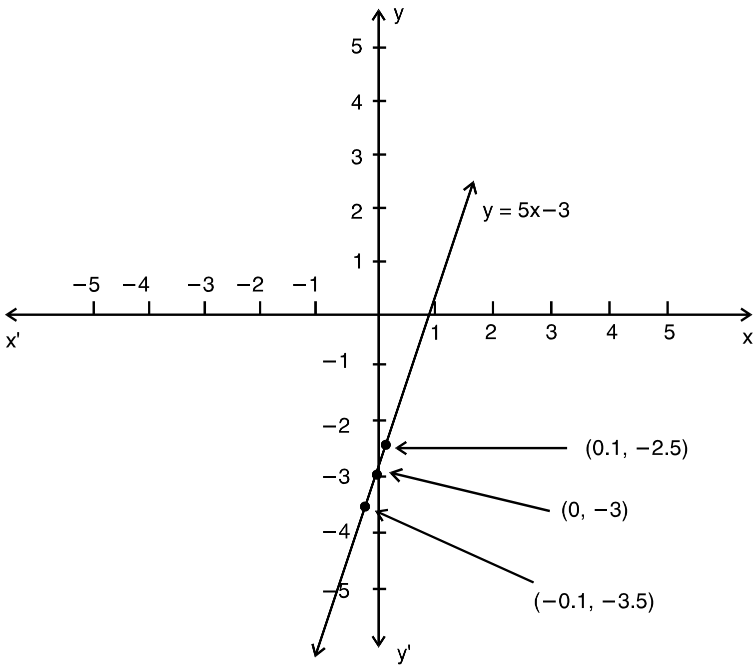



**Fig. 25.1**

The method of finding limiting values of a function at a given point by putting the values of the variable very close to that point may not always be convenient.

We, therefore, need other methods for calculating the limits of a function as x (independent variable) ends to a finite quantity, say a

**Consider an example :** Find $\lim_{x\to 3} f(x)$, where $f(x) = \dfrac{x^2-9}{x-3}$

We can solve it by the method of substitution. Steps of which are as follows :

**Remarks :** It may be noted that f (3) is not defined, however, in this case the limit of the

| | |
|---|---|
| **Step 1:** We consider a value of x close to a say x = a + h, where h is a very small positive number. Clearly, as $x \to a$, $h \to 0$ | For $f(x)=\dfrac{x^2-9}{x-3}$ we write $x=3+h$, so that as $x \to 3, h \to 0$ |
| **Step 2 :** Simplify $f(x)=f(a+h)$ | Now $f(x)=f(3+h)$ <br> $=\dfrac{(3+h)^2-9}{3+h-3}$ <br> $=\dfrac{h^2+6h}{h}$ <br> $= h + 6$ |
| **Step 3 :** Put h = 0 and get the requried result | $\therefore \lim\limits_{x\to 3} f(x)=\lim\limits_{h\to 0}(6+h)$ <br> As $x \to 0$, $h \to 0$ <br> Thus, $\lim\limits_{x\to 3} f(x)=6+0=6$ <br> by putting h = 0. |

function $f(x)$ as $x \to 3$ is 6.

Now we shall discuss other methods of finding limits of different types of functions.

**Consider the example :**

Find $\lim\limits_{x\to 1} f(x)$, where $f(x)=\begin{cases}\dfrac{x^3-1}{x^2-1}, & x \neq 1 \\ 1, & x=1\end{cases}$

Here, for $x \neq 1, f(x)=\dfrac{x^3-1}{x^2-1}=\dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

It shows that if f (x) is of the form $\dfrac{g(x)}{h(x)}$, then we may be able to solve it by the method of factors. In such case, we follow the following steps :

| | |
|---|---|
| **Step 1.** Factorise g (x) and h (x) | **Sol.** $f(x)=\dfrac{x^3-1}{x^2-1}$ $=\dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$ $(\because x\neq 1, \therefore x-1\neq 0$ and as such can be cancelled) |
| **Step 2 :** Simplify f (x) | $\therefore \quad f(x)=\dfrac{x^2+x+1}{x+1}$ |
| **Step 3 :** Putting the value of x, we get the required limit. | $\therefore \lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}=\dfrac{1+1+1}{1+1}=\dfrac{3}{2}$ Also $f(1)=1$(given) In this case, $\lim\limits_{x\to 1} f(x)\neq f(1)$ |

Thus, the limit of a function f (x) as $x\to a$ may be different from the value of the functioin at $x=a$.

Now, we take an example which cannot be solved by the method of substitutions or method of factors.

**Evaluate** $\quad \lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Here, we do the following steps :

**Step 1.** Rationalise the factor containing square root.

**Step 2.** Simplify.

**Step 3.** Put the value of x and get the required result.

**Solution :**

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{\sqrt{(1+x)^2}-\sqrt{(1-x)^2}}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\frac{1+x-1+x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$\therefore \quad \lim_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x \to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = 1$$

## 25.2 LEFT AND RIGHT HAND LIMITS

You have already seen that $x \to a$ means x takes values which are very close to 'a', i.e. either the value is greater than 'a' or less than 'a'.

In case x takes only those values which are less than 'a' and very close to 'a' then we say x is approaches 'a' from the left and we write it as $x \to a^-$. Similarly, if x takes values which are greater than 'a' and very close to 'a' then we say x is approaching 'a' from the right and we write it as $x \to a^+$.

Thus, if a function f (x) approaches a limit $\ell_1$, as x approaches 'a' from left, we say that the left hand limit of $f(x)$ as $x \to a$ is $\ell_1$.

We denote it by writing

$$\lim_{x \to a^-} f(x) = \ell_1 \qquad \text{or} \qquad \lim_{h \to 0} f(a-h) = \ell_1, h > 0$$

Similarly, if f (x) approaches the limit $\ell_2$, as x approaches 'a' from right we say, that the right hand limit of $f(x)$ as $x \to a$ is $\ell_2$.

We denote it by writing

$$\lim_{x \to a^+} f(x) = \ell_2 \qquad \text{or} \qquad \lim_{h \to 0} f(a+h) = \ell_2, h > 0$$

**Working Rules**

| Finding the right hand limit i.e., | Finding the left hand limit, i.e, |
|---|---|
| $\lim_{x \to a^+} f(x)$ | $\lim_{x \to a^-} f(x)$ |
| Put $x = a + h$ | Put $x = a - h$ |
| Find $\lim_{h \to 0} f(a+h)$ | Find $\lim_{h \to 0} f(a-h)$ |

**Note :** In both cases remember that h takes only positive values.

## 25.3 LIMIT OF FUNCTION y = f(x) AT x = a

**Consider an example :**

Find $\lim\limits_{x \to 1} f(x)$, where $f(x) = x^2 + 5x + 3$

Here $$\lim_{x \to 1^+} f(x) = \lim_{h \to 0}\left[(1+h)^2 + 5(1+h) + 3\right]$$

$$= \lim_{h \to 0}\left[1 + 2h + h^2 + 5 + 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad .....(i)$$

and $$\lim_{x \to 1^-} f(x) = \lim_{h \to 0}\left[(1-h)^2 + 5(1-h) + 3\right]$$

$$= \lim_{x \to 0}\left[1 - 2h + h^2 + 5 - 5h + 3\right]$$

$$= 1 + 5 + 3 = 9 \qquad .....(ii)$$

From (i) and (ii), $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x)$

**Now consider another example :**

Evaluate : $$\lim_{x \to 3} \frac{|x-3|}{x-3}$$

Here $$\lim_{x \to 3^+} \frac{|x-3|}{x-3} = \lim_{h \to 0} \frac{|(3+h)-3|}{[(3+h)-3]}$$

$$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} \text{ (as } h>0\text{, so } |h| = h)$$

$$= 1 \qquad .....(iii)$$

and $$\lim_{x \to 3^-} \frac{|x-3|}{x-3} = \lim_{h \to 0} \frac{|(3-h)-3|}{[(3-h)-3]}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} \quad \text{(as } h > 0\text{, so } |-h| = h)$$

$$= -1 \qquad .....(iv)$$

$\therefore$ From (iii) and (iv), $\lim\limits_{x \to 3^+} \dfrac{|x-3|}{x-3} \neq \lim\limits_{x \to 3^-} \dfrac{|x-3|}{x-3}$

Thus, in the first example right hand limit = left hand limit whereas in the second example right hand limit $\neq$ left hand limit.

Hence the left hand and the right hand limits may not always be equal.

We may conclude that

$\lim_{x\to1}(x^2+5x+3)$ exists (which is equal to 9) and $\lim_{x\to3}\frac{|x-3|}{x-3}$ does not exist.

**Note :**

I $\left.\begin{array}{l}\lim_{x\to a^+} f(x)=\ell \\ \text{and} \quad \lim_{x\to a^-} f(x)=\ell\end{array}\right\} \Rightarrow \lim_{x\to a} f(x)=\ell$

II $\left.\begin{array}{l}\lim_{x\to a^+} f(x)=\ell_1 \\ \text{and} \quad \lim_{x\to a^-} f(x)=\ell_2\end{array}\right\} \Rightarrow \lim_{x\to a} f(x)$ does not exist.

III $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ does not exist $\Rightarrow \lim_{x\to a} f(x)$ does not exist.

## 25.4 BASIC THEOREMS ON LIMITS

1. $\lim_{x\to a} cx = c\lim_{x\to a} x$, c being a constant.

To verify this, consider the function $f(x)=5x$.

We observe that in $\lim_{x\to2} 5x$, 5 being a constant is not affected by the limit.

$$\therefore \quad \lim_{x\to2} 5x = 5\lim_{x\to2} x$$
$$= 5 \times 2 = 10$$

2. $$\lim_{x\to a}\left[g(x)+h(x)+p(x)+....\right]=\lim_{x\to a} g(x)+\lim_{x\to a} h(x)+\lim_{x\to a} p(x)+........$$

where $g(x),h(x),p(x),....$ are any function.

3. $$\lim_{x\to a}\left[f(x)\cdot g(x)\right]=\lim_{x\to a} f(x)\lim_{x\to a} g(x)$$

To verify this, consider $f(x)=5x^2+2x+3$

and g (x) = x + 2.

Then $$\lim_{x\to0} f(x)=\lim_{x\to0}\left(5x^2+2x+3\right)$$

$$=5\lim_{x\to0} x^2+2\lim_{x\to0} x+3=3$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}(x+2) = \lim_{x\to 0} x + 2 = 2$$

$\therefore$
$$\lim_{x\to 0}(5x^2+2x+3) \lim_{x\to 0}(x+2) = 6 \qquad .....(i)$$

Again
$$\lim_{x\to 0}[f(x)\cdot g(x)] = \lim_{x\to 0}[(5x^2+2x+3)(x+2)]$$

$$= \lim_{x\to 0}(5x^3+12x^2+7x+6)$$

$$= 5\lim_{x\to 0} x^3 + 12\lim_{x\to 0} x^2 + 7\lim_{x\to 0} x + 6$$

$$= 6 \qquad .....(ii)$$

From (i) and (ii), $\lim_{x\to 0}[(5x^2+2x+3)(x+2)] = \lim_{x\to 0}(5x^2+2x+3) \lim_{x\to 0}(x+2)$

4. $\lim_{x\to a}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$

To verify this, consider the function $f(x) = \dfrac{x^2+5x+6}{x+2}$

we have
$$\lim_{x\to -1}(x^2+5x+6) = (-1)^2 + 5(-1) + 6 = 1-5+6 = 2$$

and
$$\lim_{x\to -1}(x+2) = -1+2 = 1$$

$$\frac{\lim_{x\to -1}(x^2+5x+6)}{\lim_{x\to -1}(x+2)} = \frac{2}{1} = 2 \qquad .....(i)$$

Also
$$\lim_{x\to -1}\frac{(x^2+5x+6)}{x+2} = \lim_{x\to -1}\frac{(x+3)(x+2)}{x+2} \quad \left[\begin{array}{l} \because x^2+5x+6 \\ = x^2+3x+2x+6 \\ = x(x+3)+2(x+3) \\ = (x+3)(x+2) \end{array}\right]$$

$$= \lim_{x\to -1}(x+3)$$

$$= -1+3 = 2 \qquad .....(ii)$$

$\therefore$ From (i) and (ii),

$$\lim_{x\to -1}\frac{x^2+5x+6}{x+2}=\frac{\lim\limits_{x\to -1}\left(x^2+5x+6\right)}{\lim\limits_{x\to -1}(x+2)}$$

We have seen above that there are many ways that two given functions may be combined to form a new function. The limit of the combined function as $x \to a$ can be calculated from the limits of the given functions. To sum up, we state below some basic results on limits, which can be used to find the limit of the functions combined with basic operations.

If $\quad \lim\limits_{x\to a} f(x)=\ell$ and $\quad \lim\limits_{x\to a} g(x)=m$, then

(i) $\quad \lim\limits_{x\to a} kf(x)=k\lim\limits_{x\to a} f(x)=k\ell \quad$ where k is a constant.

(ii) $\quad \lim\limits_{x\to a}\left[f(x)\pm g(x)\right]=\lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)=\ell\pm m$

(iii) $\quad \lim\limits_{x\to a}\left[f(x)\cdot g(x)\right]=\lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)=\ell\cdot m$

(iv) $\quad \lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\dfrac{\ell}{m}$, provided $\lim\limits_{x\to a} g(x)\neq 0$

The above results can be easily extended in case of more than two functions.

**Example 25.1** Find $\lim\limits_{x\to 1} f(x)$, where

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ 1, & x=1\end{cases}$$

**Solution :** $\quad f(x)=\dfrac{x^2-1}{x-1}=\dfrac{(x-1)(x+1)}{x-1}=(x+1) \qquad [\because x\neq 1]$

$\therefore \quad \lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}(x+1)=1+1=2$

**Note :** $\dfrac{x^2-1}{x-1}$ is not defined at x=1. The value of $\lim\limits_{x\to 1} f(x)$ is independent of the value of $f(x)$ at $x=1$.

**Example 25.2** Evaluate : $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$.

**Solution :** $\quad \lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)} = \lim_{x \to 2} \left(x^2+2x+4\right) \qquad [\because x \neq 2]$$

$$= 2^2 + 2 \times 2 + 4 = 12$$

**Example 25.3** Evaluate : $\lim\limits_{x \to 2} \dfrac{\sqrt{3-x}-1}{2-x}$.

**Solution :** Rationalizing the numerator, we have

$$\frac{\sqrt{3-x}-1}{2-x} = \frac{\sqrt{3-x}-1}{2-x} \times \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} = \frac{3-x-1}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$\therefore \qquad \lim_{x \to 2} \frac{\sqrt{3-x}-1}{2-x} = \lim_{x \to 2} \frac{2-x}{(2-x)\left(\sqrt{3-x}+1\right)}$$

$$= \lim_{x \to 2} \frac{1}{\left(\sqrt{3-x}+1\right)} = \frac{1}{\left(\sqrt{3-2}+1\right)} = \frac{1}{1+1} = \frac{1}{2}$$

**Example 25.4** Evaluate : $\lim\limits_{x \to 3} \dfrac{\sqrt{12-x}-x}{\sqrt{6+x}-3}$.

**Solution :** Rationalizing the numerator as well as the denominator, we get

$$\lim_{x \to 3} \frac{\sqrt{12-x}-x}{\sqrt{6+x}-3} = \lim_{x \to 3} \frac{\left(\sqrt{12-x}-x\right)\left(\sqrt{12-x}+x\right) \cdot \left(\sqrt{6+x}+3\right)}{\sqrt{6+x}-3\left(\sqrt{6+x}+3\right)\left(\sqrt{12-x}+x\right)}$$

$$= \lim_{x \to 3} \frac{\left(12-x-x^2\right)}{6+x-9} \cdot \lim_{x \to 3} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x}$$

$$= \lim_{x \to 3} \frac{-(x+4)(x-3)}{(x-3)} \cdot \lim_{x \to 3} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x} \qquad [\because x \neq 3]$$

$$= -(3+4) \cdot \frac{6}{6} = -7$$

**Note :** Whenever in a function, the limits of both numerator and denominator are zero, you should simplify it in such a manner that the denominator of the resulting function is not zero.

However, if the limit of the denominator is 0 and the limit of the numerator is non zero, then the limit of the function does not exist.

Let us consider the example given below :

**Example 25.5** Find $\lim_{x \to 0} \frac{1}{x}$, if it exists.

**Solution :** We choose values of x that approach 0 from both the sides and tabulate the correspondling values of $\frac{1}{x}$.

| x | −0.1 | −.01 | −.001 | −.0001 |
|---|---|---|---|---|
| $\frac{1}{x}$ | −10 | −100 | −1000 | −10000 |

| x | 0.1 | .01 | .001 | .0001 |
|---|---|---|---|---|
| $\frac{1}{x}$ | 10 | 100 | 1000 | 10000 |

We see that as $x \to 0$, the corresponding values of $\frac{1}{x}$ are not getting close to any number.

Hence, $\lim_{x \to 0} \frac{1}{x}$ does not exist. This is illustrated by the graph in Fig. 20.2

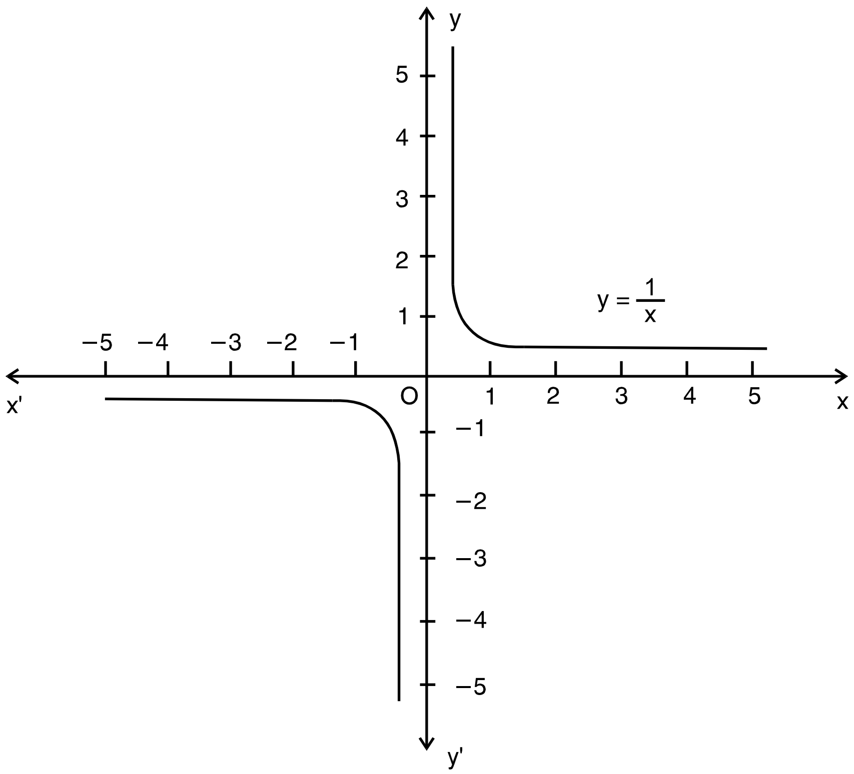



**Fig. 25.2**

**Example 25.6** Evaluate : $\lim_{x \to 0} (|x| + |-x|)$

**Solution :** Since |x| has different values for $x \geq 0$ and x<0, therefore we have to find out both left hand and right hand limits.

$$\lim_{x \to 0^-} (|x| + |-x|) = \lim_{h \to 0} (|0 - h| + |-(0 - h)|)$$

$$= \lim_{h \to 0} (|-h| + |-(-h)|)$$

$$= \lim_{h \to 0} h + h = \lim_{h \to 0} 2h = 0 \qquad \text{...(i)}$$

and
$$\lim_{x \to 0^+} (|x| + |-x|) = \lim_{h \to 0} (|0 + h| + |-(0 + h)|)$$

$$= \lim_{x \to 0} h + h = \lim_{h \to 0} 2h = 0 \qquad \text{...(ii)}$$

From (i) and (ii),

$$\lim_{x \to 0^-} (|x| + |-x|) = \lim_{h \to 0^+} [|x| + |-x|]$$

Thus ,
$$\lim_{h \to 0} [|x| + |-x|] = 0$$

**Note :** We should remember that left hand and right hand limits are specially used when (a) the functions under consideration involve modulus function, and (b) function is defined by more than one rule.

**Example 25.7** Find the vlaue of 'a' so that

$\lim_{x \to 1} f(x)$ exist, where $f(x) = \begin{cases} 3x + 5\ , x \leq 1 \\ 2x + a, x > 1 \end{cases}$

**Solution :**
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3x + 5) \qquad [\because f(x) = 3x + 5 \text{ for } x \leq 1]$$

$$= \lim_{h \to 0} [3(1 - h) + 5]$$

$$= 3 + 5 = 8 \qquad \text{.....(i)}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2x + a) \qquad [\because f(x) = 2x + a \text{ for } x > 1]$$

$$= \lim_{h \to 0} (2(1 + h) + a)$$

$$= 2 + a \qquad \text{.....(ii)}$$

We are given that $\lim_{x \to 1} f(x)$ will exists provided

$$\Rightarrow \qquad \lim_{x \to 1^-} = \lim_{x \to 1^+} f(x)$$

$\therefore$ From (i) and (ii),

2 + a = 8

$\therefore$ or, a = 6

**Example 25.8** If a function f (x) is defined as

$$f(x)=\begin{cases} x & , \quad 0 \le x < \frac{1}{2} \\ 0 & , \quad x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 \end{cases}$$

Examine the existence of $\lim_{x\to\frac{1}{2}} f(x)$.

**Solution :** Here
$$f(x)=\begin{cases} x & , \quad 0 \le x < \frac{1}{2} \qquad .....(i) \\ 0 & , \quad x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 \qquad .....(ii) \end{cases}$$

$$\lim_{x\to\left(\frac{1}{2}\right)^-} f(x) = \lim_{h\to 0} f\left(\frac{1}{2}-h\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{2}-h\right) \qquad \left[\because \frac{1}{2}-h<\frac{1}{2} \text{ and from (i)}, f\left(\frac{1}{2}-h\right)=\frac{1}{2}-h\right]$$

$$=\frac{1}{2}-0=\frac{1}{2} \qquad .....(iii)$$

$$\lim_{x\to\left(\frac{1}{2}\right)^+} f(x) = \lim_{h\to 0} f\left(\frac{1}{2}+h\right)$$

$$= \lim_{h\to 0}\left[\left(\frac{1}{2}+h\right)-1\right] \left[\because \frac{1}{2}+h>\frac{1}{2} \text{ and from (ii)}, f\left(\frac{1}{2}+h\right)=\left(\frac{1}{2}+h\right)-1\right]$$

$$=\frac{1}{2}+-1$$

$$=-\frac{1}{2} \qquad .....(iv)$$

From (iii) and (iv), left hand limit $\neq$ right hand limit

$\therefore \qquad \lim_{x\to\frac{1}{2}} f(x)$ does not exist.

## CHECK YOUR PROGRESS 25.1

1. Evaluate each of the following limits :

(a) $\lim\limits_{x\to 2}[2(x+3)+7]$ (b) $\lim\limits_{x\to 0}\left(x^2+3x+7\right)$ (c) $\lim\limits_{x\to 1}\left[(x+3)^2-16\right]$

(d) $\lim\limits_{x\to -1}\left[(x+1)^2+2\right]$ (e) $\lim\limits_{x\to 0}\left[(2x+1)^3-5\right]$ (f) $\lim\limits_{x\to 1}(3x+1)(x+1)$

2. Find the limits of each of the following functions :

(a) $\lim\limits_{x\to 5}\dfrac{x-5}{x+2}$ (b) $\lim\limits_{x\to 1}\dfrac{x+2}{x+1}$ (c) $\lim\limits_{x\to -1}\dfrac{3x+5}{x-10}$

(d) $\lim\limits_{x\to 0}\dfrac{px+q}{ax+b}$ (e) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}$ (f) $\lim\limits_{x\to -5}\dfrac{x^2-25}{x+5}$

(g) $\lim\limits_{x\to 2}\dfrac{x^2-x-2}{x^2-3x+2}$ (h) $\lim\limits_{x\to \frac{1}{3}}\dfrac{9x^2-1}{3x-1}$

3. Evaluate each of the following limits:

(a) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$ (b) $\lim\limits_{x\to 0}\dfrac{x^3+7x}{x^2+2x}$ (c) $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1}$

(d) $\lim\limits_{x\to 1}\left[\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right]$

4. Evaluate each of the following limits :

(a) $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-\sqrt{4-x}}{x}$ (b) $\lim\limits_{x\to 0}\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ (c) $\lim\limits_{x\to 3}\dfrac{\sqrt{3+x}-\sqrt{6}}{x-3}$

(d) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+x}-1}$ (e) $\lim\limits_{x\to 2}\dfrac{\sqrt{3x-2}-x}{2-\sqrt{6-x}}$

5. (a) Find $\lim\limits_{x\to 0}\dfrac{2}{x}$, if it exists. (b) Find $\lim\limits_{x\to 2}\dfrac{1}{x-2}$, if it exists.

6. Find the values of the limits given below :

(a) $\lim\limits_{x\to 0}\dfrac{x}{5-|x|}$ (b) $\lim\limits_{x\to 2}\dfrac{1}{|x+2|}$ (c) $\lim\limits_{x\to 2}\dfrac{1}{|x-2|}$

(d) Show that $\lim\limits_{x\to 5}\dfrac{|x-5|}{x-5}$ does not exist.

7. (a) Find the left hand and right hand limits of the function

$$f(x)=\begin{cases}-2x+3, & x\le 1\\ 3x-5, & x>1\end{cases} \text{ as } x\to 1$$

(b) If $f(x)=\begin{cases}x^2, & x\le 1\\ 1, & x>1\end{cases}$, find $\lim\limits_{x\to 1}f(x)$

(c) Find $\lim\limits_{x \to 4} f(x)$ if it exists, given that $f(x) = \begin{cases} 4x+3, & x < 4 \\ 3x+7, & x \ge 4 \end{cases}$

8. Find the value of 'a' such that $\lim\limits_{x \to 2} f(x)$ exists, where $f(x) = \begin{cases} ax+5, & x < 2 \\ x-1, & x \ge 2 \end{cases}$

9. Let $f(x) = \begin{cases} x, & x < 1 \\ 1, & x = 1 \\ x^2, & x > 1 \end{cases}$

Establish the existence of $\lim\limits_{x \to 1} f(x)$.

10. Find $\lim\limits_{x \to 2} f(x)$ if it exists, where

$$f(x) = \begin{cases} x-1, & x < 2 \\ 1, & x = 2 \\ x+1, & x > 2 \end{cases}$$

## 25.5 FINDING LIMITS OF SOME OF THE IMPORTANT FUNCTIONS

**(i) Prove that** $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$ **where n is a positive integer.**

**Proof :** $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = \lim\limits_{h \to 0} \dfrac{(a+h)^n - a^n}{a + h - a}$

$$= \lim_{h \to 0} \frac{\left(a^n + n\,a^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \ldots.. + h^n\right) - a^n}{h}$$

$$= \lim_{h \to 0} \frac{h\left(n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \ldots.. + h^{n-1}\right)}{h}$$

$$= \lim_{h \to 0} \left[n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + \ldots.. + h^{n-1}\right]$$

$$= n\,a^{n-1} + 0 + 0 + \ldots.. + 0$$

$$= n\,a^{n-1}$$

$$\therefore \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$$

**Note :** However, the result is true for all n

**(ii) Prove that (a)** $\lim\limits_{x \to 0} \sin x = 0$ **and** **(b)** $\lim\limits_{x \to 0} \cos x = 1$

**Proof :** Consider a unit circle with centre B, in which $\angle C$ is a right angle and $\angle ABC = x$ radians.

Now sin x = A C and cos x = BC

As x decreases, A goes on coming nearer and nearer to C.

i.e., when $x \to 0, A \to C$

or when $x \to 0, AC \to 0$

and $BC \to AB$, i.e., $BC \to 1$

$\therefore$ When $x \to 0$ $\sin x \to 0$ and $\cos x \to 1$

Thus we have

$$\lim_{x \to 0} \sin x = 0 \quad \text{and} \quad \lim_{x \to 0} \cos x = 1$$

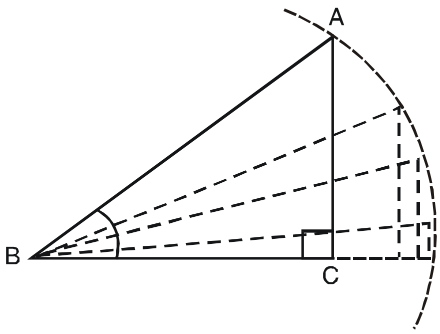


Fig. 25.3

**(iii) Prove that** $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

**Proof :** Draw a circle of radius 1 unit and with centre at the origin *O*. Let B (1,0) be a point on the circle. Let A be any other point on the circle. Draw $AC \perp OX$.

Let $\angle AOX = x$ radians, where $0 < x < \dfrac{\pi}{2}$

Draw a tangent to the circle at B meeting OA produced at D. Then $BD \perp OX$.

Area of $\Delta AOC <$ area of sector $OBA <$ area of $\Delta OBD$.

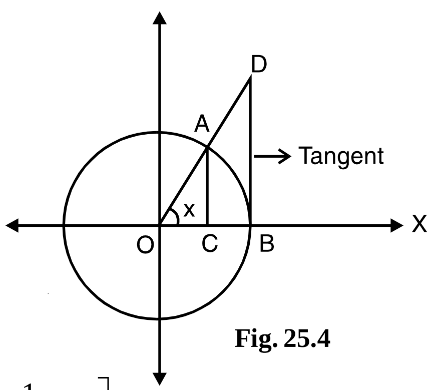


Fig. 25.4

$$\text{or } \frac{1}{2} OC \times AC < \frac{1}{2} x (1)^2 < \frac{1}{2} OB \times BD$$

$$\left[ \because \text{area of triangle} = \frac{1}{2} \text{base} \times \text{height and area of sector} = \frac{1}{2} \theta\, r^2 \right]$$

$$\therefore \quad \frac{1}{2} \cos x \sin x < \frac{1}{2} x < \frac{1}{2} \cdot 1 \cdot \tan x$$

$$\left[ \because \cos x = \frac{OC}{OA}, \sin x = \frac{AC}{OA} \text{ and } \tan x = \frac{BD}{OB}, OA = 1 = OB \right]$$

i.e., $\cos x < \dfrac{x}{\sin x} < \dfrac{\tan x}{\sin x}$ [Dividing throughout by $\dfrac{1}{2}\sin x$]

or $$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or $$\frac{1}{\cos x} > \frac{\sin x}{x} < \cos x$$

i.e., $$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$$

Taking limit as $x \to 0$, we get

$$\lim_{x\to 0} \cos x < \lim_{x\to 0} \frac{\sin x}{x} < \lim_{x\to 0} \frac{1}{\cos x}$$

or $$1 < \lim_{x\to 0} \frac{\sin x}{x} < 1 \qquad \left[\because \lim_{x\to 0} \cos x = 1 \text{ and } \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1\right]$$

**Thus,** $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

**Note :** In the above results, it should be kept in mind that the angle x must be expressed in radians.

**(iv) Prove that $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$**

**Proof :** By Binomial theorem, when $|x| < 1$, we get

$$(1+x)^{\frac{1}{x}} = \left[1 + \frac{1}{x}\cdot x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2 + + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3 + \dots\dots\dots\infty\right]$$

$$= \left[1 + 1 + \frac{(1-x)}{2!} + \frac{(1-x)(1-2x)}{3!} + \dots\dots\dots\infty\right]$$

$$\therefore \lim_{x\to 0} (1+x)^{\frac{1}{x}} = \lim_{x\to 0}\left[1 + 1 + \frac{1-x}{2!} + \frac{(1-x)(1-2x)}{3!} + \dots\dots\dots\infty\right]$$

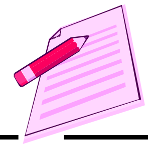
$$= \left[1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots\dots\dots\infty\right]$$

$$= e \quad \text{(By definition)}$$

**Thus** $$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

**(v) Prove that**

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{1}{x}\log(1+x) = \lim_{x\to 0} \log(1+x)^{1/x}$$





$$= \log e \left( \text{Using } \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \right)$$

$$= 1$$

**(vi) Prove that** $\displaystyle \lim_{x \to 0} \frac{e^x - 1}{x} = 1$

**Proof :** We know that $e^x = \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + .......... \right)$

$$\therefore \quad e^x - 1 = \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + .......... - 1 \right) = \left( x + \frac{x^2}{2!} + \frac{x^3}{3!} + .......... \right)$$

$$\therefore \quad \frac{e^x - 1}{x} = \frac{\left( x + \frac{x^2}{2!} + \frac{x^3}{3!} + .......... \right)}{x} \qquad \text{[Dividing throughout by x]}$$

$$= \frac{x\left( 1 + \frac{x}{2!} + \frac{x^2}{3!} + .......... \right)}{x} = \left( 1 + \frac{x}{2!} + \frac{x^2}{3!} + .......... \right)$$

$$\therefore \quad \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \left( 1 + \frac{x}{2!} + \frac{x^2}{3!} + .......... \right)$$

$$= 1 + 0 + 0 + ........ = 1$$

**Thus,** $\displaystyle \lim_{x \to 0} \frac{e^x - 1}{x} = 1$

**Example 25.9** Find the value of $\displaystyle \lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

**Solution :** We know that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \qquad .....(i)$$

$\therefore$ Putting $x = -x$ in (i), we get

$$\lim_{x \to 0} \frac{e^{-x} - 1}{-x} = 1 \qquad .....(ii)$$

Given limit can be written as

$$\lim_{x \to 0} \frac{e^x - 1 + 1 - e^{-x}}{x} \qquad \text{[Adding (i) and (ii)]}$$

$$= \lim_{x\to 0}\left[\frac{e^x-1}{x}+\frac{1-e^{-x}}{x}\right] = \lim_{x\to 0}\left[\frac{e^x-1}{x}+\frac{e^{-x}-1}{-x}\right]$$

$$= \lim_{x\to 0}\frac{e^x-1}{x}+\lim_{x\to 0}\frac{e^{-x}-1}{-x} = 1+1 = 2 \qquad \text{[ Using (i) and (ii)]}$$

Thus $\displaystyle \lim_{x\to 0}\frac{e^x-e^{-x}}{x} = 2$

**Example 25.10** Evaluate : $\displaystyle \lim_{x\to 1}\frac{e^x-e}{x-1}$.

**Solution :** Put $x = 1 + h$, where $h \to 0$

$$\lim_{x\to 1}\frac{e^x-e}{x-1} = \lim_{h\to 0}\frac{e^{1+h}-e}{h} = \lim_{h\to 0}\frac{e^1\cdot e^h-e}{h} = \lim_{h\to 0}\frac{e(e^h-1)}{h}$$

$$= e\lim_{h\to 0}\frac{e^h-1}{h} = e\times 1 = e.$$

Thus, $\displaystyle \lim_{x\to 1}\frac{e^x-e}{x-1} = e$

**Example 25.11** Evaluate : $\displaystyle \lim_{x\to 0}\frac{\sin 3x}{x}$.

**Solution :** $\displaystyle \lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot 3$ [Multiplying and dividing by 3]

$$= 3\lim_{3x\to 0}\frac{\sin 3x}{3x} \qquad [\because \text{ when } x\to 0, 3x\to 0]$$

$$= 3.1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= 3$$

Thus, $\displaystyle \lim_{x\to 0}\frac{\sin 3x}{x} = 3$

**Example 25.12** Evaluate $\displaystyle \lim_{x\to 0}\frac{1-\cos x}{2x^2}$.

**Solution :** $\displaystyle \lim_{x\to 0}\frac{1-\cos x}{2x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2x^2}$ $\left[\begin{array}{l}\because \cos 2x = 1-2\sin^2 x,\\ \therefore 1-\cos 2x = 2\sin^2 x\\ \text{or } 1-\cos x = 2\sin^2\frac{x}{2}\end{array}\right]$

$$=\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{2\times\frac{x}{2}}\right)^2 \quad \text{[Multiplying and dividing the denominator by 2]}$$

$$=\frac{1}{4}\lim_{\frac{x}{2}\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{4}\times 1=\frac{1}{4}$$

$$\therefore \quad \lim_{x\to 0}\frac{1-\cos x}{2x^2}=\frac{1}{4}$$

**Example 25.13** Find the value of $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1+\cos 2x}{(\pi-2x)^2}$.

**Solution :** Put $x=\frac{\pi}{2}+h$ $\quad \because$ when $x\to\frac{\pi}{2}, h\to 0$

$$\therefore \quad 2x=\pi+2h$$

$$\therefore \quad \lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}=\lim_{h\to 0}\frac{1+\cos 2\left(\frac{\pi}{2}+h\right)}{[\pi-(\pi+2h)]^2}$$

$$=\lim_{h\to 0}\frac{1+\cos(\pi+2h)}{4h^2}=\lim_{h\to 0}\frac{1-\cos 2h}{4h^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 h}{4h^2}=\frac{1}{2}\lim_{h\to 0}\left(\frac{\sin h}{h}\right)^2 = \frac{1}{2}\times 1=\frac{1}{2}$$

$$\therefore \quad \lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}=\frac{1}{2}$$

$$=\frac{a}{b}$$

$$\therefore \quad \lim_{x\to 0}\frac{\sin ax}{\tan bx}=\frac{a}{b}$$

## CHECK YOUR PROGRESS 25.2

1. Evaluate each of the following :

(a) $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$  (b) $\lim_{x \to 0} \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

2. Find the value of each of the following :

(a) $\lim_{x \to 1} \frac{e^{-x} - e^{-1}}{x - 1}$  (b) $\lim_{x \to 1} \frac{e - e^{x}}{x - 1}$

3. Evaluate the following :

(a) $\lim_{x \to 0} \frac{\sin 4x}{2x}$  (b) $\lim_{x \to 0} \frac{\sin x^2}{5x^2}$  (c) $\lim_{x \to 0} \frac{\sin x^2}{x}$

(d) $\lim_{x \to 0} \frac{\sin a x}{\sin b x}$

4. Evaluate each of the following :

(a) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$  (b) $\lim_{x \to 0} \frac{1 - \cos 8x}{x}$  (c) $\lim_{x \to 0} \frac{\sin 2x(1 - \cos 2x)}{x^3}$

(d) $\lim_{x \to 0} \frac{1 - \cos 2x}{3 \tan^2 x}$

5. Find the values of the following :

(a) $\lim_{x \to 0} \frac{1 - \cos a x}{1 - \cos b x}$  (b) $\lim_{x \to 0} \frac{x^3 \cot x}{1 - \cos x}$  (c) $\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

6. Evaluate each of the following :

(a) $\lim_{x \to \pi} \frac{\sin x}{\pi - x}$  (b) $\lim_{x \to 1} \frac{\cos \frac{\pi}{2} x}{1 - x}$  (c) $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

7. Evaluate the following :

(a) $\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$  (b) $\lim_{\theta \to 0} \frac{\tan 7\theta}{\sin 4\theta}$  (c) $\lim_{x \to 0} \frac{\sin 2x + \tan 3x}{4x - \tan 5x}$

## 25.6 CONTINUITY OF A FUNCTION AT A POINT

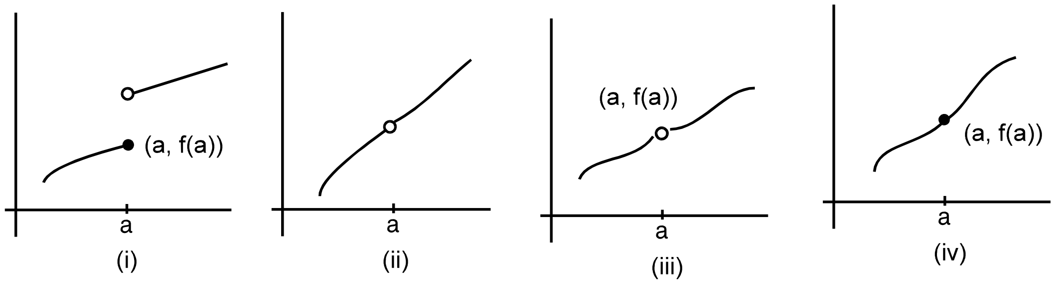



**Fig. 25.5**

Let us observe the above graphs of a function.

We can draw the graph (iv) without lifting the pencil but in case of graphs (i), (ii) and (iii), the pencil has to be lifted to draw the whole graph.

In case of (iv), we say that the function is continuous at x = a. In other three cases, the function is not continuous at x = a. i.e., they are discontinuous at x = a.

In case (i), the limit of the function does not exist at x = a.

In case (ii), the limit exists but the function is not defined at x = a.

In case (iii), the limit exists, but is not equal to value of the function at x = a.

In case (iv), the limit exists and is equal to value of the function at x = a.

**Example 25.14** Examine the continuity of the function $f(x) = x - a$ at x = a.

**Solution :** $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} [(a+h) - a]$$

$$= 0 \qquad \text{.....(i)}$$

Also $\qquad f(a) = a - a = 0 \qquad$ .....(ii)

From (i) and (ii),

$$\lim_{x \to a} f(x) = f(a)$$

Thus $f(x)$ is continuous at x = a.

**Example 25. 15** Show that $f(x) = c$ is continuous.

**Solution :** The domain of constant function c is R.Let 'a' be any arbitrary real number.

$$\therefore \qquad \lim_{x \to a} f(x) = c \text{ and } f(a) = c$$

$$\therefore \qquad \lim_{x \to a} f(x) = f(a)$$

$\therefore$ $f(x)$ is continuous at x = a. But 'a' is arbitrary. Hence f(x) = c is a constant function.

**Example 25.16** Show that $f(x) = cx + d$ is a continuous function.

**Solution :** The domain of linear function $f(x) = cx + d$ is R; and let 'a' be any arbitrary real number.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to a} [c\,(a+h)+d]$$

$$= ca + d \qquad .....(i)$$

Also $\qquad f(a) = ca + d \qquad .....(ii)$

From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ $\quad f(x)$ is continuous at $x = a$

and $\quad$ since a is any arbitrary , $\quad f(x)$ is a continuous function.

**Example 25.17** Prove that $f(x) = \sin x$ is a continuous function.

**Solution :** Let $f(x) = \sin x$

The domain of sin x is R. let 'a' be any arbitrary real number.

$$\therefore \quad \lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to 0} \sin(a+h)$$

$$= \lim_{h \to 0} [\sin a . \cos h + \cos a . \sin h]$$

$$= \sin a \lim_{h \to 0} \cos h + \cos a \lim_{h \to 0} \sin h \qquad \left[\because \lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) \text{ where k is a constant}\right]$$

$$= \sin a \times 1 + \cos a \times 0 \qquad \left[\because \lim_{x \to 0} \sin x = 0 \text{ and } \lim_{x \to 0} \cos x = 1\right]$$

$$= \sin a \qquad .....(i)$$

Also $f(a) = \sin a \qquad .....(ii)$

From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ sin x is continuous at x = a

$\because$ sin x is continuous at x = a and 'a' is an aribitary point.

Therefore, $f(x) = \sin x$ is continuous.

**Definition :**

1. A function $f(x)$ is said to be continuous in an open inteval ]a,b[ if it is continuous at every point of ]a,b[*.

2. A function $f(x)$ is said to be continuous in the closed interval [a,b] if it is continuous at every point of the open interval ]a,b[ and is continuous at the point a from the right and continuous at b from the left.

i.e. $\lim_{x \to a^+} f(x) = f(a)$

and $\lim_{x \to b^-} f(x) = f(b)$

* *In the open interval ]a,b[ we do not consider the end points a and b.*

## CHECK YOUR PROGRESS 25.3

1. Examine the continuity of the functions given below :

   (a) $f(x) = x - 5$ at $x = 2$ (b) $f(x) = 2x + 7$ at $x = 0$

   (c) $f(x) = \frac{5}{3}x + 7$ at $x = 3$ (d) $f(x) = px + q$ at $x = -q$

2. Show that f (x)=2a+3b is continuous, where a and b are constants.
3. Show that $5x + 7$ is a continuous function
4. (a) Show that cos x is a continuous function.

   (b) Show that cot x is continuous at all points of its domain.

5. Find the value of the constants in the functions given below :

   (a) $f(x) = px - 5$ and $f(2) = 1$ such that $f(x)$ is continuous at $x = 2$.

   (b) $f(x) = a + 5x$ and $f(0) = 4$ such that $f(x)$ is continuous at $x = 0$.

   (c) $f(x) = 2x + 3b$ and $f(-2) = \frac{2}{3}$ such that f(x) is continuous at $x = -2$.

## 25.7 DISCONTINUITY OF A FUNCTION AT A POINT

So far, we have considered only those functions which are continuous. Now we shall discuss some examples of functions which may or may not be continuous.

**Example 25.18** Show that the function $f(x) = e^x$ is a continuous function.

**Solution :** Domain of $e^x$ is R. Let $a \in R$ . where 'a ' is arbitrary.

$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$ , where h is a very small number.

$$= \lim_{h \to 0} e^{a+h} = \lim_{h \to 0} e^a \cdot e^h = e^a \lim_{h \to 0} e^h = e^a \times 1 \qquad .....(i)$$

$$= e^a \qquad .....(ii)$$

Also $f(a) = e^a$

$\therefore$ From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a

Since a is arbitary, $e^x$ is a continuous function.

**Example 25.19** By means of graph discuss the continuity of the function $f(x) = \frac{x^2 - 1}{x - 1}$.

**Solution :** The grah of the function is shown in the adjoining figure. The function is discontinuous as there is a gap in the graph at x = 1.

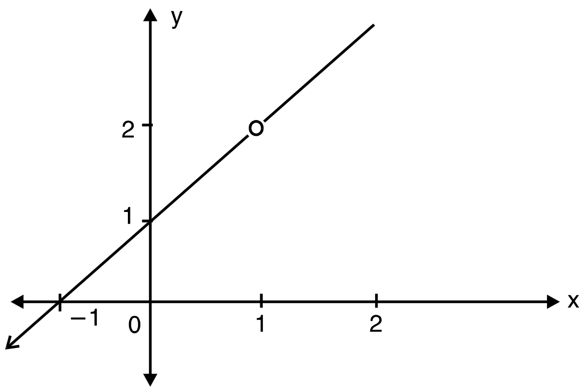



**Fig. 25.6**

## CHECK YOUR PROGRESS 25.4

1. (a) Show that $f(x) = e^{5x}$ is a continuous function.

   (b) Show that $f(x) = e^{\frac{-2}{3}x}$ is a continuous function.

   (c) Show that $f(x) = e^{3x+2}$ is a continuous function.

   (d) Show that $f(x) = e^{-2x+5}$ is a continuous function.

2. By means of graph, examine the continuity of each of the following functions :

   (a) f (x) = x+1.  (b) $f(x) = \frac{x+2}{x-2}$

   (c) $f(x) = \frac{x^2 - 9}{x + 3}$  (d) $f(x) = \frac{x^2 - 16}{x - 4}$

## 25.8 PROPERTIES OF CONTINUOUS FUNCTIONS

(i) Consider the function f (x) = 4. Graph of the function f (x) = 4 is shown in the Fig. 20.7. From the graph, we see that the function is continuous. In general, all constant functions are continuous.

(ii) If a function is continuous then the constant multiple of that function is also continuous.

Consider the function $f(x)=\frac{7}{2}x$. We know that x is a constant function. Let 'a' be an arbitrary real number.

$$\lim_{x\to a} f(x)=\lim_{h\to 0} f(a+h)$$

$$=\lim_{h\to 0}\frac{7}{2}(a+h)$$

$$=\frac{7}{2}a \qquad .....(i)$$

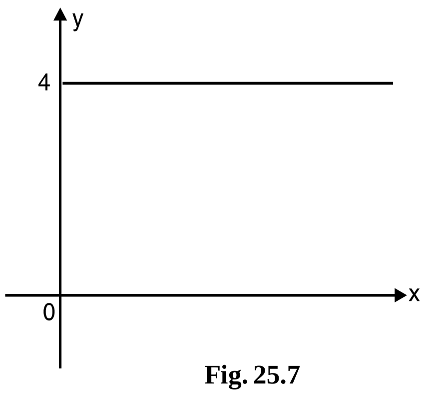



**Fig. 25.7**

Also $\qquad f(a)=\frac{7}{2}a \qquad .....(ii)$

$\therefore$ From (i) and (ii),

$$\lim_{x\to a} f(x)=f(a)$$

$\therefore$ $f(x)=\frac{7}{2}x$ is continuous at $x = a$.

As $\frac{7}{2}$ is constant, and x is continuous function at $x = a$, $\frac{7}{2}x$ is also a continuous function at $x = a$.

(iii) Consider the function $f(x)=x^2+2x$. We know that the function $x^2$ and 2x are continuous.

Now
$$\lim_{x\to a} f(x)=\lim_{h\to 0} f(a+h)$$

$$=\lim_{h\to 0}\left[(a+h)^2+2(a+h)\right]$$

$$=\lim_{h\to 0}\left[a^2+2ah+h^2+2a+2ah\right]$$

$$=a^2+2a \qquad .....(i)$$

Also $\qquad f(a)=a^2+2a \qquad .....(ii)$

$\therefore$ From (i) and (ii), $\lim_{x\to a} f(x)=f(a)$

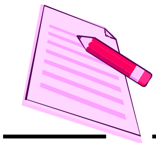
$\therefore$ f(x) is continuous at $x = a$.

Thus we can say that if $x^2$ and 2x are two continuous functions at $x = a$ then $(x^2+2x)$ is also continuous at $x = a$.

(iv) Consider the function $f(x)=(x^2+1)(x+2)$. We know that $(x^2+1)$ and $(x+2)$ are two continuous functions.

Also $\qquad f(x)=(x^2+1)(x+2)$

$$=x^3+2x^2+x+2$$





As $x^3, 2x^2, x$ and 2 are continuous functions, therefore.

$x^3 + 2x^2 + x + 2$ is also a continuous function.

$\therefore$ We can say that if $\left(x^2 + 1\right)$ and (x+2) are two continuous functions then $\left(x^2 + 1\right)(x+2)$ is also a continuous function.

(v) Consider the function $f(x) = \dfrac{x^2 - 4}{x + 2}$ at $x = 2$. We know that $\left(x^2 - 4\right)$ is continuous at $x = 2$. Also $(x + 2)$ is continuous at $x = 2$.

Again
$$\lim_{x \to 2} \frac{x^2 - 4}{x + 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x+2}$$
$$= \lim_{x \to 2} (x - 2)$$
$$= 2 - 2 = 0$$

Also
$$f(2) = \frac{(2)^2 - 4}{2 + 2}$$
$$= \frac{0}{4} = 0$$

$\therefore$ $\lim\limits_{x \to 2} f(x) = f(2)$. Thus $f(x)$ is continuous at $x = 2$.

$\therefore$ If $x^2 - 4$ and $x + 2$ are two continuous functions at $x = 2$, then $\dfrac{x^2 - 4}{x + 2}$ is also continuous.

(vi) Consider the function $f(x) = |x - 2|$. The function can be written as

$$f(x) = \begin{cases} -(x-2), x < 2 \\ (x-2), x \geq 2 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h), \ h > 0$$
$$= \lim_{h \to 0} [(2 - h) - 2]$$
$$= 2 - 2 = 0$$

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2 + h), \ h > 0 \qquad \text{......(i)}$$
$$= \lim_{x \to 2} [(2 + h) - 2]$$
$$= 2 - 2 = 0 \qquad \text{.....(ii)}$$

Also
$$f(2) = (2 - 2) = 0 \qquad \text{.....(iii)}$$

$\therefore$ From (i), (ii) and (iii), $\lim\limits_{x \to 2} f(x) = f(2)$

Thus, $|x - 2|$ is continuous at $x = 2$.

After considering the above results, we state below some properties of continuous functions.

If f (x) and g (x) are two functions which are continuous at a point x = a, then

(i) C f (x) is continuous at x = a, where C is a constant.

(ii) $f(x) \pm g(x)$ is continuous at x = a.

(iii) $f(x) \cdot g(x)$ is continuous at x = a.

(iv) f (x)/g (x) is continuous at x = a, provided g (a) $\neq$ 0.

(v) |f(x)| is continuous at x = a.

**Note :** Every constant function is continuous.

## 25.9 IMPORTANT RESULTS ON CONTINUITY

By using the properties mentioned above, we shall now discuss some important results on continuity.

(i) Consider the function $f(x) = px + q, x \in R$ (i)

The domain of this functions is the set of real numbers. Let a be any arbitary real number. Taking limit of both sides of (i), we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} (px + q) = pa + q \quad [= \text{value of } p\ x + q \text{ at } x = a.]$$

$\therefore$ px +q is continuous at x = a.

Similarly, if we consider $f(x) = 5x^2 + 2x + 3$, we can show that it is a continuous function.

In general $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} + a_n x^n$

where $a_0, a_1, a_2 .....a_n$ are constants and n is a non-negative integer,

we can show that $a_0, a_1 x, a_2 x^2, .....a_n x^n$ are all continuos at a point x = c (where c is any real number) and by property (ii), their sum is also continuous at x = c.

$\therefore$ f (x) is continuous at any point c.

Hence every polynomial function is continuous at every point.

(ii) Consider a function $f(x) = \dfrac{(x+1)(x+3)}{(x-5)}$, $f(x)$ is not defined when $x - 5 = 0$ i.e, at x = 5.

Since (x + 1) and (x + 3) are both continuous, we can say that (x + 1) (x + 3) is also continuous. [Using property iii]

$\therefore$ Denominator of the function f (x), i.e., (x – 5) is also continuous.

$\therefore$ Using the property (iv), we can say that the function $\dfrac{(x+1)(x+3)}{(x-5)}$ is continuous at all points except at $x = 5$.

In general if $f(x) = \dfrac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial functions and $q(x) \neq 0$, then f (x) is continuous if p (x) and q (x) both are continuous.

**Example 25.20** Examine the continuity of the following function at $x = 2$.

$$f(x) = \begin{cases} 3x-2 & \text{for } x < 2 \\ x+2 & \text{for } x \geq 2 \end{cases}$$

**Solution :** Since f (x) is defined as the polynomial function $3x-2$ on the left hand side of the point $x = 2$ and by another polynomial function $x + 2$ on the right hand side of $x = 2$, we shall find the left hand limit and right hand limit of the function at $x = 2$ separately.

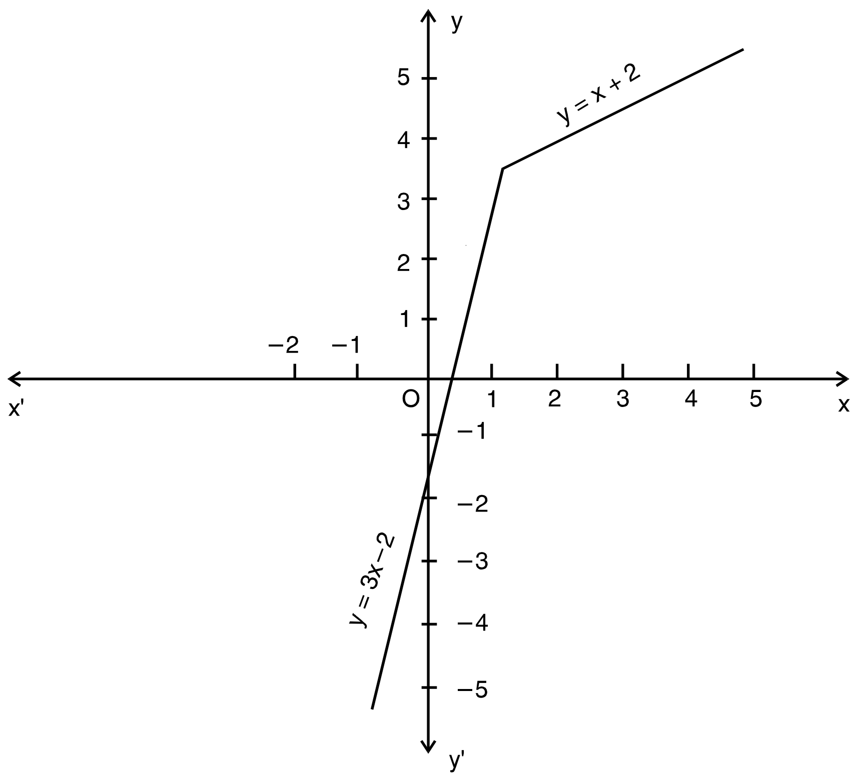



**Fig. 25.8**

Left hand limit $= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2} (3x-2) = 3 \times 2 - 2 = 4$

Right hand limit at $x = 2$;

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (x+2) = 4$$

Since the left hand limit and the right hand limit at x = 2 are equal, the limit of the function f (x) exists at x =2 and is equal to 4 i.e., $\lim_{x \to 2} f(x) = 4$.

Also f (x) is defined by (x +2) at x = 2

$\therefore$ $$f(2) = 2 + 2 = 4.$$

Thus, $$\lim_{x \to 2} f(x) = f(2)$$

Hence f (x) is continuous at x = 2.

**Example 25.21**

(i) Draw the graph of f (x) = |x|.

(ii) Discusss the continuity of f (x) at x = 0.

**Solution :** We know that for $x \geq 0, |x| = x$ and for $x < 0, |x| = -x$. Hence f (x) can be written as.

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

(i) The graph of the function is given in Fig 20.9

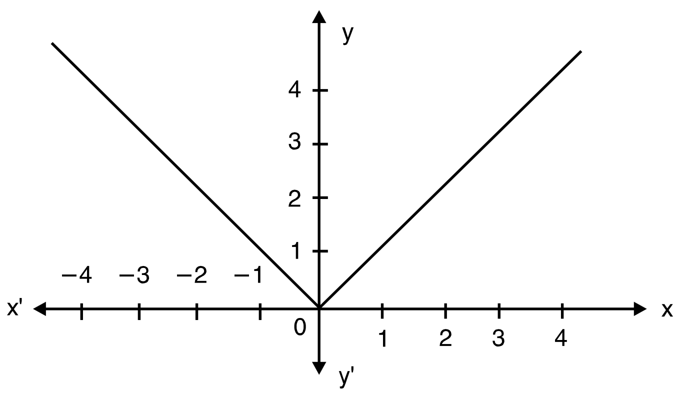



**Fig. 25.9**

(ii) Left hand limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$

Right hand limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0} x = 0$

Thus, $$\lim_{x \to 0} f(x) = 0$$

Also, $$f(0) = 0$$

$\therefore$ $$\lim_{x \to 0} f(x) = f(0)$$

Hence the function $f(x)$ is continuous at $x=0$.

**Example 25.22** Examine the continuity of $f(x) = |x-b|$ at $x=b$.

**Solution :** We have $f(x) = |x-b|$. This function can be written as

$$f(x) = \begin{cases} -(x-b), x < b \\ (x-b), x \geq b \end{cases}$$

Left hand limit $= \lim\limits_{x \to b^-} f(x) = \lim\limits_{h \to 0} f(b-h)$

$$= \lim_{h \to 0}[-(b-h-b)]$$

$$= \lim_{h \to 0} h = 0 \qquad .....(i)$$

Right hand limit $= \lim\limits_{x \to b^+} f(x) = \lim\limits_{h \to 0} f(b+h)$

$$= \lim_{h \to 0}[(b+h)-b]$$

$$= \lim_{h \to 0} h = 0 \qquad .....(ii)$$

Also, $f(b) = b - b = 0$ .....(iii)

From (i), (ii) and (iii), $\lim\limits_{x \to b} f(x) = f(b)$

Thus, f(x) is continuous at $x = b$.

**Example 25.23** If $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

find whether f(x) is continuous at $x = 0$ or not.

**Solution :** Here $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

Left hand limit $= \lim\limits_{x \to 0^-} \dfrac{\sin 2x}{x} = \lim\limits_{h \to 0} \dfrac{\sin 2(0-h)}{0-h} = \lim\limits_{h \to 0} \dfrac{-\sin 2h}{-h}$

$$= \lim_{h \to 0}\left(\frac{\sin 2h}{2h} \times \frac{2}{1}\right) = 1 \times 2 = 2 \qquad .....(i)$$

Right hand limit $= \lim\limits_{x \to 0^+} \dfrac{\sin 2x}{x} = \lim\limits_{h \to 0} \dfrac{\sin 2(0+h)}{0+h} = \lim\limits_{h \to 0} \dfrac{\sin 2h}{2h} \times \dfrac{2}{1}$

$= 1 \times 2 = 2$ (ii)

Also $f(0) = 2$ (Given) (iii)

From (i) to (iii),

$$\lim_{x\to 0} f(x) = 2 = f(0)$$

Hence f(x) is continuous at x = 0.

**Signum Function :** The function f (x)=sgn(x) (read as signum x) is defined as

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Find the left hand limit and right hand limit of the function from its graph given below:

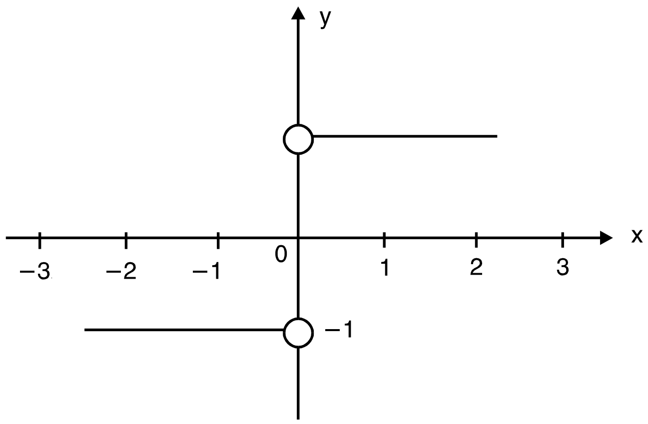



**Fig. 25.11**

From the graph, we see that as $x \to 0^+$, $f(x) \to 1$ and as $(x) \to 0^-$, $f(x) \to -1$

Hence, $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = -1$

As these limits are not equal, $\lim_{x\to 0} f(x)$ does not exist. Hence f (x) is discontinuous at x =0.

**Greatest Integer Function :** Let us consider the function f (x)=[x] where [x] denotes the greatest integer less than or equal to x. Find whether f (x) is continuous at

(i) $x = \frac{1}{2}$ (ii) x =1

To solve this, let us take some arbitrary values of x say 1.3, 0.2,–0.2..... By the definition of greatest integer function,

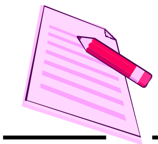
$$[1.3] = 1, [1.99] = 1, [2] = 2, [0.2] = 0, [-0.2] = -1, [-3.1] = -4, \text{ etc.}$$

In general :

for $-3 \le x < -2$, $[x] = -3$

for $-2 \le x < -1$, $[x] = -2$

for $-1 \le x < 0$, $[x] = -1$





for $0 \le x < 1$, $[x] = 0$

for $1 \le x < 2$, $[x]=1$ and so on.

The graph of the function f (x) = [x] is given in Fig. 25.12

(i) From graph

$$\lim_{x \to \frac{1}{2}^-} f(x) = 0, \quad \lim_{x \to \frac{1}{2}^+} f(x) = 0,$$

$\therefore$ $$\lim_{x \to \frac{1}{2}} f(x) = 0$$

Also $$f\left(\frac{1}{2}\right) = [0.5] = 0$$

Thus $$\lim_{x \to \frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$$

Hence f (x) is continuous at

$$x = \frac{1}{2}$$

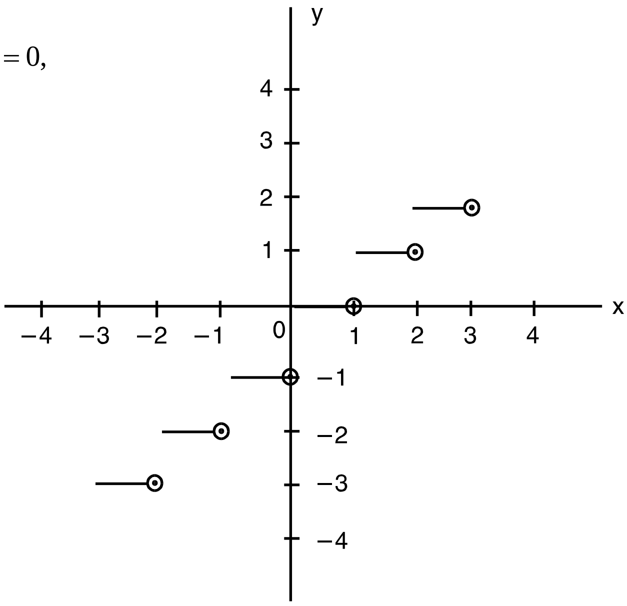



**Fig. 25.12**

(ii) $$\lim_{x \to 1^-} f(x) = 0, \quad \lim_{x \to 1^+} f(x) = 1$$

Thus $\lim_{x \to 1} f(x)$ does not exist.

Hence, f (x) is discontinuous at x = 1.

**Note :** The function f (x) = [x] is also known as Step Function.

**Example 25.24** At what points is the function $\frac{x-1}{(x+4)(x-5)}$ continuous?

**Solution :** Here $f(x) = \frac{x-1}{(x+4)(x-5)}$

The function in the numerator i.e., x – 1 is continuous. The function in the demoninator is (x+4) (x – 5) which is also continuous.

But f (x) is not defined at the points – 4 and 5.

$\therefore$ The function f (x) is continuous at all points except – 4 and 5 at which it is not defined.

In other words, f (x) is continuous at all points of its domain.

## CHECK YOUR PROGRESS 25.5

1. (a) If $f(x) = 2x + 1$, when $x \ne 1$ and f(x)=3 when x = 1, show that the function f (x) continuous at x =1.

(b) If $f(x)=\begin{cases}4x+3, & x\neq 2\\ 3x+5, & x=2\end{cases}$, find whether the function f is continuous at $x=2$.

(c) Determine whether f (x) is continuous at x = 2, where

$$f(x)=\begin{cases}4x+3, & x\le 2\\ 8-x, & x>2\end{cases}$$

(d) Examine the continuity of f (x) at x = 1, where

$$f(x)=\begin{cases}x^2, & x\le 1\\ x+5, & x>1\end{cases}$$

(e) Determine the values of k so that the function

$$f(x)=\begin{cases}kx^2, & x\le 2\\ 3, & x>2\end{cases}\quad \text{is con tinuous at } x=2.$$

2. Examine the continuity of the following functions :

(a) $f(x)=|x-2|$ at $x=2$ (b) $f(x)=|x+5|$ at $x=-5$

(c) $f(x)=|a-x|$ at $x=a$

(d) $f(x)=\begin{cases}\dfrac{|x-2|}{x-2}, & x\neq 2\\ 1, & x=2\end{cases}$ at $x=2$

(e) $f(x)=\begin{cases}\dfrac{|x-a|}{x-a}, & x\neq a\\ 1, & x=a\end{cases}$ at $x=a$

3. (a) If $f(x)=\begin{cases}\sin 4x, & x\neq 0\\ 2, & x=0\end{cases}$, at $x=0$

(b) If $f(x)=\begin{cases}\dfrac{\sin 7x}{x}, & x\neq 0\\ 7, & x=0\end{cases}$, at $x=0$

(c) For what value of a is the function

$$f(x)=\begin{cases}\dfrac{\sin 5x}{3x}, & x\neq 0\\ a, & x=0\end{cases}\quad \text{continuous at } x=0\ ?$$

4. (a) Show that the function f (x) is continuous at x = 2, where

$$f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2}, & \text{for } x\neq 2\\ 3, & \text{for } x=2\end{cases}$$

(b) Test the continuity of the function f (x) at x = 1, where

$$f(x)=\begin{cases}\dfrac{x^2-4x+3}{x-1} & \text{for } x\neq 1\\ -2 & \text{for } x=1\end{cases}$$

(c) For what value of k is the following function continuous at x = 1?

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1} & \text{when } x\neq 1\\ k & \text{when } x=1\end{cases}$$

(d) Discuss the continuity of the function f (x) at x = 2, when

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{for } x\neq 2\\ 7, & x=2\end{cases}$$

5. (a) If $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$, find whether f is continuous at x = 0.

(b) Test the continuity of the function f (x) at the origin.

where $$f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 1, & x=0\end{cases}$$

6. Find whether the function f (x)=[x] is continuous at

(a) $x=\dfrac{4}{3}$ (b) x = 3 (c) $x=-1$ (d) $x=\dfrac{2}{3}$

7. At what points is the function f (x) continuous in each of the following cases ?

(a) $f(x)=\dfrac{x+2}{(x-1)(x-4)}$ (b) $f(x)=\dfrac{x-5}{(x+2)(x-3)}$ (c) $f(x)=\dfrac{x-3}{x^2+5x-6}$

(d) $f(x)=\dfrac{x^2+2x+5}{x^2-8x+16}$

## LET US SUM UP

- If a function f(x) approaches $l$ when x approches a, we say that $l$ is the limit of f (x). Symbolically, it is written as

$$\lim_{x\to a} f(x)=\ell$$

- If $\lim_{x\to a} f(x)=\ell$ and $\lim_{x\to a} g(x)=m$, then

(i) $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = k\ell$

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = \ell \pm m$

(iii) $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = \ell m$

(iv) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{\ell}{m}$, provided $\lim_{x \to a} g(x) \neq 0$

- **LIMIT OF IMPORTANT FUNCTIONS**

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x \to 0} \sin x = 0$

(iii) $\lim_{x \to 0} \cos x = 1$

(iv) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(v) $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

(vi) $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$

(vii) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

## SUPPORTIVE WEB SITES

http://www.youtube.com/watch?v=HB8CzZEd4xw

http://www.zweigmedia.com/RealWorld/Calcsumm3a.html

http://www.intuitive-calculus.com/limits-and-continuity.html

## TERMINAL EXERCISE

Evaluate the following limits :

1. $\lim_{x \to 1} 5$

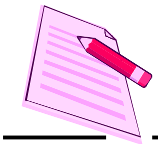
2. $\lim_{x \to 0} \sqrt{2}$

3. $\lim_{x \to 1} \frac{4x^5 + 9x + 7}{3x^6 + x^3 + 1}$

4. $\lim_{x \to -2} \frac{x^2 + 2x}{x^3 + x^2 - 2x}$

5. $\lim_{x \to 0} \frac{(x + k)^4 - x^4}{k(k + 2x)}$

6. $\lim_{x \to 0} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x}$





7. $\lim_{x \to -1} \left[ \frac{1}{x+1} + \frac{2}{x^2-1} \right]$

8. $\lim_{x \to 1} \frac{(2x-3)\sqrt{x}-1}{(2x+3)(x-1)}$

9. $\lim_{x \to 2} \frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$

10. $\lim_{x \to 1} \left[ \frac{1}{x-1} - \frac{2}{x^2-1} \right]$

11. $\lim_{x \to \pi} \frac{\sin x}{\pi - x}$

12. $\lim_{x \to a} \frac{x^2-(a+1)x+a^2}{x^2-a^2}$

Find the left hand and right hand limits of the following functions :

13. $f(x) = \begin{cases} -2x+3 \text{ if } x \le 1 \\ 3x-5 \quad \text{if } x > 1 \end{cases} \text{as } x \to 1$

14. $f(x) = \frac{x^2-1}{|x+1|} \text{as } x \to 1$

Evaluate the following limits :

15. $\lim_{x \to 1^-} \frac{|x+1|}{x+1}$

16. $\lim_{x \to 2^+} \frac{|x-2|}{x-2}$

17. $\lim_{x \to 2^-} \frac{x-2}{|x-2|}$

18. If $f(x) = \frac{(x+2)^2-4}{x}$, prove that $\lim_{x \to 0} f(x) = 4$ though $f(0)$ is not defined.

19. Find k so that $\lim_{x \to 2} f(x)$ may exist where $f(x) = \begin{cases} 5x+2, x \le 2 \\ 2x+k, x > 2 \end{cases}$

20. Evaluate $\lim_{x \to 0} \frac{\sin 7x}{2x}$

21. Evauate $\lim_{x \to 0} \left[ \frac{e^x + e^{-x} - 2}{x^2} \right]$

22. Evaluate $\lim_{x \to 0} \frac{1-\cos 3x}{x^2}$

23. Find the value of $\lim_{x \to 0} \frac{\sin 2x + 3x}{2x + \sin 3x}$

24. Evaluate $\lim_{x \to 1} (1-x) \tan \frac{\pi x}{2}$

25. Evaluate $\lim\limits_{\theta \to 0} \frac{\sin 5\theta}{\tan 8\theta}$

Examine the continuity of the following :

26. $f(x)\begin{cases} 1+3x \text{ if } x > -1 \\ 2 \text{ if } x \le -1 \end{cases}$

at $x = -1$

27. $f(x) = \begin{cases} \frac{1}{x} - x, 0 < x < \frac{1}{2} \\ \frac{1}{2}, x = \frac{1}{2} \\ \frac{3}{2} - x, \frac{1}{2} < x < 1 \end{cases}$

at $x = \frac{1}{2}$

28. For what value of k, will the function

$$f(x) = \begin{cases} \frac{x^2 - 16}{x - 4} \text{ if } x \ne 4 \\ k \text{ if } x = 4 \end{cases}$$

be continuous at $x = 4$ ?

29. Determine the points of discontinuity, if any, of the following functions :

(a) $\frac{x^2+3}{x^2+x+1}$ (b) $\frac{4x^2+3x+5}{x^2-2x+1}$

(b) $\frac{x^2+x+1}{x^2-3x+1}$ (d) $f(x) = \begin{cases} x^4 - 16, & x \ne 2 \\ 16, & x = 2 \end{cases}$

30. Show that the function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos, & x \ne 0 \\ 2, & x = 0 \end{cases}$ is continuous at $x = 0$

31. Determine the value of 'a', so that the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{a \cos x}{\pi - 2x}, & x \ne \frac{\pi}{2} \\ 5, & x = \frac{\pi}{2} \end{cases}$$

is continuous.

Notes

## ANSWERS

### CHECK YOUR PROGRESS 25.1

1. (a) 17 (b) 7 (c) 0 (d) 2
   (e)- 4 (f) 8
2. (a) 0 (b) $\frac{3}{2}$ (c) $-\frac{2}{11}$ (d) $\frac{q}{b}$ (e) 6
   (f) $-10$ (g) 3 (h) 2
3. (a) 3 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{1}{2}$
4. (a) $\frac{1}{2}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{1}{2\sqrt{6}}$ (d) 2 (e) $-1$
5. (a) Does not exist (b) Does not exist
6. (a) 0 (b) $\frac{1}{4}$ (c) does not exist
7. (a) $1, -2$ (b) 1 (c) 19
8. $a = -2$
10. limit does not exist

### CHECK YOUR PROGRESS 25.2

1. (a) 2 (b) $\frac{e^2 - 1}{e^2 + 1}$
2. (a) $-\frac{1}{e}$ (b) $-e$
3. (a) 2 (b) $\frac{1}{5}$ (c) 0 (d) $\frac{a}{b}$
4. (a) $\frac{1}{2}$ (b) 0 (c) 4 (d) $\frac{2}{3}$
5. (a) $\frac{a^2}{b^2}$ (b) 2 (c) $\frac{1}{2}$
6. (a) 1 (b) $\frac{\pi}{2}$ (c) 0
7. (a) $\frac{5}{3}$ (b) $\frac{7}{4}$ (c) -5

## CHECK YOUR PROGRESS 25.3

1. (a) Continuous (b) Continuous
   (c) Continuous (d) Continuous
5. (a) p =3 (b) a = 4
   (c) $b = \frac{14}{9}$

## CHECK YOUR PROGRESS 25.4

2. (a) Continuous
   (b) Discontinuous at x = 2
   (c) Discontinuous at $x = -3$
   (d) Discontinuous at x = 4

## CHECK YOUR PROGRESS 25.5

1. (b) Continuous (c) Discontinuous
   (d) Discontinuous (e) $k = \frac{3}{4}$
2 (a) Continuous (c) Continuous,
   (d) Discontinuous (e) Discontinuous
3 (a) Discontinuous (b) Continuous (c) $\frac{5}{3}$
4 (b) Continuous (c) k = 2
   (d) Discontinuous
5. (a) Discontinuous (b) Discontinuous
6 (a) Continuous (b) Discontinuous
   (c) Discontinuous (d) Continuous
7. (a) All real number except 1 and 4
   (b) All real numbers except –2 and 3
   (c) All real number except –6 and 1
   (d) All real numbers except 4

## TERMINAL EXERCISE

1. 5 2. $\sqrt{2}$ 3. 4 4. $-\frac{1}{3}$

5. $2x^2$ 6. 1 7. $-\frac{1}{2}$ 8. $-\frac{1}{10}$

| | | | |
|---|---|---|---|
| 9. | $-8$ | 10. | $\frac{1}{2}$ |
| 11. | $1$ | 12. | $\frac{a-1}{2a}$ |
| 13. | $1, -2$ | 14. | $-2, 2$ |
| 15. | $-1$ | 16. | $1$ |
| 17. | $-1$ | 19. | $k = 8$ |
| 20. | $\frac{7}{2}$ | 21. | $1$ |
| 22. | $\frac{9}{2}$ | 23. | $1$ |
| 24. | $\frac{2}{\pi}$ | 25. | $\frac{5}{8}$ |

26. Discontinuous

27. Discontinuous

28. $k = 8$

29. (a) No (b) $x = 1$

(c) $x = 1$, $x = 2$ (d) $x = 2$

31. 10