30

INTEGRATION

311en30

In the previous lesson, you have learnt the concept of derivative of a function. You have also learnt the application of derivative in various situations.

Consider the reverse problem of finding the original function, when its derivative (in the form of a function) is given. This reverse process is given the name of integration. In this lesson, we shall study this concept and various methods and techniques of integration.

OBJECTIVES

After studying this lesson, you will be able to :

- explain integration as inverse process (anti-derivative) of differentiation;
- find the integral of simple functions like x^n, sin x, cos x,

  $\sec^2 x, \operatorname{cosec}^2 x, \sec x \tan x, \operatorname{cosec} x \cot x, \frac{1}{x}, e^x$ etc.;
- state the following results :

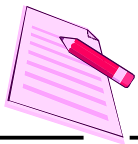
  (i) $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

  (ii) $\int [\pm k f(x)]\, dx = \pm k \int f(x)\, dx$
- find the integrals of algebraic, trigonometric, inverse trigonometric and exponential functions;
- find the integrals of functions by substitution method.
- evaluate integrals of the type

  $$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{a^2 - x^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c},$$

  $$\int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \frac{(px + q)\, dx}{ax^2 + bx + c}, \int \frac{(px + q)\, dx}{\sqrt{ax^2 + bx + c}}$$
- derive and use the result

  $$\int \frac{f'(x)}{f(x)} = \ell n |f(x)| + C$$
- state and use the method of integration by parts;



- evaluate integrals of the type :

  $\int \sqrt{x^2 \pm a^2}\,dx,\ \int \sqrt{a^2 - x^2}\,dx,\ \int e^{ax} \sin bx\ dx,\ \int e^{ax} \cos bx\ dx,$

  $\int (px+q)\sqrt{ax^2+bx+c}\ dx,\ \int \sin^{-1} x\ dx,\ \int \cos^{-1} x\ dx,$

  $\int \sin^n x \cos^m x\ dx,\ \int \frac{dx}{a+b\sin x},\ \int \frac{dx}{a+b\cos x}$

- derive and use the result

  $$\int e^x [f(x) + f'(x)]dx = e^x f(x) + c;$$ and

- integrate rational expressions using partial fractions.

EXPECTED BACKGROUND KNOWLEDGE

- Differentiation of various functions
- Basic knowledge of plane geometry
- Factorization of algebraic expression
- Knowledge of inverse trigonometric functions

30.1 INTEGRATION

Integration literally means summation. Consider, the problem of finding area of region ALMB as shown in Fig. 30.1.

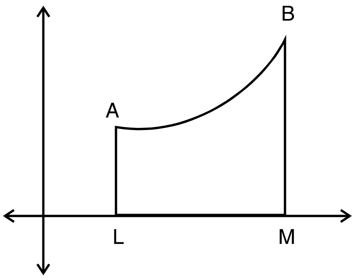


Fig. 30.1

We will try to find this area by some practical method. But that may not help every time. To solve such a problem, we take the help of integration (summation) of area. For that, we divide the figure into small rectangles (See Fig.30.2).

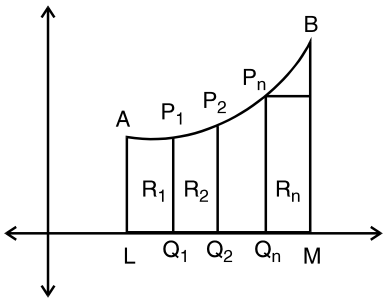


Fig. 30.2

Unless these rectangles are having their width smaller than the smallest possible, we cannot find the area.

This is the technique which Archimedes used two thousand years ago for finding areas, volumes, etc. The names of Newton (1642-1727) and Leibnitz (1646-1716) are often mentioned as the creators of present day of Calculus.

The integral calculus is the study of integration of functions. This finds extensive applications in Geometry, Mechanics, Natural sciences and other disciplines.

In this lesson, we shall learn about methods of integrating polynomial, trigonometric, exponential and logarithmic and rational functions using different techniques of integration.

30.2. INTEGRATION AS INVERSE OF DIFFERENTIATION

Consider the following examples :

(i) $\frac{d}{dx}\left(x^2\right) = 2x$ (ii) $\frac{d}{dx}(\sin x) = \cos x$ (iii) $\frac{d}{dx}\left(e^x\right) = e^x$

Let us consider the above examples in a different perspective

(i) 2x is a function obtained by differentiation of x^2.

$\Rightarrow x^2$ is called the antiderivative of $2x$

(ii) cos x is a function obtained by differentiation of sin x

$\Rightarrow \sin x$ is called the antiderivative of cos x

(iii) Similarly, e^x is called the antiderivative of e^x

Generally we express the notion of antiderivative in terms of an operation. This operation is called the operation of integration. We write

1. Integration of 2 x is x^2
2. Integration of cos x is sin x
3. Integration of e^x is e^x

The operation of integration is denoted by the symbol $\int$.

Thus

1. $\int 2x\,dx = x^2$ 2. $\int \cos x\,dx = \sin x$ 3. $\int e^x\,dx = e^x$

Remember that dx is symbol which together with symbol $\int$ denotes the operation of integration.

The function to be integrated is enclosed between $\int$ and dx.

Definition : If $\frac{d}{dx}[f(x)] = f'(x)$, then f (x) is said to be an integral of $f'(x)$ and is written as $\int f'(x)dx = f(x)$

The function f '(x) which is integrated is called the integrand.

Constant of integration

$$\text{If } y = x^2, \text{ then } \frac{dy}{dx} = 2x$$

$$\therefore \quad \int 2x\,dx = x^2$$

Now consider $\frac{d}{dx}\left(x^2 + 2\right)$ or $\frac{d}{dx}\left(x^2 + c\right)$ where c is any real constant . Thus, we see that integral of 2x is not unique. The different values of $\int 2x\,dx$ differ by some constant. Therefore, $\int 2x\,dx = x^2 + C$, where c is called the constant of integration.

Thus $\int e^x\,dx = e^x + C$, $\int \cos x\,dx = \sin x + c$

In general $\int f'(x)\,dx = f(x) + C$. The constant c can take any value.

We observe that the derivative of an integral is equal to the integrand.

Note : $\int f(x)\,dx$, $\int f(y)\,dy$, $\int f(z)\,dz$ but not like $\int f(z)\,dx$

	Integral	**Verification**
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where n is a constant and $n \neq -1$.	$\because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = x^n$
2.	$\int \sin x\,dx = -\cos x + C$	$\because \frac{d}{dx}(-\cos x + C) = \sin x$
3.	$\int \cos x\,dx = \sin x + C$	$\because \frac{d}{dx}(\sin x + C) = \cos x$
4.	$\int \sec^2 x\,dx = \tan x + C$	$\because \frac{d}{dx}(\tan x + C) = \sec^2 x$
5.	$\int \operatorname{cosec}^2 x\,dx = -\cot x + C$	$\because \frac{d}{dx}(-\cot x + C) = \operatorname{cosec}^2 x$
6.	$\int \sec x \tan x\,dx = \sec x + C$	$\because \frac{d}{dx}(\sec x + C) = \sec x \tan x$
7.	$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$	$\because \frac{d}{dx}(-\operatorname{cosec} x + C) = \operatorname{cosec} x \cot x$
8.	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$	$\because \frac{d}{dx}\left(\sin^{-1} x + C\right) = \frac{1}{\sqrt{1-x^2}}$

9. $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$ $\quad\because \frac{d}{dx}\left(\tan^{-1}x + C\right) = \frac{1}{1+x^2}$

10. $\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x + C$ $\quad\because \frac{d}{dx}\left(\sec^{-1}x + C\right) = \frac{1}{x\sqrt{x^2-1}}$

11. $\int e^x dx = e^x + C$ $\quad\because \frac{d}{dx}\left(e^x + C\right) = e^x$

12. $\int a^x dx = \frac{a^x}{\log a} + C$ $\quad\because \frac{d}{dx}\left(\frac{a^x}{\log a} + C\right) = a^x = \frac{1}{x}$ if $x > 0$

13. $\int \frac{1}{x}dx = \log|x| + C$ $\quad\because \frac{d}{dx}\left(\log|x| + C\right)$

WORKING RULE

1. To find the integral of x^n, increase the index of x by 1, divide the result by new index and add constant C to it.

2. $\int \frac{1}{f(x)}dx$ will be very often written as $\int \frac{dx}{f(x)}$.

CHECK YOUR PROGRESS 30.1

1. Write any five different values of $\int x^{\frac{5}{2}}dx$

2. Write indefinite integral of the following :

(a) x^5 (b) $\cos x$ (c) 0

3. Evaluate :

(a) $\int x^6 dx$ (b) $\int x^{-7}dx$ (c) $\int \frac{1}{x}dx$ (d) $\int 3^x 5^{-x}dx$

(e) $\int \sqrt[3]{x}\,dx$ (f) $\int x^{-9}\,dx$ (g) $\int \frac{1}{\sqrt{x}}\,dx$ (h) $\int \sqrt[9]{x^{-8}}\,dx$

4. Evaluate :

(a) $\int \frac{\cos\theta}{\sin^2\theta}d\theta$ (b) $\int \frac{\sin\theta}{\cos^2\theta}d\theta$

(c) $\int \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} d\theta$ (d) $\int \frac{1}{\sin^2\theta} d\theta$

30.4 PROPERTIES OF INTEGRALS

If a function can be expressed as a sum of two or more functions then we can write the integral of such a function as the sum of the integral of the component functions, e.g. if $f(x) = x^7 + x^3$, then

$$\int f(x)dx = \int [x^7 + x^3]dx = \int x^7dx + \int x^3dx = \frac{x^8}{8} + \frac{x^4}{4} + C$$

So, in general the integral of the sum of two functions is equal to the sum of their integrals.

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Similarly, if the given function

$$f(x) = x^7 - x^2$$

we can write it as $\int f(x)dx = \int (x^7 - x^2)dx = \int x^7dx - \int x^2dx$

$$= \frac{x^8}{8} - \frac{x^3}{3} + C$$

The integral of the difference of two functions is equal to the difference of their integrals.

i.e. $$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

If we have a function f(x) as a product of a constant (k) and another function $[g(x)]$

i.e. $f(x) = kg(x)$, then we can integrate f(x) as

$$\int f(x)dx = \int kg(x)dx = k\int g(x)dx$$

Integral of product of a constant and a function is product of that constant and integral of the function.

i.e. $$\int kf(x)dx = k\int f(x)dx$$

Example 30.1 Evaluate :

(ii) $\int 4^x dx$ (ii) $\int (2^x)(3^{-x})dx$

Solution :(i) $\int 4^x dx = \frac{4^x}{\log 4} + C$

(ii) $$\int (2^x)(3^{-x})\,dx = \int \frac{2^x}{3^x}\,dx = \int \left(\frac{2}{3}\right)^x dx = \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$$

Remember in (ii) it would not be correct to say that

$$\int 2^x 3^{-x}\,dx = \int 2^x\,dx \int 3^{-x}\,dx$$

Because $$\int 2^x\,dx \int 3^{-x}\,dx = \frac{2^x}{\log 2}\left(\frac{3^{-x}}{\log 3}\right) + C \neq \frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)} + C$$

Therefore, **integral of a product of two functions is not always equal to the product of the integrals.** We shall deal with the integral of a product in a subsequent lesson.

Example 30.2 Evaluate :

(i) $\int \frac{dx}{\cos^n x}$, when n = 0 and n = 2 (ii) $\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\,d\theta$

Solution :

(i) When n = 0, $$\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^o x}$$

$$= \int \frac{dx}{1} = \int dx$$

Now $\int dx$ can be written as $\int x^o dx$.

$\therefore$ $$\int dx = \int x^o dx = \frac{x^{o+1}}{0+1} + C = x + C$$

When n = 2,

$$\int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^2 x}$$

$$= \int \sec^2 x\,dx$$

$$= \tan x = C$$

(ii) $$\int -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{-1}{\sin^2\theta}\,d\theta = -\int \operatorname{cosec}^2\theta\,d\theta$$

$$= \cot\theta + C$$

Example 30.3 Evaluate :

(i) $\int (\sin x + \cos x)\, dx$ (ii) $\int \frac{x^2+1}{x^3}\, dx$

(iii) $\int \frac{1-x}{\sqrt{x}}\, dx$ (iv) $\int \left(\frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx$

Solution : (i) $\int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx = -\cos x + \sin x + C$

(ii) $\int \frac{x^2+1}{x^3}\, dx = \int \left(\frac{x^2}{x^3} + \frac{1}{x^3} \right) dx = \int \frac{1}{x}\, dx + \int \frac{1}{x^3}\, dx$

$$= \log|x| + \frac{x^{-3+1}}{-3+1} + C = \log|x| - \frac{1}{2x^2} + C$$

(iii) $\int \frac{1-x}{\sqrt{x}}\, dx = \int \left(\frac{1}{\sqrt{x}} - \frac{x}{\sqrt{x}} \right) dx = \int \left(x^{-\frac{1}{2}} - x^{\frac{1}{2}} \right) dx$

$$= 2\sqrt{x} - \frac{2}{3} x^{3/2} + C$$

(iv) $\int \left(\frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx = \int \frac{dx}{1+x^2} - \int \frac{dx}{\sqrt{1-x^2}}$

$$= \tan^{-1} x - \sin^{-1} x + C$$

Example 30.4 Evaluate :

(i) $\int \sqrt{1 - \sin 2\theta}\, d\theta$ (ii) $\int \left(4e^x - \frac{3}{x\sqrt{x^2-1}} \right) dx$

(iii) $\int (\tan x + \cot x)^2\, dx$ (iv) $\int \left(\frac{x^6 - 1}{x^2 - 1} \right) dx$

Solution : (i) $\sqrt{1 - \sin 2\theta} = \sqrt{\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}$

$$\left[\because \sin^2\theta + \cos^2\theta = 1 \right]$$

$$= \sqrt{(\cos\theta - \sin\theta)^2} = \pm(\cos\theta - \sin\theta)$$

(sign is selected depending upon the value of θ)

(a) If $\sqrt{1 - \sin 2\theta} = \cos\theta - \sin\theta$

then $\int \sqrt{1 - \sin 2\theta}\, d\theta = \int (\cos\theta - \sin\theta)\, d\theta$

$$= \int \cos\theta \, d\theta - \int \sin\theta \, d\theta = \sin\theta + \cos\theta + C$$

(b) If $\int \sqrt{1-\sin 2\theta}\, d\theta = \int (-\cos\theta + \sin\theta)\, d\theta = -\int \cos\theta\, d\theta + \int \sin\theta\, d\theta$

$$= -\sin\theta - \cos\theta + C$$

(ii) $\int \left(4e^x - \frac{3}{x\sqrt{x^2-1}} \right) dx = \int 4e^x dx - \int \frac{3}{x\sqrt{x^2-1}} dx$

$$= 4\int e^x dx - 3\int \frac{dx}{x\sqrt{x^2-1}} = 4e^x - 3\sec^{-1} x + C$$

(iii) $\int (\tan x + \cot x)^2 dx = \int \left(\tan^2 x + \cot^2 x + 2\tan x \cot x \right) dx$

$$= \int \left(\tan^2 x + \cot^2 x + 2 \right) dx$$

$$= \int \left(\tan^2 x + 1 + \cot^2 x + 1 \right) dx$$

$$= \int \left(\sec^2 x + \operatorname{cosec}^2 x \right) dx$$

$$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx$$

$$= \tan x - \cot x + C$$

(iv) $\int \left(\frac{x^6 - 1}{x^2 + 1} \right) dx = \int \left(x^4 - x^2 + 1 - \frac{2}{x^2+1} \right) dx$ (dividing $x^6 - 1$ by $x^2 + 1$)

$$= \int x^4 dx - \int x^2 dx + \int dx - 2\int \frac{dx}{x^2+1}$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1} x + C$$

Example 30.5 Evaluate :

(i) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^3 dx$ (ii) $\int \left(\frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}} \right) dx$

Solution :

(i) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^3 dx = \int \left(x^{3/2} + 3x\frac{1}{\sqrt{x}} + 3\sqrt{x}\,\frac{1}{x} + \frac{1}{x^{3/2}} \right) dx$

$$= \int x^{3/2} dx + 3\int \sqrt{x}\, dx + 3\int \frac{1}{\sqrt{x}} dx + \int \frac{dx}{x^{3/2}}$$

$$= \frac{x^{5/2}}{\frac{5}{2}} + 3\frac{x^{3/2}}{\frac{3}{2}} + 3\frac{x^{1/2}}{\frac{1}{2}} - \frac{2}{\sqrt{x}} + C$$

MODULE - VIII
Calculus

Notes

$$= \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + C$$

(ii) $$\int\left(\frac{4e^{5x} - 9e^{4x} - 3}{e^{3x}}\right)dx = \int\frac{4e^{5x}}{e^{3x}}dx - \int\frac{9e^{4x}}{e^{3x}}dx - \int\frac{3dx}{e^{3x}}$$

$$= 4\int e^{2x}\,dx - 9\int e^{x}\,dx - 3\int e^{-3x}\,dx$$

$$= 2e^{2x} - 9e^{x} + e^{-3x} + C$$

CHECK YOUR PROGRESS 30.2

1. Evaluate :

(a) $\int\left(x + \frac{1}{2}\right)dx$ (b) $\int\frac{-x^2}{1 + x^2}dx$

(c) $\int\left(10x^9 - \sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$ (d) $\int\left(\frac{5 + 3x - 6x^2 - 7x^4 - 8x^6}{x^6}\right)dx$

(e) $\int\frac{x^4}{1 + x^2}dx$ (f) $\int\left(\sqrt{x} + \frac{2}{\sqrt{x}}\right)^2 dx$

2. Evaluate :

(a) $\int\frac{dx}{1 + \cos 2x}$ (b) $\int\tan^2 x dx$ (c) $\int\frac{2\cos x}{\sin^2 x}dx$

(d) $\int\frac{dx}{1 - \cos 2x}$ (e) $\int\frac{\sin x}{\cos^2 x}dx$ (f) $\int(\operatorname{cosec} x - \cot x)\operatorname{cosec} x\,dx$

3. Evaluate :

(a) $\int\sqrt{1 + \cos 2x}\,dx$ (b) $\int\sqrt{1 - \cos 2x}\,dx$ (c) $\int\frac{1}{1 - \cos 2x}dx$

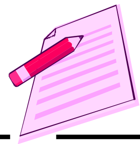
4. Evaluate :

(a) $\int\sqrt{x + 2}\,dx$

30.5 TECHNIQUES OF INTEGRATION

11.5.1 Integration By Substitution

This method consists of expressing $\int f(x)\,dx$ in terms of another variable so that the resultant function can be integrated using one of the standard results discussed in the previous lesson. First, we will consider the functions of the type $f(ax + b)$, $a \neq 0$ where f (x) is a standard function.



Example 30.6 Evaluate :

(i) $\int \sin(ax+b)\,dx$

Solution : (i) $\int \sin(ax+b)\,dx$

Put $ax+b=t$.

Then $a=\dfrac{dt}{dx}$ or $dx=\dfrac{dt}{a}$

$\therefore$ $\int \sin(ax+b)\,dx = \int \sin t \dfrac{dt}{a}$ (Here the integration factor will be replaced by dt.)

$$= \frac{1}{a}\int \sin t\,dt = \frac{1}{a}(-\cos t)+C = -\frac{\cos(ax+b)}{a}+C$$

Example 30.7 Evaluate :

(i) $\int (ax+b)^n\,dx$, where $n \neq -1$ (ii) $\int \dfrac{1}{(ax+b)}\,dx$

Solution : (i) $\int (ax+b)^n\,dx$, where $n \neq -1$

Put $ax+b=t \Rightarrow a=\dfrac{dt}{dx}$ or $dx=\dfrac{dt}{a}$

$\therefore$ $\int (ax+b)^n\,dx = \dfrac{1}{a}\int t^n dt = \dfrac{1}{a}\cdot\dfrac{t^{n+1}}{(n+1)}+C$

$$= \frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1}+C \qquad \text{where } n \neq -1$$

(ii) $\int \dfrac{1}{(ax+b)}\,dx$

Put $ax+b=t \Rightarrow dx=\dfrac{1}{a}dt$

$\therefore$ $\int \dfrac{1}{(ax+b)}\,dx = \int \dfrac{1}{a}\cdot\dfrac{dt}{t} = \dfrac{1}{a}\log|t|+C$

$$= \frac{1}{a}\log|ax+b|+C$$

Example 30.8 Evaluate :

(i) $\int e^{5x+7} dx$

Solution : (i) $\int e^{5x+7} dx$

Put $\quad 5x + 7 = t \quad \Rightarrow \quad dx = \frac{dt}{5}$

$$\therefore \quad \int e^{5x+7} dx = \frac{1}{5}\int e^t dt = \frac{1}{5} e^t + C$$

$$= \frac{1}{5} e^{5x+7} + C$$

Likewise $\quad \int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$

Similarly, using the substitution $ax + b = t$, the integrals of the following functions will be :

$$\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{(ax+b)} dx = \frac{1}{a} \log |ax+b| + C$$

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + C$$

$$\int \sec(ax+b)\tan(ax+b)\, dx = \frac{1}{a}\sec(ax+b) + C$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\, dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$$

Example 30.9 Evaluate :

(i) $\int \sin^2 x\, dx$ (ii) $\int \sin^3 x\, dx$ (iii) $\int \cos^3 x\, dx$ (iv) $\int \sin 3x \sin 2x\, dx$

Solution : We use trigonometrical identities and express the functions in terms of sines and cosines of multiples of x

(i) $\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$

$= \frac{1}{2}\int (1-\cos 2x)\, dx = \frac{1}{2}\int 1 dx - \frac{1}{2}\int \cos 2x \, dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

(ii) $\int \sin^3 x \, dx = \int \frac{3\sin x - \sin 3x}{4} dx \qquad \left[\because \sin 3x = 3\sin x - 4\sin^3 x\right]$

$= \frac{1}{4}\int (3\sin x - \sin 3x)\, dx = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$

(iii) $\int \cos^3 x \, dx = \int \frac{\cos 3x + 3\cos x}{4} dx \qquad \left[\because \cos 3x = 4\cos^3 x - 3\cos x\right]$

$= \frac{1}{4}\int (\cos 3x + 3\cos x)\, dx = \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right] + C$

(iv) $\int \sin 3x \sin 2x \, dx = \frac{1}{2}\int 2\sin 3x \sin 2x \, dx$

$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx = \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right] + C$

CHECK YOUR PROGRESS 30.3

1. Evaluate :

(a) $\int \sin(4-5x)\, dx$ (b) $\int \sec^2(2+3x)\, dx$

(c) $\int \sec\left(x+\frac{\pi}{4}\right) dx$ (d) $\int \cos(4x+5)\, dx$

(e) $\int \sec(3x+5)\tan(3x+5)\, dx$

(f) $\int \operatorname{cosec}(2+5x)\cot(2+5x)\, dx$

2. Evaluate :

(a) $\int \frac{dx}{(3-4x)^4}$ (b) $\int (x+1)^4 \, dx$ (c) $\int (4-7x)^{10} \, dx$

(d) $\int (4x-5)^3 \, dx$ (e) $\int \frac{1}{3x-5} dx$ (f) $\int \frac{1}{\sqrt{5-9x}} dx$

(g) $\int (2x+1)^2\,dx$ (h) $\int \frac{1}{x+1}\,dx$

3. Evaluate :

(a) $\int e^{2x+1}dx$ (b) $\int e^{3-8x}dx$ (c) $\int \frac{1}{e^{(7+4x)}}\,dx$

4. Evaluate :

(a) $\int \cos^2 x\,dx$ (b) $\int \sin^3 x\cos^3 x\,dx$

(c) $\int \sin 4x\,\cos 3x\,dx$ (d) $\int \cos 4x\,\cos 2x\,dx$

30.5.2 Integration of Function of The Type $\frac{f'(x)}{f(x)}$

To evaluate $\int \frac{f'(x)}{f(x)}\,dx$, we put f (x) = t. Then f ' (x) dx = dt.

$$\therefore \quad \int \frac{f'(x)}{f(x)}\,dx = \int \frac{dt}{t} = \log|t| + C = \log|f(x)| + C$$

Integral of a function, whose numerator is derivative of the denominator, is equal to the logarithm of the denominator.

Example 30.10 Evaluate :

(i) $\int \frac{2x}{x^2+1}\,dx$ (ii) $\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)}$

Solution :

(i) Now 2x is the derivative of x^2+1.

$\therefore$ By applying the above result, we have

$$\int \frac{2x}{x^2+1}\,dx = \log\left|x^2+1\right| + C$$

(ii) $\frac{1}{2\sqrt{x}}$ is the derivative of $3+\sqrt{x}$

$$\int \frac{dx}{2\sqrt{x}\left(3+\sqrt{x}\right)} = \log\left|3+\sqrt{x}\right| + C$$

Example 30.11 Evaluate :

(i) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$ (ii) $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$

Solution :

(i) $e^x + e^{-x}$ is the derivative of $e^x - e^{-x}$

$$\therefore \quad \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \log\left|e^x - e^{-x}\right| + C$$

Alternatively,

For $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$,

Put $e^x - e^{-x} = t.$

Then $\left(e^x + e^{-x}\right) dx = dt$

$$\therefore \quad \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int \frac{dt}{t} = \log|t| + C = \log\left|e^x - e^{-x}\right| + C$$

(ii) $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$

Here $e^{2x} - 1$ is not the derivative of $e^{2x} + 1$. But if we multiply the numerator and denominator by e^{-x}, the given function will reduce to

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \log\left|e^x + e^{-x}\right| + C$$

$$\therefore \quad \int \frac{e^{2x} - 1}{e^{2x} + 1} dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} = \log\left|e^x + e^{-x}\right| + C$$

$$\left[\because \left(e^x - e^{-x}\right) \text{ is the derivative of } \left(e^x + e^{-x}\right)\right]$$

CHECK YOUR PROGRESS 30.4

1. Evaluate :

(a) $\int \frac{x}{3x^2 - 2} dx$ (b) $\int \frac{2x + 1}{x^2 + x + 1} dx$ (c) $\int \frac{2x + 9}{x^2 + 9x + 30} dx$

(d) $\int \frac{x^2 + 1}{x^3 + 3x + 3} dx$ (e) $\int \frac{2x + 1}{x^2 + x - 5} dx$ (f) $\int \frac{dx}{\sqrt{x}\left(5 + \sqrt{x}\right)}$

(g) $\int \frac{dx}{x\left(8 + \log x\right)}$

2. Evaluate :

(a) $\int \frac{e^x}{2 + be^x} dx$ (b) $\int \frac{dx}{e^x - e^{-x}}$

30.5.3 INTEGRATION BY SUBSTITUTION

(i) $\int \tan x \, dx$ (ii) $\int \sec x \, dx$

Solution :

(i) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$

$= -\log\left|\cos x\right| + C$ ($\because -\sin x$ is derivative of $\cos x$)

$= \log\left|\frac{1}{\cos x}\right| + C$ or $= \log\left|\sec x\right| + C$

$\therefore$ $\int \tan x \, dx = \log\left|\sec x\right| + C$

Alternatively,

$$\int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x} = -\int \frac{-\sin x \, dx}{\cos x}$$

Put $\cos x = t.$

Then $-\sin x \, dx = dt$

$\therefore$ $\int \tan x \, dx = -\int \frac{dt}{t} = -\log\left|t\right| + C = -\log\left|\cos x\right| + C$

$= \log\left|\frac{1}{\cos x}\right| + C = \log\left|\sec x\right| + C$

(ii) $\int \sec x \, dx$

sec x can not be integrated as such because sec x by itself is not derivative of any function. But this is not the case with $\sec^2 x$ and sec x tan x. Now $\int \sec x \, dx$ can be written as

$$\int \sec x \frac{\left(\sec x + \tan x\right)}{\left(\sec x + \tan x\right)} dx$$

$$= \int \frac{\left(\sec^2 x + \sec x \tan x\right)}{\sec x + \tan x} dx$$

Put $\sec x + \tan x = t.$

Then $\left(\sec x \tan x + \sec^2 x\right) dx = dt$

$$\therefore \quad \int \sec x \, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$$

Example 30.13 Evaluate $\int \frac{1}{a^2 - x^2} dx$

Solution : Put $x = a \sin\theta \quad \Rightarrow \quad dx = a\cos\theta \, d\theta$

$$\therefore \quad \int \frac{1}{a^2 - x^2} dx = \int \frac{a\cos\theta}{a^2 - a^2\sin^2\theta} d\theta$$

$$= \frac{1}{a}\int \frac{\cos\theta}{1 - \sin^2\theta} d\theta = \frac{1}{a}\int \frac{1}{\cos\theta} d\theta = \frac{1}{a}\int \sec\theta \, d\theta$$

$$= \frac{1}{a}\log|\sec\theta + \tan\theta| + C = \frac{1}{a}\log\left|\frac{1 + \sin\theta}{\cos\theta}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{1 + \frac{x}{a}}{\sqrt{1 - \frac{x^2}{a^2}}}\right| + C = \frac{1}{a}\log\left|\frac{a + x}{\sqrt{a^2 - x^2}}\right| + C = \frac{1}{a}\log\left|\frac{\sqrt{a + x}}{\sqrt{a - x}}\right| + C$$

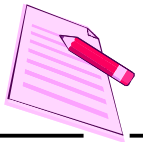
$$= \frac{1}{a}\log\left|\left(\frac{a + x}{a - x}\right)^{\frac{1}{2}}\right| + C = \frac{1}{2a}\log\left|\frac{a + x}{a - x}\right| + C$$

Example 30.14 Evaluate : $\int \frac{1}{x^2 - a^2} dx$

Solution : Put $x = a\sec\theta \quad \Rightarrow \quad dx = a\sec\theta\tan\theta \, d\theta$

$$\therefore \quad \int \frac{1}{x^2 - a^2} dx = \int \frac{a\sec\theta\tan\theta \, d\theta}{a^2\sec^2\theta - a^2}$$

$$= \frac{1}{a}\int \frac{\sec\theta\tan\theta}{\tan^2\theta} d\theta \qquad \left(\tan^2\theta = \sec^2\theta - 1\right)$$



$$= \frac{1}{a}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{a}\int \frac{1}{\sin\theta}\,d\theta = \frac{1}{a}\int \operatorname{cosec}\theta\,d\theta$$

$$= \frac{1}{a}\log\left|\operatorname{cosec}\theta - \cot\theta\right| + C = \frac{1}{a}\log\left|\frac{1-\cos\theta}{\sin\theta}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{1-\frac{a}{x}}{\sqrt{1-\frac{a^2}{x^2}}}\right| + C = \frac{1}{a}\log\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + C$$

$$= \frac{1}{a}\log\left|\frac{\sqrt{x-a}}{\sqrt{x+a}}\right| + C = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

Example 30.15 $\int \frac{1}{a^2+x^2}\,dx$

Solution : Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\,d\theta$

$$\therefore \quad \int \frac{1}{a^2+x^2}\,dx = \int \frac{a\sec^2\theta}{a^2\left(1+\tan^2\theta\right)}\,d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C \qquad \left(\frac{x}{a} = \tan\theta \Rightarrow \tan^{-1}\frac{x}{a} = \theta\right)$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Example 30.16 $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

$$\therefore \quad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}\,d\theta$$

$$= \int \frac{a\cos\theta}{a\cos\theta}\,d\theta = \int d\theta = \theta + C$$

$$= \sin^{-1}\frac{x}{a} + C$$

Example 30.17 $\int \frac{1}{\sqrt{x^2 - a^2}} dx$

Solution : Let $\quad x = a \sec\theta \quad \Rightarrow \quad dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \quad \int \frac{1}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta \tan\theta}{a\sqrt{\sec^2\theta - 1}} d\theta$$

$$= \int \sec\theta d\theta = \log|\sec\theta + \tan\theta| + C$$

$$= \log\left|\frac{x}{a} + \frac{1}{a}\sqrt{x^2 - a^2}\right| + C$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Example 30.18 $\int \frac{1}{\sqrt{a^2 + x^2}} dx$

Solution : Put $x = a\tan\theta \quad \Rightarrow \quad dx = a\sec^2\theta\, d\theta$

$$= \int \sec\theta\, d\theta$$

$$= \log|\sec\theta + \tan\theta| + C = \log\left|\frac{1}{a}\sqrt{a^2 + x^2} + \frac{x}{a}\right| + C$$

$$= \log\left|\sqrt{a^2 + x^2} + x\right| + C$$

Example 30.19 $\int \frac{x^2 + 1}{x^4 + 1} dx$

Solution : Since x^2 is not the derivative of $x^4 + 1$, therefore, we write the given integral as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx$$

Let $\quad x - \frac{1}{x} = t$. Then $\quad \therefore \quad \left(1 + \frac{1}{x^2}\right) dx = dt$

Also $\quad x^2 - 2 + \frac{1}{x^2} = t^2 \quad \Rightarrow \quad x^2 + \frac{1}{x^2} = t^2 + 2$

$$\therefore \quad \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{dt}{t^2+2} = \int \frac{dt}{(t)^2+\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + C$$

Example 30.20 $\int \frac{x^2-1}{x^4+1}dx$

Solution : $\int \frac{x^2-1}{x^4+1}dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

Put $x+\frac{1}{x} = t$. Then $\left(1-\frac{1}{x^2}\right)dx = dt$

Also $x^2+2+\frac{1}{x^2} = t^2 \quad \Rightarrow \quad x^2+\frac{1}{x^2} = t^2-2$

$$\therefore \quad \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{dt}{t^2-2} = \int \frac{dt}{(t)^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

Example 30.21 $\int \frac{x^2}{x^4+1}dx$

Solution : In order to solve it, we will reduce the given integral to the integrals given in Examples 11.19 and 11.20.

i.e., $\int \frac{x^2}{x^4+1}dx = \frac{1}{2}\int \left[\frac{x^2+1}{x^4+1}+\frac{x^2-1}{x^4+1}\right]dx$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx + \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right] + C$$

Example 30.22 $\int \frac{1}{x^4+1}dx$

Solution : We can reduce the given integral to the following form

$$\frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right] + C$$

Example 30.23 (a) $\int \frac{1}{x^2-x+1}dx$ (b) $\int \frac{x^2-1}{x^4+x^2+1}dx$

Solution : (a) $\int \frac{1}{x^2-x+1}dx = \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}dx$

$$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

(b) $$\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

Put $x + \frac{1}{x} = t. \quad \Rightarrow \quad \left(1 - \frac{1}{x^2}\right) dx = dt$

Also $x^2 + 2 + \frac{1}{x^2} = t^2 \quad \Rightarrow \quad x^2 + 1 + \frac{1}{x^2} = t^2 - 1$

$$\therefore \quad \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx = \int \frac{dt}{t^2 - 1} = \frac{1}{2} \log \left| \frac{t - 1}{t + 1} \right| + C$$

$$= \frac{1}{2} \log \left| \frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} \right| + C$$

Example 30.24 $\int \sqrt{\tan x}\, dx$

Solution : Let $\tan x = t^2 \quad \Rightarrow \quad \sec^2 x\, dx = 2t\, dt$

$$\Rightarrow \quad dx = \frac{2t}{\sec^2 x} dt = \frac{2t}{1 + t^4} dt$$

$$\therefore \quad \int \sqrt{\tan x} dx = \int t \left(\frac{2t}{1 + t^4} \right) dt = \int \frac{2t^2}{1 + t^4} dt$$

$$= \int \left(\frac{t^2 + 1}{t^4 + 1} + \frac{t^2 - 1}{t^4 + 1} \right) dt = \int \frac{t^2 + 1}{t^4 + 1} dt + \int \frac{t^2 - 1}{t^4 + 1} dt$$

Example 30.25 $\int \sqrt{\cot x}\, dx$

Solution : Let $\cot x = t^2 \quad \Rightarrow \quad -\text{cosec}^2 x\, dx = 2t\, dt$

$$\Rightarrow \quad dx = \frac{-2t}{\text{cosec}^2 x} dt = -\frac{2t}{t^4 + 1} dt$$

$$\therefore \quad \int \sqrt{\cot x}\, dx = -\int t \left(\frac{2t}{t^4 + 1} \right) dt$$

$$= -\int \frac{2t^2}{t^4+1}\,dt = -\int \left(\frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1} \right) dt$$

Proceed according to Examples 11.19 and 11.20 solved before.

Example 30.26 $\int \left(\sqrt{\tan x} + \sqrt{\cot x} \right) dx$

Let $\quad \sin x - \cos x = t \Rightarrow (\cos x + \sin x)\,dx = dt$

Also $\quad 1 - 2\sin x \cos x = t^2 \Rightarrow \qquad 1 - t^2 = 2\sin x \cos x$

$$\Rightarrow \qquad \frac{1-t^2}{2} = \sin x \cos x$$

$$\therefore \qquad \int \frac{\sin x - \cos x}{\sqrt{\cos x \sin x}}\,dx = \int \frac{dt}{\sqrt{\frac{1-t^2}{2}}} = \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\,\sin^{-1}[\sin x - \cos x] + C$$

(Using the result of Example 26.25)

Example 30.27 Evaluate :

(a) $\displaystyle\int \frac{dx}{\sqrt{8+3x-x^2}}$ (b) $\displaystyle\int \frac{dx}{x(1-2x)}$

Solution :

(a) $$\int \frac{dx}{\sqrt{8+3x-x^2}} = \int \frac{dx}{\sqrt{8-(x^2-3x)}}$$

$$= \int \frac{dx}{\sqrt{8-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}}$$

$$= \sin^{-1}\left[\frac{\left(x-\frac{3}{2}\right)}{\frac{\sqrt{41}}{2}} \right] + C$$

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}} \right) + C$$

(b) $\displaystyle\int \frac{dx}{x(1-2x)} = \int \frac{dx}{\sqrt{x-2x^2}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{x}{2}-x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{1}{16}-\left[x^2-\frac{x}{2}+\frac{1}{16}\right]}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\frac{1}{4}\right)^2-\left(x-\frac{1}{4}\right)^2} = \frac{1}{\sqrt{2}}\sin^{-1}\left\{\frac{\left(x-\frac{1}{4}\right)}{\left(\frac{1}{4}\right)}\right\}+C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}(4x-1)+C$$

CHECK YOUR PROGRESS 30.5

1. Evaluate :

(a) $\displaystyle\int \frac{x^2}{x^2-9}dx$ (b) $\displaystyle\int \frac{e^x}{e^{2x}+1}dx$ (c) $\displaystyle\int \frac{x}{1+x^4}dx$

(d) $\displaystyle\int \frac{dx}{\sqrt{16-9x^2}}$ (e) $\displaystyle\int \frac{dx}{1+3\sin^2 x}$ (f) $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$

(g) $\displaystyle\int \frac{dx}{3x^2+6x+21}$ (h) $\displaystyle\int \frac{dx}{\sqrt{5-4x-x^2}}$ (i) $\displaystyle\int \frac{dx}{x\sqrt{3x^2-12}}$

(j) $\displaystyle\int \frac{d\theta}{\sin^4\theta+\cos^4\theta}$ (k) $\displaystyle\int \frac{e^x dx}{\sqrt{1+e^{2x}}}$ (l) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$

(m) $\displaystyle\int \frac{dx}{\sqrt{2ax-x^2}}$ (n) $\displaystyle\int \frac{3x^2}{\sqrt{9-16x^6}}dx$ (o) $\displaystyle\int \frac{(x+1)}{\sqrt{x^2+1}}dx$

(p) $\displaystyle\int \frac{dx}{\sqrt{9+4x^2}}$ (q) $\displaystyle\int \frac{\sin\theta}{\sqrt{4\cos^2\theta-1}}d\theta$ (r) $\displaystyle\int \frac{\sec^2 x}{\sqrt{\tan^2 x-4}}dx$

(s) $\displaystyle\int \frac{1}{(x+2)^2+1}dx$ (t) $\displaystyle\int \frac{1}{\sqrt{16x^2+25}}dx$

30.6 INTEGRATION BY PARTS

In differentiation you have learnt that

$$\frac{d}{dx}(fg) = f\frac{d}{dx}(g)+g\frac{d}{dx}(f)$$

or $$f\frac{d}{dx}(g)=\frac{d}{dx}(fg)-g\frac{d}{dx}(f) \qquad (1)$$

Also you know that $\int \frac{d}{dx}(fg)\,dx = fg$

Integrating (1). we have

$$\int f\frac{d}{dx}(g)\,dx=\int\frac{d}{dx}(fg)\,dx-\int g\frac{d}{dx}(f)\,dx$$

$$= fg-\int g\frac{d}{dx}(f)\,dx$$

if we take $$f=u(x):\frac{d}{dx}(g)=v(x).$$

(2) become $\int u(x)v(x)\,dx$

$$=u(x)\cdot\int v(x)\,dx-\int\left[\frac{d}{dx}(u(x))\int v(x)\,dx\right]dx$$

=I function $\times$ integral of II function $-\int$ [differential coefficient of function $\times$ integralof II function]dx

A B

Here the important factor is the choice of I and II function in the product of two functions because either can be I or II function. For that the indicator will be part 'B' of the result above.

The first function is to be chosen such that it reduces to a next lower term or to a constant term after subseqent differentiations.

In questions of integration like

$$x\sin x,\ x\cos^2 x,\ x^2e^x$$

(i) algebraic function should be taken as the first function

(ii) If there is no algebraic function then look for a function which simplifies the product in 'B' as above; the choice can be in order of preference like choosin first function

(i) an inverse function (ii) a logarithmic function

(iii) a trigonometric function (iv) an exponential function.

The following examples will give a practice to the concept of choosing first function.

		I function	**II function**
1.	$\int x\cos x\,dx$	x (being algebraic)	cos x

2.	$\int x^2 e^x dx$	x^2 (being algebraic)	e^x
3.	$\int x^2 \log dx$	log x	x^2
4.	$\int \frac{\log x}{(1+x^2)} dx$	log x	$\frac{1}{(1+x)^2}$
5.	$\int x \sin^{-1} x \, dx$	$\sin^{-1} x$	x
6.	$\int \log x \, dx$	log x	1 (In single function of logarithm and inverse trigonometric we take unity as II function)
7.	$\sin^{-1} x \, dx$	$\sin^{-1} x$	1

Example 30.28 Evaluate :

$$\int x^2 \sin x \, dx$$

Solution: Taking algebraic function x^2 as function and sin x as II function, we herv.

$$\int_{I}^{x^2} \sin x \, dx = x^2 \int \sin x - \int \left[\frac{d}{dx}(x^2) \int \sin x \, dx \right] dx$$

$$= -x^2 \cos x - 2\int x(-\cos x) \, dx$$

$$= -x^2 \cos x + 2\int x \cos x \, dx \qquad (1)$$

again $\int x \cos x \, dx = x \sin x + \cos x + c \qquad (2)$

Substituting (2) in (1), we have

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2[x \sin x + \cos x] + C$$

$$= -x^2 \cos x + 2x \sin x + \cos x + C$$

Example 30.29 Evaluate :

$$\int x^2 \log x \, dx$$

Solution: In order of preference log x is to be taken as I function.

$$\therefore \quad \int \underset{\text{I}}{\log x}\, \underset{\text{II}}{x^2}\, dx = \frac{x^3}{3}\log x - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{x^3}{3}\log x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\log x - \frac{1}{-3}\left(\frac{x^3}{3}\right) + C$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C$$

Example 30.30 Evaluate :

$$\int \sin^{-1} x\, dx$$

Solution: $\int \sin^{-1} x\, dx = \int \sin^{-1} x \cdot 1 \cdot dx$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $1 - x^2 = t \quad \Rightarrow \quad -2x\,dx = dt \quad \Rightarrow \quad x\,dx = \frac{-1}{2}\,dt$

$$\therefore \quad \int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} + C = -\sqrt{1-x^2} + C$$

$$\int \sin^{-1} x\, dx = x\sin^{-1} x + \sqrt{1-x^2} + C$$

CHECK YOUR PROGRESS 30.6

Evaluate:

1. (a) $\int x\sin x\,dx$ (b) $\int (1+x^2)\cos 2x\,dx$ (c) $\int x\sin 2x\,dx$

2. (a) $\int x\tan^2 x\,dx$ (b) $\int x^2\sin^2 x\,dx$

3. (a) $\int x^3\log 2x\,dx$ (b) $(1-x^2)\log x\,dx$ (c) $\int (\log x)^2\,dx$

4. (a) $\int \frac{\log x}{x^n}\,dx$ (b) $\int \frac{\log(\log x)}{x}\,dx$

5. (a) $\int x^2 e^{3x} dx$ (b) $\int x\, e^{3x} dx$

6. (a) $\int x(\log x)^2\, dx$

7. (a) $\int \sec^{-1} x\, dx$ (b) $\int x \cot^{-1} x\, dx$

30.7 INTEGRAL OF THE FORM

$$\int e^x \left[f(x) + f'(x) \right] dx$$

where f' (x) is the differentiation of f (x). In such type of integration while integrating by parts the solution will be $e^x\left(f(x)\right) + C$.

For example, consider

$$\int e^x \left[\tan x + \log \sec x\right] dx$$

Let $\int(x) = \log \sec x,$ then $f'(x) = \dfrac{\sec x \tan x}{\sec x} = \tan x$

So (1) can be rewritten as

$$\int e^x \left[f'(x) + f(x) \right] dx = e^x\left(f(x)\right) + C - e^x \log \sec x + C$$

Alternatively, you can evaluate it as under:

$$\int e^x \left[\tan x + \log \sec x\right] dx = \int \underset{\text{I}}{e^x} \underset{\text{II}}{\tan x}\, dx + \int e^x \log \sec x\, dx$$

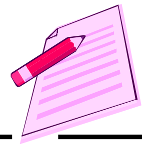
$$= e^x \log \sec x - \int e^x \log \sec x\, dx + \int e^x \log \sec x\, dx$$

$$= e^x \log \sec x + C$$

Example 30.31 Evaluate the following:

(a) $\int e^x \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right) dx$ (b) $\int e^x \left(\dfrac{1 + x\log x}{x}\right) dx$

(c) $\int \dfrac{x e^x}{(x+1)^2} dx$ (d) $\int e^x \left[\dfrac{1 + \sin x}{1 + \cos x}\right] dx$



Notes

Solution:

(a) $$\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\int e^x\left[\frac{1}{x}+\frac{d}{dx}\left(\frac{1}{x}\right)\right]dx=e^x\left(\frac{1}{x}\right)$$

(b) $$\int e^x\left(\frac{1+x\log x}{x}\right)dx=\int e^x\left(\frac{1}{x}+\log x\right)dx$$

$$=\int e^x\left(\log x+\frac{d}{dx}(\log x)\right)dx \quad =e^x\log x+C$$

(c) $$\int\frac{xe^x}{(x+1)^2}dx=\int\frac{x+1-1}{(x+1)^2}e^x\,dx \quad =\int e^x\left(\frac{1}{x+1}-\frac{1}{(x+1)^2}\right)dx$$

$$=\int e^x\left(\frac{1}{x+1}-\frac{d}{dx}\left(\frac{1}{(x+1)}\right)\right)dx$$

$$=e^x\left(\frac{1}{x+1}\right)+C$$

(d) $$=\int e^x\left[\frac{1+\sin x}{1+\cos x}\right]dx=\int e^x\left[\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right]dx$$

$$=\int e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right]dx$$

$$=\int e^x\left[\tan\frac{x}{2}+\frac{d}{x}\left(\tan\frac{x}{2}\right)\right]dx$$

$$=e^x\tan\frac{x}{2}+C$$

Example 30.32 Evaluate the following:

(a) $\int\sec^3 x\,dx$ (b) $\int e^x\sin x\,dx$

Solution:

(a) $$\int \sec^3 x\, dx$$

Let $$I = \int \sec x \cdot \sec^2 x\, dx$$

$$= \sec x \cdot \tan x - \int \sec x \tan x \cdot \tan x\, dx$$

$\therefore$ $$I = \sec x \tan x - \int \left(\sec^3 x - \sec x\right) dx \qquad \left(\because \tan^2 x = \sec^2 x - 1\right)$$

or $$I = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

or $$2I = \sec x \tan x + \int \sec x\, dx$$

or $$I = \sec x \tan x + \log\left|\sec x + \tan x\right| + C_1$$

or $$I = \frac{1}{2}\left[\sec x \tan x + \log\left|\sec x + \tan x\right|\right] + C$$

(b) $$\int e^x \sin x\, dx$$

Let $$I = \int e^x \sin x\, dx$$

$$= e^x\left(-\cos x\right) - \int e^x\left(-\cos x\right) dx \;\; = -e^x \cos x + \int e^x \cos x\, dx$$

$$= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x\, dx\right)$$

$\therefore$ $$I = -e^x \cos x + e^x \sin x - 1$$

or

$$2I = -e^x \cos x + e^x \sin x$$

or $$I = \frac{e^x}{2}\left(\sin x - \cos x\right) + C$$

Example 30.33 Evaluate:

$$\int \sqrt{a^2 - x^2}\, dx$$

Solution:

Let $$I = \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2} \cdot 1\, dx$$

Integrating by parts only and taking 1 as the second function, we have

$$I=\left(\sqrt{a^2-x^2}\right)x-\int\frac{1}{2\sqrt{a^2-x^2}}(-2x)\cdot x\,dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}dx\;=x\sqrt{a^2-x^2}+\int\frac{a^2-\left(a^2-x^2\right)}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}+a^2\int\frac{1}{\sqrt{a^2-x^2}}dx-\int\sqrt{a^2-x^2}\,dx$$

∴ $$I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)-1$$

or $$2I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)$$

or $$I=\frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]+C$$

Similarly, $$\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

∴ $$\int\sqrt{a^2+x^2}\,dx=\frac{x\sqrt{a^2+x^2}}{2}-\frac{a^2}{2}\log\left|x+x\sqrt{a^2+x^2}\right|+C$$

Example 30.34 Evaluate:

(a) $\int\sqrt{16x^2+25}dx$ (b) $\int\sqrt{16-x^2}dx$ (c) $\int\sqrt{1+x-2x^2}dx$

Solution:

(a) $$\int\sqrt{16x^2+25}dx=4\int\sqrt{x^2+\frac{25}{16}}dx=4\int\sqrt{x^2+\left(\frac{5}{4}\right)^2}\,dx$$

Using the formula for $\int\sqrt{\left(x^2+a^2\right)}dx$ we get,

$$\int\sqrt{16x^2+25}dx=\left[\frac{x}{2}\sqrt{x^2+\frac{25}{16}}+\frac{25}{32}\log\left|x+\sqrt{x^2+\frac{25}{16}}\right|\right]+C$$

$$=\frac{x}{8}\sqrt{16x^2+25}+\frac{25}{8}\log\left|4x+\sqrt{16x^2+25}\right|+C$$

(b) Using the formula for $\int \sqrt{\left(a^2 - x^2\right)}\, dx$ *we get*,

$$\int \sqrt{16 - x^2}\, dx = \int \sqrt{(4)^2 - x^2}\, dx = \frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4} + C$$

(c)

$$\int \sqrt{1 + x - 2x^2}\, dx = \sqrt{2}\int \sqrt{\frac{1}{2} + \frac{x}{2} - x^2}\, dx$$

$$= \sqrt{2}\int \sqrt{\left[\frac{1}{2} - \left(x^2 - \frac{x}{2} + \frac{1}{16}\right) + \frac{1}{16}\right]}\, dx$$

$$= \sqrt{2}\int \sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}\, dx$$

$$= \sqrt{2}\left[\frac{x - \frac{1}{4}}{2}\sqrt{\frac{9}{16} - \left(x - \frac{1}{4}\right)^2} + \frac{9}{16 \times 2}\sin^{-1}\frac{x - \frac{1}{4}}{\frac{3}{4}}\right] + C$$

$$= \sqrt{2}\left[\frac{4x - 1}{8} \cdot \frac{1}{\sqrt{2}}\sqrt{1 + x - 2x^2} + \frac{9}{32}\sin^{-1}\frac{4x - 1}{3}\right] + C$$

$$= \frac{4x - 1}{8}\sqrt{1 + x - 2x^2} + \frac{9\sqrt{2}}{32}\sin^{-1}\frac{4x - 1}{3} + C$$

CHECK YOUR PROGRESS 30.7

Evaluate:

1. (a) $\int e^x \sec x \left[1 + \tan x\right] dx$ (b) $\int e^x \left[\sec x + \log\left|\sec x + \tan x\right|\right] \mathrm{dx}$

2. (a) $\int \frac{x - 1}{x^2} e^x\, dx$ (b) $\int e^x \left(\sin^{-1} x - \frac{1}{\sqrt{1 - x^2}}\right) dx$

3. $\int e^x \frac{(x - 1)}{(x + 1)^3}\, dx$

4. $\int \frac{xe^x}{(x + 1)^2}\, dx$

5. $\int \frac{x + \sin x}{1 + \cos x}\, dx$

6. $\int e^x \sin 2x\, dx$

30.8 INTEGRATION BY USING PARTIAL FRACTIONS

By now we are equipped with the various techniques of integration.

But there still may be a case like $\frac{4x+5}{x^2+x-6}$, where the substitution or the integration by parts may not be of much help. In this case, we take the help of another technique called **techmique of integrayion using partial functions.**

Any proper rational fraction $\frac{p(x)}{q(x)}$ can be expressed as the sum of rational functions, each having a single factor of q(x). Each such fraction is known as **partial fraction** and the process of obtaining them is called decomposition or resolving of the given fraction into partial fractions.

For example, $\frac{3}{x+2}+\frac{5}{x-1}=\frac{8x+7}{(x+2)(x-1)}=\frac{8x+7}{x^2+x-2}$

Here $\frac{3}{x+2},\frac{5}{x-1}$ are called partial fractions of $\frac{8x+7}{x^2+x-2}$.

If $\frac{f(x)}{g(x)}$ is a proper fraction and g (x)can be resolved into real factors then,

(a) corresponding to each non repeated linear factor ax + b, there is a partial fraction of the form

(b) for $(ax+b)^2$ we take the sum of two partial fractions · as

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

For $(ax+b)^3$ we take the sum of three partial fractions as

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$$

and so on.

(c) For non-fractorisable quadratic polynomial ax^2+bx+c there is a partial fraction

$$\frac{Ax+B}{ax^2+bx+c}$$

Therefore, if g (x) is a proper fraction $\frac{f(x)}{g(x)}$ and can be resolved into real factors, $\frac{f(x)}{g(x)}$ can be written in the follwoing form:

Factor in the denominator	corresponding partial fraction
ax+b	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}$
$(ax+b)^3$	$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}$

where A,B,C,D are arbitary constants.

The rational functions which we shall consider for integration will be those whose denominators can be fracted into linear and quadratic factors.

Example 30.35 Evaluate:

$$\int \frac{2x+5}{x^2-x-2}dx$$

Solution: $\frac{2x+5}{x^2-x-2}=\frac{2x+5}{(x-2)(x+1)}$

Let $\frac{2x+5}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$

Multiplyping both sides by $(x-2)(x+1)$,we have

$$2x+5=A(x+1)+B(x-2)$$

Putting x = 2, weget 9 =3A or A = 3

Putting x = -1, we get 3 = -3B or B = -1

substituting these values in(1),we have

$$\frac{2x+5}{(x-2)(x+1)} = \frac{3}{x-2} - \frac{1}{x+1}$$

$$\Rightarrow \quad \int \frac{2x+5}{x^2-x-2}dx = \int \frac{3}{x-2}dx - \int \frac{1}{x+1}dx$$

$$= 3\log|x-2| - \log|x+1| + C$$

Example 30.36 Evaluate:

$$\int \frac{x^3+x+1}{x^2-1}dx$$

Solution: $I = \int \frac{x^3+x+1}{x^2-1}dx$

Now $\quad \frac{x^2+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1} = x + \frac{2x+1}{(x+1)(x-1)}$

$$\therefore \quad I = \int \left(x + \frac{2x+1}{(x+1)(x-1)} \right) dx$$

Let $\quad \frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \qquad (2)$

$$\Rightarrow \quad 2x+1 = A(x-1) + B(x+1)$$

Putting x = 1, we get $B = \frac{3}{2}$

Putting x = -1, we get $A = \frac{1}{2}$

Substituting the values of A and B in (2) and integrating, we have

$$\int \frac{2x+1}{(x^2-1)}dx = \frac{1}{2}\int \frac{1}{(x^2+1)}dx + \frac{3}{2}\int \frac{1}{x-1}dx$$

$$= \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| \qquad (3)$$

$\therefore$ From (1) and (3), we have

$$I = \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

Example 30.37 Evaluate:

$$\int \frac{8}{(x-2)(x^2+4)}dx$$

Solution:

(a) $$\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

(As x^2+4 is not factorisable into linear factors)

Multiplying both sides by $(x+2)(x^2+4)$, we have

$$8 = A(x^2+4) + (Bx+C)(x+2)$$

On comparing the corresponding coeffcients of powers of x on both sides, we get

$$\left.\begin{aligned} 0 &= A+B \\ 0 &= 2B+C \\ 8 &= 4A+2C \end{aligned}\right\} \Rightarrow A=1, B=-1, C=2$$

$$\therefore \int \frac{8}{(x+2)(x^2+4)}dx = \int \frac{1}{x+2}dx - \int \frac{x-2}{x^2-4}dx$$

$$= \int \frac{1}{x+2}dx = \frac{1}{2}\int \frac{2x}{x^2+4}dx + 2\int \frac{dx}{x^4+4}$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + 2\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\frac{x}{2} + C$$

Example 30.38 Evaluate:

$$\int \frac{2\sin 2\theta - \cos\theta}{4-\cos^2\theta - 4\sin\theta}d\theta$$

Solution:

Let $$\mathrm{I} = \int \frac{2\sin 2\theta - \cos\theta}{4-\cos^2\theta - 4\sin\ \theta}d\theta = \int \frac{(4\sin\ \theta - 1)\cos\theta\, d\theta}{3+\sin^2\theta - 4\sin\ \theta}$$

Let $\sin\theta = t$, then $\cos\theta\, d\theta = dt$

$$\therefore \qquad I = \int \frac{4t-1}{3+t^2-4t}\,dt$$

Let $\dfrac{4t-1}{3-t^2-4t} = \dfrac{A}{t-3} + \dfrac{B}{t-1}$ Thus $4t-1 = A(t-1) + b(t-3)$

Put $t = 1\ then B = -\dfrac{3}{2}$ Put $t = 3\ then\ A = \dfrac{11}{2}$

$$\therefore \qquad I = \frac{11}{2}\int\left(\frac{1}{t-3}\right)dt - \frac{3}{2}\int\frac{dt}{t-1} = \frac{11}{2}\log|t-3| - \frac{3}{2}\log|t-1| + C$$

$$= \frac{11}{2}\log|\sin\theta - 3| - \frac{3}{2}\log|\sin\theta - 1| + C$$

$$= -\frac{1}{3}\int\frac{dt}{1+t} + \frac{1}{6}\int\frac{(2t-1)\,dt}{t^2-t+1} + \frac{1}{2}\int\frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log\left|t^2-t+1\right| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log\left|t^2-t+1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log\left|\tan^2\theta - \tan\theta + 1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$$

CHECK YOUR PROGRESS 30.8

Evaluate the following:

1. (a) $\int\sqrt{4x^2-5}\,dx$ (b) $\int\sqrt{x^2+3x}\,dx$ (c) $\sqrt{3-2x-2x^2}\,dx$

2. (a) $\int\dfrac{x+1}{(x-2)(x-3)}\,dx$ (b) $\int\dfrac{x}{x^2-16}\,dx$

3. (a) $\int\dfrac{x^2}{x^2-4}\,dx$ (b) $\int\dfrac{2x^2+x+1}{(x-1)^2(x+2)}\,dx$

4. $\int \frac{x^2+x+1}{(x-1)^3}dx$

5. (a) $\int \frac{\sin x}{\sin 4x}dx$ (b) $\int \frac{1-\cos x}{\cos x(1+\cos x)}dx$

LET US SUM UP

- Integration is the inverse of differentiation
- Standard form of some inddefinite integrals

(a) $\int x^n dx = \frac{x^{n+1}}{n+1}+C\ (n \neq -1)$

(b) $\int \frac{1}{x}dx = \log|x|+C$

(c) $\int \sin x\, ax = -\cos x + C$

(d) $\int \cos x\ dx = \sin x + C$

(e) $\int \sec^2 x\, dx = \tan x + C$

(f) $\int \cos ec^2 x\, dx = -\cot x + C$

(g) $\int \sec x \tan x dx = \sec x + C$

(h) $\int \cos ecx \cot x\, dx = -\cos ecx + C$

(i) $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$

(j) $\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$

(k) $\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x + C$

(l) $\int e^x dx = e^x + C$

(m) $\int a^x\ dx = \frac{a^x}{\log}+C\ (a>0\, and\, a \neq 1)$

- Properties of indefinite integrals

(a) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

(b) $\int kf(x)\,dx \quad = k\int f(x)\,dx$

(i) $\int (ax+b)^n dx \quad = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C \ (n \neq -1)$

(ii) $\int \frac{1}{ax+b}dx \quad = \frac{1}{a}\log|ax+b| + C$

(iii) $\int \sin(ax+b)\,dx \quad = \frac{-1}{a}\cos(ax+b) + C$

(iv) $\int \cos(ax+b)\,dx \quad = \frac{1}{a}\sin(ax+b) + C$

(v) $\int \sec^2(ax+b)\,dx \quad = \frac{1}{a}\tan(ax+b) + C$

(vi) $\int cosec^2(ax+b)\,dx \quad = \frac{1}{a}\cot(ax+b) + C$

(vii) $\int \sec(ax+b)\tan(ax+b)\,dx \quad = \frac{1}{a}\sec(ax+b) + C$

(viii) $\int cosec(ax+b)\cot(ax+b)\,dx = \frac{1}{a}cosec(ax+b) + C$

(ix) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$

(i) $\int \tan x\,dx \quad = -\log|\cos x| + C = \log|\sec x| + C$

(ii) $\int \cot x\,dx \quad = \log|\sin x| + C$

(iii) $\int \sec x\,dx \quad = \log|\sec x + \tan x| + C$

(iv) $\int cosec\,x\,dx \quad = \log|cosec\,x - \cot x| + C$

• (i) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(ii) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a-x}{a+x}\right| + C$

(iii) $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(iv) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$

(v) $$\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right| + C$$

(vi) $$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$$

- Integral of the product of two functions

I function $\times$ Integral of II function $-\int$[Derivative of I function $\times$ Integral of II function]dx

- $$\int e^x\left[f(x)+f(x)\right]dx = e^x f(x) + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]+C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + C$$

- Rational fractions are of following two types:

(i) Proper, where degree of variable of numerator < denominator.

(ii) Improper, where degree of variable of numerator $\geq$ denominator.

- If g (x) is a proper fraction $\frac{f(x)}{g(x)}$ can be resolved into real factors, then $\frac{f(x)}{g(x)}$ can be written in the following form :

Factors in denominator	Corresponding partial fraction
a x + b	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$
$(ax+b)^3$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}$

where A,B,C, D are arbitary constants.

SUPPORTIVE WEB SITES

http://www.bbc.co.uk/education/asguru/maths/12methods/04integration/index.shtml

http://en.wiktionary.org/wiki/integration

http://www.sosmath.com/calculus/integration/byparts/byparts....

TERMINAL EXERCISE

Integrate the following functions w.r.t.x:

1. $\dfrac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}$

2. $\sqrt{1+\sin 2x}$

3. $\dfrac{\cos 2x}{\cos^2 x\sin^2 x}$

4. $(\tan x-\cot x)^2$

5. $\dfrac{4}{1+x^2}-\dfrac{1}{\sqrt{1-x^2}}$

6. $\dfrac{2\sin^2 x}{1+\cos 2x}$

7. $\dfrac{2\cos^2 x}{1-\cos 2x}$

8. $\left(\sin\dfrac{x}{2}+\cos\dfrac{x}{2}\right)^2$

9. $\left(\cos\dfrac{x}{2}-\sin\dfrac{x}{2}\right)^2$

10. $\cos(7x-\pi)$

11. $\sin(3x+4)$

12. $\cos^2(2x+b)$

13. $\displaystyle\int\frac{dx}{\sin x-\cos x}$

14. $\displaystyle\int\frac{1}{(1+x^2)\tan^{-1}x}dx$

15. $\displaystyle\int\frac{\cos ec}{\log\left(\tan\frac{x}{2}\right)}dx$

16. $\displaystyle\int\frac{\cot}{3+4\log\sin x}dx$

17. $\displaystyle\int\frac{dx}{\sin 2x\log\tan x}$

18. $\displaystyle\int\frac{e^x+1}{e^x-1}dx$

19. $\displaystyle\int\sec^4 x\tan x\,dx$

20. $\displaystyle\int e^2\sin e^x dx$

21. $\displaystyle\int\frac{x\,dx}{\sqrt{2x^2+3}}$

22. $\displaystyle\int\frac{\sec^2 x}{\sqrt{\tan x}}dx$

23. $\displaystyle\int\sqrt{25-9x^2}\,dx$

24. $\displaystyle\int\sqrt{2ax-x^2dx}$

25. $\displaystyle\int\sqrt{3x^2+4dx}$

26. $\displaystyle\int\sqrt{1+9x^2}\,dx$

27. $\displaystyle\int\frac{x^2dx}{\sqrt{x^2-a^2}}$

28. $\displaystyle\int\frac{dx}{\sin^2 x+4\cos^2 x}$

29. $\displaystyle\int\frac{dx}{2+\cos x}$

30. $\displaystyle\int\frac{dx}{x^2-6x+13}$

31. $\displaystyle\int\frac{dx}{1+3\sin^2 x}$

32. $\displaystyle\int\frac{x^2}{x^2-a^2}dx$

33. $\displaystyle\int\frac{dx}{x\sqrt{9+x^4}}$

34. $\displaystyle\int\frac{\sin}{\sin 3x}dx$

35. $\displaystyle\int\frac{dx}{1-4\cos^2 x}$

36. $\displaystyle\int\sec^2(ax+b)\,dx$

37. $\int \frac{dx}{x(2+\log x)}$

38. $\int \frac{x^5}{1+x^6} dx$

39. $\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$

40. $\int \frac{\cot x}{\log \sin x} dx$

41. $\int \frac{\sec^2 x}{a + b\tan x} dx$

42. $\int \frac{\sin x}{1+\cos} dx$

43. $\int \cos^2 x \, dx$

44. $\int \sin^3 x \, dx$

45. $\int \sin 5x \sin 3x \, dx$

46. $\int \sin^2 x \cos^3 x \, dx$

47. $\int \sin^4 x \, dx$

48. $\int \frac{1}{1+\sin x} dx$

49. $\int \tan^3 x \, dx$

50. $\int \frac{\cos x - \sin x}{1+\sin 2x} dx$

51. $\int \frac{\operatorname{cosec}^2 x}{1+\cot x} dx$

52. $\int \frac{1+x+\cos 2x}{x^2+\sin 2x+2x} dx$

53. $\int \frac{\sec\theta \operatorname{cosec}\theta \, d\theta}{\log \tan\theta}$, 54. $\int \frac{\cot\theta \, d\theta}{\log \sin\theta}$

55. $\int \frac{dx}{1+4x^2}$

56. $\int \frac{1-\tan\theta}{1+\tan\theta} d\theta$

57. $\int \frac{1}{x^2} e^{\frac{-1}{x}} dx$

58. $\int \frac{\sin x \cos x \, dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

59. $\int \frac{dx}{\sin x + \cos x}$

60. $\int e^x \left(\cos^{-1} x - \frac{1}{\sqrt{1-x^2}} \right) dx$

61. $\int e^x \left(\frac{\sin x + \cos x}{\cos^2 x} \right) dx$

62. $\int \tan^{-1} \sqrt{\frac{1-\cos x}{1+\cos x}} dx$

63. $\int \cos \left[2\cot^{-1} \left(\sqrt{\frac{1-x}{1+x}} \right) \right] dx$

64. $\int \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} dx$

65. $\int \sqrt{x} \log x \, dx$

66. $\int e^x (1+x) \log(xe^x) \, dx$

67. $\int \frac{\log x}{(1+x)^2} dx$

68. $\int e^x \sin^2 x \, dx$

69. $\int \cos(\log x) \, dx$

70. $\int \log(x+1) \, dx$

71. $\int \frac{x^2+1}{(x-1)^2(x+3)} dx$

72. $\int \frac{\sin\theta \cos\theta}{\cos^2\theta - \cos\theta - 2} dx$

73. $\int \frac{dx}{x(x^5+1)}$

74. $\int \frac{x^2+1}{(x^2+2)(2x^2+1)} dx$

75. $\int \frac{\log x}{x(1+\log x)(2+\log x)} dx$

76. $\int \frac{dx}{1-e^x}$

ANSWERS

CHECK YOUR PROGRESS 30.1

1. $\frac{2}{7}x^{\frac{7}{2}}+1, \frac{2}{7}x^{\frac{7}{2}}+2, \frac{2}{7}x^{\frac{7}{2}}+3, \frac{2}{7}x^{\frac{7}{2}}+4, \frac{2}{7}x^{\frac{7}{2}}+5$

2. (a) $\frac{x^6}{6}+C$ (b) $\sin x + C$ (c) 0

3. (a) $\frac{x^7}{7}+C$ (b) $\frac{1}{6x^6}+C$ (c) $\log|x|+C$

   (d) $\frac{\left(\frac{3}{5}\right)^x}{\log\left(\frac{3}{5}\right)}+C$ (e) $\frac{3}{4}x^{\frac{4}{3}}+C$ (f) $\frac{-1}{8x^8}+C$

   (g) $2\sqrt{x}+C$ (h) $9x^{\frac{1}{9}}+C$

4. (a) $-\operatorname{cosec}\theta + C$ (b) $\sec\theta + C$

   (c) $\tan\theta + C$ (d) $-\cot\theta + C$

CHECK YOUR PROGRESS 30.2

1. (a) $\frac{x^2}{2}+\frac{1}{2}x+C$ (b) $-x+\tan^{-1}x+C$

   (c) $x^{10}-\frac{2}{3}x^{\frac{3}{2}}+2\sqrt{x}+C$ (d) $-\frac{1}{x^5}-\frac{3}{4x^4}+\frac{2}{3x^3}+\frac{7}{x}-8x+C$

   (e) $\frac{x^3}{3}-x-\tan^{-1}x+C$ (f) $\frac{x^2}{2}+4x+4\log x+C$

2. (a) $\frac{1}{2}\tan x + C$ (b) $\tan x - x + C$

   (c) $-2\operatorname{cosec} x + C$ (d) $-\frac{1}{2}\cot x + C$

   (e) $-\sec x + C$ (f) $-\cot x + \operatorname{cosec} x + C$

3. (a) $\sqrt{2}\sin x + C$ (b) $-\sqrt{2}\cos x + C$

(c) $-\frac{1}{2}\cot x + C$

4. (a) $\frac{2}{3}(x+2)^{\frac{3}{2}} + C$

CHECK YOUR PROGRESS 30.3

1. (a) $\frac{1}{5}\cos(4-5x) + C$ (b) $\frac{1}{3}\tan(2+3x) + C$

(c) $\log\left|\sec\left(x+\frac{\pi}{4}\right) + \tan\left(x+\frac{\pi}{4}\right)\right| + C$

(d) $\frac{1}{4}\sin(4x+5) + C$ (e) $\frac{1}{3}\sec(3x+5) + C$

(f) $-\frac{1}{5}\operatorname{cosec}(3+5x) + C$

2. (a) $\frac{1}{12(3-4x)^3} + C$ (b) $\frac{1}{5}(x+1)^5 + C$

(c) $-\frac{1}{77}(4-7x)^{11} + C$ (d) $\frac{1}{16}(4x-5)^4 + C$

(e) $\frac{1}{3}\log|3x-5| + C$ (f) $-\frac{2}{9}\sqrt{5-9x} + C$

(g) $\frac{1}{6}(2x+1)^3 + C$ (h) $\log|x+1| + C$

3. (a) $\frac{1}{2}e^{2x+1} + C$ (b) $-\frac{1}{8}e^{3-8x} + C$

(c) $-\frac{1}{4e^{(7+4x)}} + C$

4. (a) $\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$ (b) $\frac{1}{32}\left(-\frac{3}{2}\cos 2x + \frac{1}{6}\cos 6x\right) + C$

(c) $\frac{1}{2}\left(-\frac{\cos 7x}{7} - \cos x\right) + C$ (d) $\frac{1}{2}\left(\frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right) + C$

CHECK YOUR PROGRESS 30.4

1. (a) $\frac{1}{6}\log\left|3x^2-2\right|+C$ (b) $\log\left|x^2+x+1\right|+C$

(c) $\log\left|x^2+9x+30\right|+C$ (d) $\frac{1}{3}\log\left|x^3+3x+3\right|+C$

(e) $\log\left|x^2+x-5\right|+C$ (f) $2\log\left|5+\sqrt{x}\right|+C$

(g) $\log\left|8+\log x\right|+C$

2. (a) $\frac{1}{b}\log\left|a+be^x\right|+C$ (b) $\tan^{-1}\left(e^x\right)+C$

CHECK YOUR PROGRESS 30.5

1. (a) $x+\frac{3}{2}\log\left|\frac{x-3}{x+3}\right|+C$ (b) $\tan^{-1}\left(e^x\right)+C$

(c) $\frac{1}{2}\tan^{-1}\left(x^2\right)+C$ (d) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right)+C$

(e) $\frac{1}{2}\tan^{-1}\left(2\tan x\right)+C$ (f) $\sin^{-1}\left(\frac{x+1}{2}\right)+C$

(g) $\frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$ (h) $\sin^{-1}\left(\frac{x+2}{3}\right)+C$

(i) $\frac{1}{2\sqrt{3}}\sec^{-1}\frac{x}{2}+C$ (j) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2\theta-1}{\sqrt{2}\tan\theta}\right)+C$

(k) $\log\left|e^x+\sqrt{1+e^{2x}}\right|+C$ (l) $\sin^{-1}x-\sqrt{1-x^2}+C$

(m) $\sin^{-1}\left(\frac{x-a}{a}\right)+C$ (n) $\frac{1}{4}\sin^{-1}\left(\frac{4}{3}x^3\right)+C$

(o) $\sqrt{x^2+1}+\log\left|x+\sqrt{x^2+1}\right|+C$

(p) $\frac{1}{2}\log\left|\frac{2x+\sqrt{9+4x^2}}{2}\right|+C$

(q) $-\frac{1}{2}\log\left|2\cos\theta+\sqrt{4\cos^2\theta-1}\right|+C$

(r) $\log\left|\tan x+\sqrt{\tan^2 x-4}\right|+C$

(s) $\tan^{-1}\left(\frac{x+2}{1}\right)+C$ (t) $\frac{1}{4}\log\left|x+\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right|+C$

CHECK YOUR PROGRESS 30.6

1. (a) $-x\cos x+\sin+C$

(b) $\frac{1}{2}\left(1+x^2\right)\sin 2x+\frac{x\cos 2x}{2}-\frac{\sin 2x}{4}+C$

(c) $\frac{-x\cos 2x}{2}+\frac{1}{2}\frac{\sin 2x}{2}+C$

2. (a) $x\tan x-\log\left|\sec x\right|-x+C$

(b) $\frac{1}{6}x^3-\frac{1}{4}x^2\sin 2x-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C$

3. (a) $\frac{x^4\log 2x}{4}-\frac{x^4}{16}+C$ (b) $\left(x-\frac{x^3}{3}\right)\log x-x+\frac{x^3}{9}+C$

(c) $x\left(\log x\right)^2-2x\log x+2x+C$

4. (a) $\frac{x^{1-n}}{1-n}\log x-\frac{x^{1-n}}{\left(1-n\right)^2}+C$ (b) $\log x.\left[\log\left(\log x\right)-1\right]+C$

5. (a) $e^{3x}\left[\frac{x^2}{3}-\frac{2x}{9}+\frac{2}{27}\right]+C$ (b) $x\frac{e^{4x}}{4}-x\frac{e^{4x}}{16}+C$

6. (a) $\frac{x^2}{2}\left[\left(\log x\right)^2-\log x+\frac{1}{2}\right]+C$

7. (a) $x\sec^{-1}x-\log\left|x+\sqrt{x^2-1}\right|+C$

(b) $\frac{x^2}{2}\cot^{-1}x+\frac{x}{2}+\frac{1}{2}\cot^{-1}x+C$

CHECK YOUR PROGRESS 30.7

1. (a) $e^x \sec x0 + C$ (b) $e^x \log|\sec x + \tan x| + C$

2. (a) $\frac{1}{x}e^x + C$ (b) $e^x \sin^{-1} x + C$

3. $\frac{e^x}{(1+x)^2} + C$

4. $\frac{e^X}{1+x} + C$

5. $x \tan \frac{x}{2} + C$

6. $\frac{1}{5}e^x(\sin 2x - 2\cos 2x) + C$

CHECK YOUR PROGRESS 30.8

1. (a) $x\sqrt{x^2 - \frac{5}{4}} - \frac{5}{4}\log\left|x + \sqrt{x^2 - \frac{5}{4}}\right| + C$

(b) $\frac{(2x+3)}{4}\sqrt{x^2 + 3x} - \frac{9}{8}\log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^{2+}3x}\right| + C$

(c) $\frac{1}{4}(2x+1)\sqrt{3 - 2x - 2x^2} + \frac{7}{4\sqrt{2}}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C$

2. (a) $4\log|x-3| - 3\log|x-2| + C$

(b) $\frac{1}{2}\log|x-4| + \log|x+4| + C$

3. (a) $\frac{x^2}{2} - 2\left[\log|x-2|| + \log||x+2|\right] + c$

(b) $\frac{11}{9}\log|x-1| + \frac{7}{9}\log(x+2) - \frac{4}{3(x-1)} + C$

4. $\log|x-1| - \frac{3}{(x-1)} - \frac{3}{2(x-1)^2} + C$

5. (a) $\frac{1}{8}\log|1 - \sin x| - \frac{1}{8}|1 + \sin x|$

$-\frac{1}{4\sqrt{2}}\log\left|1 - \sqrt{2}\sin x\right| + \frac{1}{4\sqrt{2}}\log\left|1 + \sqrt{2}\sin x\right| + C$

(b) $\log\left|\sec x + \tan x\right| - 2\tan\frac{x}{2} + C$

TERMINAL EXERCISE

1. $\sec x - \operatorname{cosec} x + C$

2. $\sin x - \cos x + C$

3. $-\cot x - \tan x + C$

4. $\tan x - \cot x - 4x + C$

5. $4\tan^{-1} x - \sin^{-1} x + C$

6. $\tan x - x + C$

7. $-\cot x - x + C$

8. $x - \cos x + C$

9. $x + \cos x + C$

10. $\frac{\sin(7x-\pi)}{7} + C$

11. $\frac{-\cos(3x+4)}{3} + C$

12. $\frac{\tan(2x+b)}{2} + C$

13. $\frac{1}{\sqrt{2}}\log\left|\operatorname{cosec}\left(x-\frac{\pi}{4}\right) - \cot\left(x-\frac{\pi}{4}\right)\right| + C$

14. $\log\left|\tan^{-1} x\right| + C$

15. $\log\left|\log\tan\frac{x}{2}\right| + C$

16. $\frac{1}{4}\log\left|3 + 4\log\sin x\right| + C$

17. $\frac{1}{2}\log\left|\log\tan x\right| + C$

18. $2\log\left|e^{\frac{x}{2}} - e^{\frac{-x}{2}}\right| + C$

19. $\frac{1}{4}\sec^4 x + C$

20. $-\cos e^x + C$

21. $\frac{\sqrt{2x^2+3}}{2} + C$

22. $2\sqrt{\tan x} + C$

23. $\frac{1}{6}x\sqrt{(25-9x^2)} + \frac{25}{6}\sin^{-1}\left(\frac{3}{5}x\right) + C$

24. $\frac{1}{2}(x-a)\sqrt{2ax-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x-a}{a}\right) + C$

25. $\frac{x\sqrt{3x^2+4}}{2} + \frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{3}x + \sqrt{x^2+4}}{2}\right| + C$

26. $\frac{x\sqrt{9x^2+1}}{2}+\frac{1}{6}\log\left|3x+\sqrt{1+9x^2}\right|+C$

27. $\left[\frac{1}{2}x\sqrt{x^2-a^2}+\frac{1}{2}a^2\log\left|x+\sqrt{x^2-a^2}\right|\right]+C$

28. $\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right)+C$

29. $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right]+C$

30. $\frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+C$

31. $\frac{1}{2}\tan^{-1}(2\tan x)+C$

32. $x+\frac{a}{2}\log\left|\frac{x-a}{x+a}\right|+C$

33. $\frac{1}{12}\log\left|\frac{\sqrt{9+x^4}-3}{\sqrt{9+x^4}+3}\right|+C$

34. $\frac{2}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|+C$

35. $\frac{1}{2\sqrt{2}}\log\left|\frac{\tan x-\sqrt{2}}{\tan x+\sqrt{2}}\right|+C$

36. $\frac{1}{a}\tan(ax+b)+C$

37. $\log\left|(2+\log x)\right|+C$

38. $\frac{1}{6}\log(1+x^6)+C$

39. $\log\left|\sin x+\cos x\right|+C$

40. $\log\left|\log(\sin x)\right|+C$

41. $\frac{1}{b}\log\left|a+b\tan x\right|+C$

42. $-\log\left|1+\cos x\right|+C$

43. $\frac{1}{2}\frac{\sin 2x}{2}+\frac{1}{2}x+C$

44. $-\cos x+\frac{\cos^3 x}{3}+C$

45. $\frac{1}{2}\frac{\sin 2x}{2}-\frac{1}{2}\frac{\sin 8x}{8}+C$

46. $\frac{1}{3}\sin^3 x-\frac{\sin^5 x}{5}+C$

47. $\frac{1}{32}[12x-8\sin 2x+\sin 4x]+C$

48. $\tan x-\sec x+C$

49 $\frac{\tan^2 x}{2}+\log\left|\cos x\right|+C$.

50. $\frac{-1}{\cos x+\sin x}+C$

51. $\log\left|\frac{1}{1+\cot x}\right|+C$

52. $\frac{1}{2}\log\left|x^2+\sin 2x+2x\right|+C$

53 $\log\left|\tan\theta\right|+C$.

54. $\log\left|\log\sin\theta\right|+C$

55. $\frac{1}{2}\tan^{-1}2x$

56. $\log\left|\cos\theta+\sin\theta\right|+C$

57. $e^{\frac{1}{x}}+C$

58. $\frac{1}{2\left(a^2-b^2\right)}\log\left|a^2\sin^2 x+b^2\cos^2 x\right|+C$

59. $\frac{1}{\sqrt{2}}\log\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|+C$

60. $e^x\cos^{-1}x+C$

61. $e^x\sec x+C$

62. $\frac{1}{4}x^2+C$

63. $-\frac{1}{2}x^2+C$

64. $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}+\frac{1}{2}\log\left|1-x^2\right|+C$

65. $\frac{2}{3}x^{\frac{3}{2}}\left(\log x-\frac{2}{3}\right)+C$

66. $xe^x\left[\log\left(xe^x\right)-1\right]+C$

67. $-\frac{1}{1+x}\log|x|+\log|x|-\log|x+1|+c$

68. $\frac{1}{2}e^x-\frac{e^x}{10}\left(2\sin 2x+\cos 2x\right)+C$

69. $\frac{x}{2}\left[\cos\left(\log x\right)+\sin\left(\log x\right)\right]+C$

70. $x\log|x+1|-x+\log|x+1|+C$

71. $\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+\frac{5}{8}\log|x+3|+C$

72. $-\frac{2}{3}\log|\cos\theta-2|-\frac{1}{3}\log|\cos\theta+1|+C$

73. $\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+C$

74. $\frac{1}{3\sqrt{2}}\left[\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\tan^{-1}\left(\sqrt{2}x\right)\right]+C$

75. $\log\left|\frac{(2+\log x)^2}{1+\log x}\right|+C$

76. $\log\left|\frac{e^x}{1-e^x}\right|+C$